

Estimation of the Generalized Rayleigh Distribution Parameters

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Abstract. This paper presents estimations of the generalized Rayleigh distribution model based on grouped and censored data. The maximum likelihood method is used to derive point and asymptotic confidence estimates of the unknown parameters. The results obtained in this paper generalize some of those available in the literature. Finally, we test whether the current model fits a set of real data better than other models.

Key Words : *Maximum likelihood estimate, grouped data.*

1. INTRODUCTION

The problem of estimating the unknown parameters in statistical distributions used to study a certain phenomenon is one of the important problems facing constantly those who are interested in applied statistics. This paper considers the estimations of the unknown parameters of the generalized Rayleigh distribution. This distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences.

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There are several types of data that arise in every day life. Among these, there are simple data, grouped data, truncated data, censored data, progressively censored data, and so on.

The exponential distribution is one of the most frequently used distribution in reliability theory and applications. This is because it has simple mathematical expressions. The mean and its confidence limit for this distribution have been estimated based on grouped and censored data by Seo and Yum (1993) and Chen and Mi (1996), respectively. Nevertheless, the exponential distribution does not always fit well a given set of real data. The reason is that it has a constant failure rate. Abuammoh and Sarhan (2007) used grouped and censored data to estimate the parameters of the generalized exponential distribution. They found that it fits a set of real data better than the exponential distribution.

Other distributions have been used in reliability theory. Some were taken from the twelve different forms of distributions introduced by Burr (1942) to model data. Among those different distributions, Burr-Type X and Burr-Type XII received the most attention. There is a thorough analysis of Burr-Type XII distribution in Rodriguez (1977). see also Wingo (1993) for a nice account of it.

The two-parameter (scale and shape) Burr-Type X distribution is also called generalized Rayleigh distribution. Several aspects of the one-parameter (scale parameter equals one) generalized Rayleigh distribution were studied by Sartawi and Abu-Salih (1991), Jaheen (1995, 1996), Ahmad et al. (1997), Raqab (1998) and Surles and Padgett (1998). Recently Surles and Padgett (2001) observed that the two-parameter generalized Rayleigh distribution can be used quite effectively in modeling strength and general lifetime data. Kundu and Raqab (2005) used different methods to estimate the parameters of the generalized Rayleigh on simple data.

Our aim in this paper is to estimate the unknown parameters of the generalized Rayleigh distribution based on grouped and censored data. The maximum likelihood method is used to derive point and asymptotic confidence estimates of the unknown parameters. The results obtained in this paper generalize some of those available in the literature. Finally, we test whether the current model fits a set of real data better than other models.

The rest of the paper is organized as follows. In Section 2, we present the definition and some characteristics of the generalized Rayleigh distribution. Point and interval estimations of the unknown parameters are discussed in Section 3. In Section 4 an illustrative example is presented to explain how the generalized Rayleigh distribution fits a set of real data better than other distributions. A conclusion is drawn in Section 5.

2. THE DISTRIBUTION

Let T be a random variable with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. Setting $\theta = (\alpha, \beta)$, the cumulative distribution function of T is given by

$$F(t; \theta) = \left(1 - e^{-\beta t^2}\right)^\alpha, \quad t \geq 0.$$

The distribution of T is the so-called generalized Rayleigh distribution. Its survival function is given by

$$S(t; \theta) = 1 - \left(1 - e^{-\beta t^2}\right)^\alpha, \quad t \geq 0, \tag{2.1}$$

while the probability density function (pdf) is given by

$$f(t; \theta) = 2\alpha\beta t e^{-\beta t^2} \left(1 - e^{-\beta t^2}\right)^{\alpha-1}, \quad t \geq 0. \tag{2.2}$$

Figure 2.1 shows the variations of the pdf for $\beta = 1$ and $\alpha = 0.25$ (left) and $\alpha = 1$ (right).

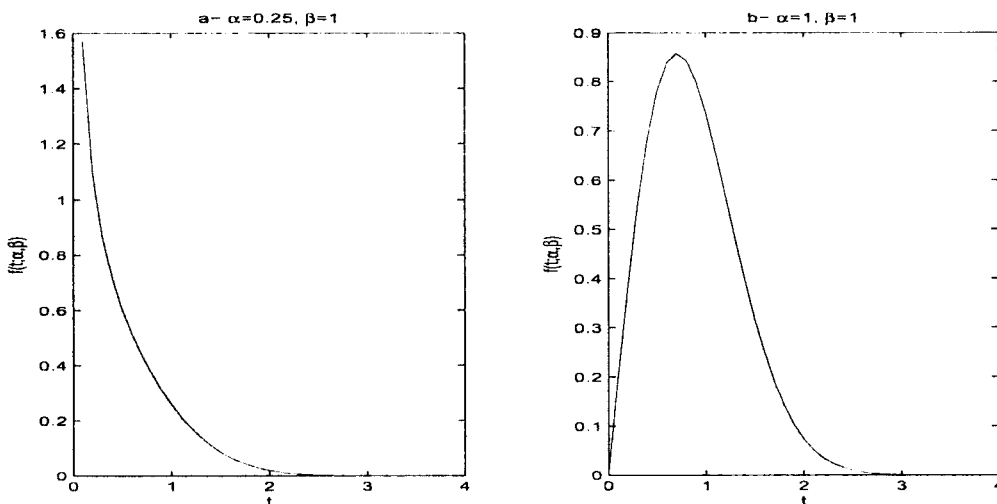


Figure 2.1. Probability density function of the generalized Rayleigh distribution.

In fact it was shown by Raqab and Kundu (2006) that the pdf of a generalized Rayleigh distribution is a decreasing function for $\alpha \leq 0.5$ and it is a right-skewed unimodal function for $\alpha > 0.5$. The hazard rate function is given by

$$h(t; \theta) = \frac{2\alpha\beta t e^{-\beta t^2} \left(1 - e^{-\beta t^2}\right)^{\alpha-1}}{1 - \left(1 - e^{-\beta t^2}\right)^\alpha}, \quad t \geq 0. \tag{2.3}$$

Figure 2.2 shows the variations of the hazard rate function for $\beta = 1$ and $\alpha = 0.25$ (left) and $\alpha = 1$ (right).

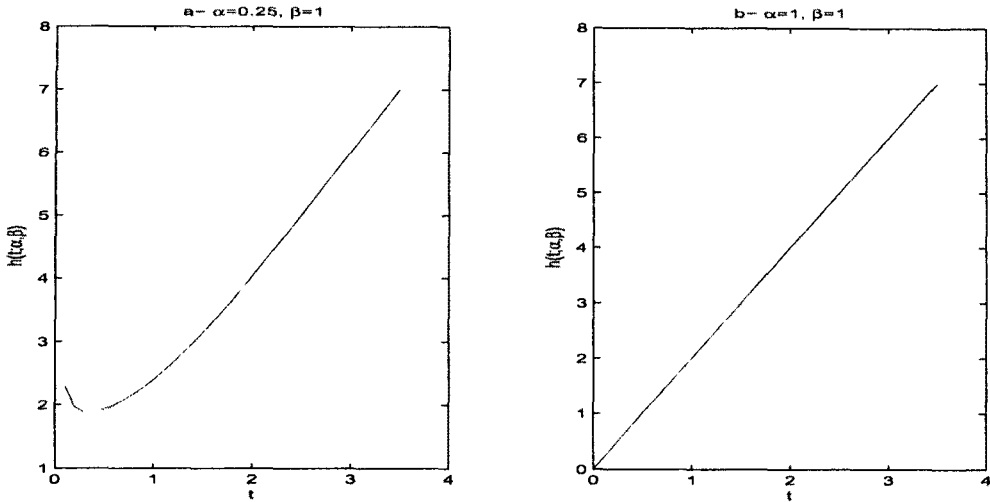


Figure 2.2. Hazard rate function of the generalized Rayleigh distribution.

It was also shown by Raqab and Kundu (2006) that the hazard rate function of a generalized Rayleigh distribution is bathtub type for $\alpha \leq 0.5$ and it is an increasing function for $\alpha > 0.5$. The mean time to failure of the generalized Rayleigh distribution is given by

$$MTTF = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \sum_{i=1}^{\infty} \binom{\alpha}{i} (-1)^{i+1} i^{-\frac{1}{2}}. \tag{2.4}$$

3. PARAMETERS ESTIMATION

Consider n independent and identical components put on test. Assume that the lifetimes of these components follow the generalized Rayleigh distribution expressed in the previous section. Let $\mathbf{t} = (t_1, \dots, t_k)$, where $t_1 < \dots < t_k$, denote the predetermined inspection times with t_k representing the completion time of the test. Also, let $t_0 = 0$ and $t_{k+1} = \infty$. For $i = 1, \dots, k$, denote by n_i the number of failures recorded in the time interval $(t_{i-1}, t_i]$ and by n_{k+1} the number of censored units, that is units that have not failed by the end of test.

3.1 Maximum Likelihood Estimators Using the data described above, the likelihood function is given by

$$L(\mathbf{t}; \theta) = C \prod_{i=1}^k [P\{t_{i-1} < T \leq t_i\}]^{n_i} [P\{T > t_k\}]^{n_{k+1}}, \tag{3.1}$$

where $C = \frac{n!}{\prod_{\ell=1}^{k+1} n_{\ell}!}$ is a constant with respect to the parameters α and β . Since,

$$P\{t_{i-1} < T \leq t_i\} = F(t_i) - F(t_{i-1}),$$

and

$$P\{T > t_k\} = 1 - F(t_k),$$

therefore, (3.1) can be written as

$$L(\mathbf{t}; \theta) = C [S(t_k)]^{n_{k+1}} \prod_{i=1}^k [S(t_{i-1}) - S(t_i)]^{n_i}. \tag{3.2}$$

Substituting from (2.1) into (3.2), one gets

$$L(\mathbf{t}; \theta) = C \left[1 - \left(1 - e^{-\beta t_k^2}\right)^\alpha\right]^{n_{k+1}} \prod_{i=1}^k \left[\left(1 - e^{-\beta t_i^2}\right)^\alpha - \left(1 - e^{-\beta t_{i-1}^2}\right)^\alpha\right]^{n_i}. \tag{3.3}$$

Hence, the log likelihood function is given by

$$\mathcal{L}(\mathbf{t}; \theta) = \ln C + n_{k+1} \ln \left[1 - \left(1 - e^{-\beta t_k^2}\right)^\alpha\right] + \sum_{i=1}^k n_i \ln \left[\left(1 - e^{-\beta t_i^2}\right)^\alpha - \left(1 - e^{-\beta t_{i-1}^2}\right)^\alpha\right]. \tag{3.4}$$

We let

$$A_i(\beta) = \begin{cases} 0, & i = 0, \\ 1 - e^{-\beta t_i^2}, & i = 1, \dots, k, \\ 1, & i = k + 1, \end{cases}$$

and note, for future use, that

$$\frac{dA_i(\beta)}{d\beta} = t_i^2 e^{-\beta t_i^2}, \quad i = 1, \dots, k.$$

The log likelihood function (3.4) becomes

$$\mathcal{L}(\mathbf{t}; \theta) = \ln C + \sum_{i=1}^{k+1} n_i \ln \left\{ \left[A_i(\beta)\right]^\alpha - \left[A_{i-1}(\beta)\right]^\alpha \right\}. \tag{3.5}$$

We also let

$$D^{(i)}(\theta) = \left[A_i(\beta)\right]^\alpha - \left[A_{i-1}(\beta)\right]^\alpha, \quad i = 1, \dots, k + 1,$$

so that the log likelihood function (3.5) can be written in a more compact way as

$$\mathcal{L}(\mathbf{t}; \theta) = \ln C + \sum_{i=1}^{k+1} n_i \ln D^{(i)}(\theta). \tag{3.6}$$

For ease of notation, we will denote, for any function $f(x, y)$, the first partial derivatives by f_x, f_y , and the second partial derivatives by $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. To write the log normal equations, compute the first partial derivatives of the log likelihood function as follows:

$$\mathcal{L}_\alpha = \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_\alpha^{(i)}}{D^{(i)}(\theta)}, \quad (3.7)$$

$$\mathcal{L}_\beta = \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_\beta^{(i)}}{D^{(i)}(\theta)}, \quad (3.8)$$

where

$$\mathcal{D}_\alpha^{(i)} = \begin{cases} [A_1(\beta)]^\alpha \ln A_1(\beta), & i = 1, \\ [A_i(\beta)]^\alpha \ln A_i(\beta) - [A_{i-1}(\beta)]^\alpha \ln A_{i-1}(\beta), & i = 2, \dots, k+1, \end{cases}$$

and

$$\mathcal{D}_\beta^{(i)} = \begin{cases} \alpha \{t_1^2 e^{-\beta t_1^2} [A_1(\beta)]^{\alpha-1}\}, & i = 1, \\ \alpha \{t_i^2 e^{-\beta t_i^2} [A_i(\beta)]^{\alpha-1} - t_{i-1}^2 e^{-\beta t_{i-1}^2} [A_{i-1}(\beta)]^{\alpha-1}\}, & i = 2, \dots, k+1. \end{cases}$$

The maximum likelihood estimators of α and β can be found by solving the system of equations $\mathcal{L}_\alpha = \mathcal{L}_\beta = 0$. Although the proposed estimators cannot be expressed in closed forms, they can easily be obtained through the use of an appropriate numerical technique.

3.2 Asymptotic Confidence Bounds

Applying the usual large sample approximation, the maximum likelihood estimators of θ can be treated as being approximately bivariate normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$(\hat{\theta} - \theta) \rightarrow N_2(0, \mathbf{I}^{-1}(\hat{\theta})), \quad (3.9)$$

where $\mathbf{I}^{-1}(\hat{\theta})$ is the variance-covariance matrix of the unknown parameters $\theta = (\alpha, \beta)$. The elements of the 2×2 matrix \mathbf{I}^{-1} , $I_{ij}(\hat{\theta})$, $i, j = 1, 2$, can be approximated by $I_{ij}(\hat{\theta})$, where

$$I_{ij}(\hat{\theta}) = -\mathcal{L}_{\theta_i \theta_j} \Big|_{\theta = \hat{\theta}}. \quad (3.10)$$

From (3.7) and (3.8), the second partial derivatives of the log likelihood function are found to be

$$\begin{aligned} \mathcal{L}_{\alpha\alpha} &= \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_{\alpha\alpha}^{(i)} D^{(i)}(\theta) - [\mathcal{D}_{\alpha}^{(i)}]^2}{[D^{(i)}(\theta)]^2}, \\ \mathcal{L}_{\beta\beta} &= \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_{\beta\beta}^{(i)} D^{(i)}(\theta) - [\mathcal{D}_{\beta}^{(i)}]^2}{[D^{(i)}(\theta)]^2}, \\ \mathcal{L}_{\alpha\beta} &= \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_{\alpha\beta}^{(i)} D^{(i)}(\theta) - \mathcal{D}_{\alpha}^{(i)} \mathcal{D}_{\beta}^{(i)}}{[D^{(i)}(\theta)]^2}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_{\alpha\alpha}^{(i)} &= \begin{cases} [A_1(\beta)]^{\alpha-1} \{A_1(\beta) \ln^2 A_1(\beta) + t_1^2 e^{-\beta t_1^2}\}, & i = 1, \\ [A_i(\beta)]^{\alpha-1} \{A_i(\beta) \ln^2 A_i(\beta) + t_i^2 e^{-\beta t_i^2}\} + \\ - [A_{i-1}(\beta)]^{\alpha-1} \{A_{i-1}(\beta) \ln^2 A_{i-1}(\beta) + t_{i-1}^2 e^{-\beta t_{i-1}^2}\}, & i = 2, \dots, k+1, \end{cases} \\ \mathcal{D}_{\beta\beta}^{(i)} &= \begin{cases} -\alpha \{t_1^4 e^{-\beta t_1^2} [1 - \alpha(1 - A_1(\beta))] [A_1(\beta)]^{\alpha-2}\}, & i = 1, \\ -\alpha \{t_i^4 e^{-\beta t_i^2} (1 - \alpha e^{-\beta t_i^2}) [A_i(\beta)]^{\alpha-2} + \\ - t_{i-1}^4 e^{-\beta t_{i-1}^2} (1 - \alpha e^{-\beta t_{i-1}^2}) [A_{i-1}(\beta)]^{\alpha-2}\}, & i = 2, \dots, k+1, \end{cases} \\ \mathcal{D}_{\alpha\beta}^{(i)} &= \begin{cases} t_1^2 e^{-\beta t_1^2} [A_1(\beta)]^{\alpha-1} \{1 + \alpha \ln [A_1(\beta)]\}, & i = 1 \\ t_i^2 e^{-\beta t_i^2} [A_i(\beta)]^{\alpha-1} \{1 + \alpha \ln [A_i(\beta)]\} + \\ + t_{i-1}^2 e^{-\beta t_{i-1}^2} [A_{i-1}(\beta)]^{\alpha-1} \{1 + \alpha \ln [A_{i-1}(\beta)]\}, & i = 2, \dots, k+1. \end{cases} \end{aligned}$$

Therefore, the approximate 100(1 - γ)% two-sided confidence intervals for α and β are, respectively, given by

$$\hat{\alpha} \pm Z_{\gamma/2} \sqrt{\mathbf{I}_{11}^{-1}(\hat{\theta})}, \quad \hat{\beta} \pm Z_{\gamma/2} \sqrt{\mathbf{I}_{22}^{-1}(\hat{\theta})}.$$

Here, $Z_{\gamma/2}$ is the upper (γ/2)th percentile of a standard normal distribution.

4. ILLUSTRATIVE EXAMPLE

In this section we use a set of real data presented in Nelson (1982), which reports a set of cracking data on 167 independent and identically parts in a machine. The test duration was 63.48 months and 8 unequally spaced inspections were conducted to obtain the number of cracking parts in each interval. The data were

$$(t_1, \dots, t_8) = (6.12, 19.92, 29.64, 35.40, 39.72, 45.24, 52.32, 63.48),$$

and

$$(n_1, \dots, n_9) = (5, 16, 12, 18, 18, 2, 6, 17, 73).$$

We assume that these data follow different distributions. Besides the generalized Rayleigh distribution, $\text{GRD}(\alpha, \beta)$, the distributions considered here are:

1. Exponential with parameter α , $\text{ED}(\alpha)$. Its survival function is

$$S(t) = e^{-\alpha t}, \quad t \geq 0, \alpha > 0.$$

2. Generalized exponential with parameters α, β , $\text{GED}(\alpha, \beta)$. Its survival function is

$$S(t) = 1 - (1 - e^{-\alpha t})^\beta, \quad t \geq 0, \alpha, \beta > 0.$$

First we compute the maximum likelihood estimator(s) for the parameter(s) included in each distribution. Then we compare these distributions based on two different criteria. The criteria used are: the log-likelihood function and the Kolmogorov-Smirnov (K-S) test statistic. Finally, we compute the asymptotic confidence interval of the parameters of the underlying distribution and the maximum likelihood estimate of the MTTF of each distribution.

Table 4.1 shows the MLE of the parameter(s) and MTTF of the distributions considered and the associated Log-Likelihood function value.

Table 4.1. The MLE of the parameter(s), MTTF and the Log-Likelihood.

Distribution	MLE		Log-Likelihood
	parameters	MTTF	
$\text{ED}(\alpha)$	$\hat{\alpha} = 1.2097 \times 10^{-2}$	82.666	-316.671
$\text{GED}(\alpha, \beta)$	$\hat{\alpha} = 2.0285 \times 10^{-2}, \hat{\beta} = 1.7839$	69.551	-309.740
$\text{GRD}(\alpha, \beta)$	$\hat{\alpha} = 0.684, \hat{\beta} = 1.425 \times 10^{-4}$	62.229	-291.954

Based on the values of the Log-likelihood functions, we can conclude that the $\text{GRD}(0.684, 1.425 \times 10^{-4})$ fits this data much better than $\text{ED}(1.2097 \times 10^{-2})$ and $\text{GED}(2.0285 \times 10^{-2}, 1.7839)$.

Figure 4.1 shows the empirical hazard rate and fitted hazard rate functions of the $\text{ED}(1.2097 \times 10^{-2})$, $\text{GED}(2.0285 \times 10^{-2}, 1.7839)$ and $\text{GRD}(0.684, 1.425 \times 10^{-4})$.

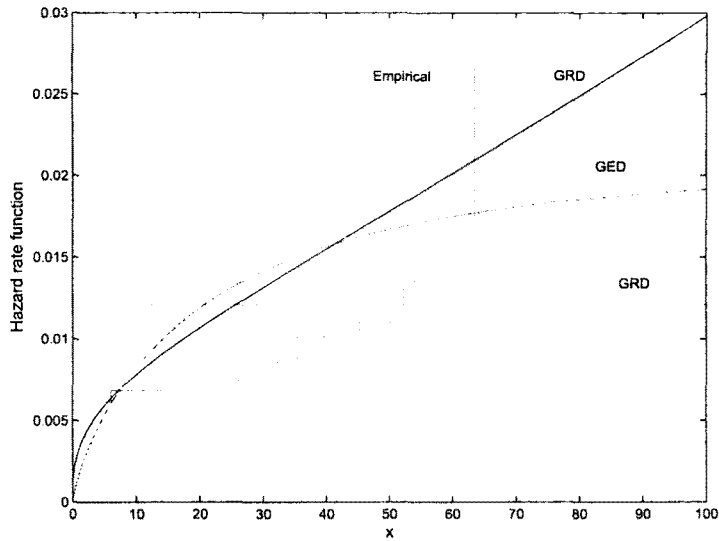


Figure 4.1. The empirical and estimated hazard rate functions.

The variance covariance matrix of $GRD(0.684, 1.425 \times 10^{-4})$ is computed as

$$I^{-1}(\hat{\theta}) = \begin{bmatrix} 3.13 \times 10^{-3} & 2.293 \times 10^{-7} \\ 2.293 \times 10^{-7} & 3.705 \times 10^{-10} \end{bmatrix}.$$

Thus, the variances of the MLE of α and β become $Var(\hat{\alpha}) = 3.13 \times 10^{-3}$ and $Var(\hat{\beta}) = 3.705 \times 10^{-10}$. Therefore, the 95% C.I of α and β , respectively, are

$$[0.574, 0.794] \quad \text{and} \quad [1.048 \times 10^{-4}, 1.803 \times 10^{-4}].$$

The survival function of the data is estimated using non-parametric and parametric methods. In the case of the non-parametric estimation, we used Kaplan-Meier method. For the parametric estimations, we used the three distributions mentioned above. Figure 4.2 shows the results of these estimations.

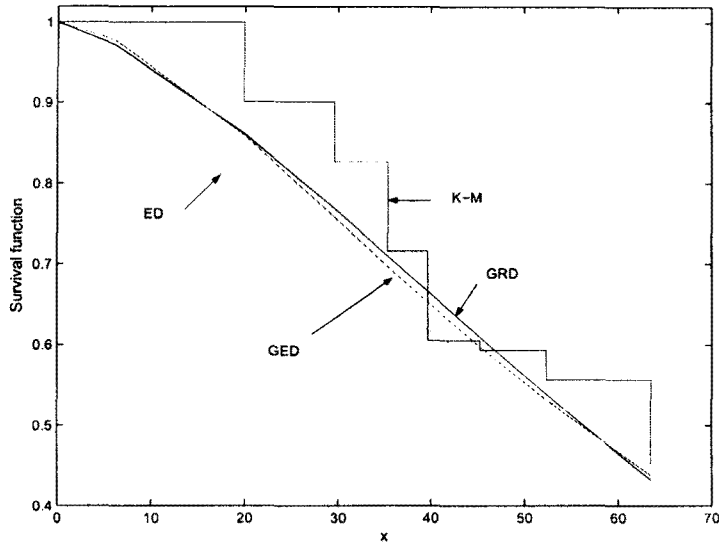


Figure 4.2. The empirical and fitted survival functions.

Also, we computed the Kolmogorov-Smirnov (K-S) test statistics of the empirical distribution function and the fitted distributions for the data set. The values of these statistics are summarized in Table 4.2 below.

Table 4.2. The K-S distances.

Distribution	ED	GED	GRD
K-S	0.214	0.144	0.105

Based on the values of K-S, we get the same conclusion that the GRD(0.684, 1.425×10^{-4}) fits this data much better than ED(1.2097×10^{-2}) and GED(2.0285×10^{-2} , 1.7839).

5. CONCLUSION

The parameter estimation of the generalized Rayleigh distribution with two unknown parameters α and β , GRD(α, β), based on grouped and censored data is discussed in this paper. The maximum likelihood technique has been used to derive both point and interval estimations of the two unknown parameters α and β . We tested the GRD(α, β) against both the generalized exponential distribution with two parameters α, β , GED(α, β), and exponential distribution with parameter α , ED(α), using a set of real data from Nelson (1982). Based on the two criteria (the values of the log-likelihood function and K-S test statistic), we found that the GRD(0.684, 1.425×10^{-4}) fits the data better than the ED(1.2097×10^{-2}) and GED(2.0285×10^{-2} , 1.7839).

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