

HAMILTONIAN OF A SECOND ORDER TWO-LAYER EARTH MODEL

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(Received February 4, 2007; Accepted June 1, 2007)

ABSTRACT

This paper deals with the theory for rotational motion of a two-layer Earth model (an inelastic mantle and liquid core) including the dissipation in the mantle-core boundary (CMB) along with tidal effects produced by Moon and Sun. An analytical solution being derived using Hori's perturbation technique at a second order Hamiltonian. Numerical nutation series will be deduced from the theory.

Key Words: Earth's rotation – nutation series – celestial mechanics

I. INTRODUCTION

Although the Earth is almost an axis-symmetric body, the influence of its triaxiality on the nutation is not negligible. The influence of this effect on nutation was calculated firstly by Kinoshita (1977) for the biggest terms and by Kinoshita and Souchay (1990) up to 0.005 milliarcseconds.

Smith & Dahlen (1981) extended their calculation of the theoretical period of the Chandler Wobble to account for the non-hydrostatic portion of the Earth's equatorial bulge and the effect of the fluid core upon the lengthening of the period due to the pole tide. Backus (1988) has shown that for an axis symmetric magnetic field B whose sources lie in the region $r < a$, the Stokes stream function p in $r \geq a$ can be computed from the Gauss coefficients of B on the sphere $S(a)$ where $r=a$. Bodri & Bodri (1977) calculated numerically the resonant amplification of diurnal Earth tides for waves with frequencies close to the resonant frequency of quasi-diurnal free nutation. Their method was applied to the Earth models of Wang (1972) and of Gilbert & Dziewonski (1975).

More of that, Folgueira et al. (1998), Souchay et al. (1999) and Bretagnon et al. (1997) have evaluated the coefficient of nutation due to higher parts of geopotential, obtaining the diurnal and semidiurnal terms. Selim (2004) has shown the effects of rotation of the Earth along with tidal effects produced by Moon and Sun, where the coupling between different effects was not considered. Notes that all these works assume rigid Earth. A simpler Earth model constituted by 2-layers, whose relative motion gives rise to a dissipation at the mutual boundary. The mantle is assumed to be deformed in an elastic way, the displacement vector solved by an expression as Takeuchi (1951), through consideration of time delay, weak inelasticity can be solved. A core is considered as fluid according to a tisserand form through its motion.

Dehant & Zschau (1989) computed the effects of mantle inelasticity on the Earth gravity response to the lunisolar attraction has been computed for the two most commonly used earth models: a spherical nonrotating Earth and an elliptical uniformly rotating Earth. The corresponding results were compared. In view of dynamical formation of structures at the core-mantle boundary, Hansen & Stemmer (2006) apply fully three dimensional models in cartesian and spherical geometry, in order to study the evolving patterns at the lower boundary in convection with strongly temperature-dependent viscosity at high Rayleigh number. Getino & Ferrandiz (2001) calculated the nutation series of a 2-layer Earth model (at a first order), Ferrandiz et al (2004) calculated precession for a free core nutation (FCN), they mentioned the computation of precession due to secular part of a second order term of the transformed Hamiltonian. In this work, we will deduce a second order Hamiltonian of a 2-layer Earth model including the tidal effect produced by the attraction of Moon and Sun. Drive the generating function of a complete second order transformation, using Hori's (1966) perturbation technique through a suitable set of canonical variables. A complete analytical and numerical solutions of nutation series will be deduce, after calculations of some basic Earth parameters and coefficients.

II. Expression of The Hamiltonian

The Hamiltonian, in form of (CMB), with our approximation is written as

$$H = T_{MC} + T_{def.} + V_o + V_t \quad (1)$$

where:

T_{MC} : is the kinetic energy for free rotation , MC refers to mantle-core .

$T_{def.}$: is the increase in kinetic energy corresponding to tidal deformation.

V_o : is the potential energy corresponding to the rigid case'

V_t : is the additional potential energy due to the redistribution of mass caused by tidal deformation.

we write the Hamiltonian with its different orders as:

$$H = H_o + H_1 + H_2 \quad (2)$$

where H_o is corresponding to the free motion(unperturbed part) and the other parts corresponding to the perturbed one, for a forced perturbations.

i.e

$$H_o = T_{MC} \quad (3)$$

$$H_1 = V_o + T_{def} \quad (4)$$

and

$$\begin{aligned} H_2 = V_t = & \frac{Gm^*}{r^{*3}} D_t \left[\frac{2t_{33} - t_{11} - t_{22}}{2} P_2(\sin \delta) + (t_{13} \cos \alpha + t_{23} \sin \alpha) P_2^1(\sin \delta) \right. \\ & \left. + \left(\frac{t_{11} - t_{22}}{4} \cos 2\alpha + \frac{1}{2} t_{12} \sin 2\alpha \right) P_2^2(\sin \delta) \right] \end{aligned} \quad (5)$$

which we deal with in our evaluations.

III. EVALUATION OF GENERATING FUNCTION

According to last Hamiltonian, we perform a second order analytical integration of this Hamiltonian by using Hori's perturbation method(1966), by following the procedure:

$$H = H_o + H_1 + H_2$$

$$H^* = H_o^* + H_1^* + H_2^*$$

$$H_2^* = H_{2sec} \quad (6)$$

$$H_{2p} = (H_2 - H_2^*) \quad (7)$$

$$\widetilde{H}_2 = H_{2p} + \frac{1}{2}((H_1 + H_1^*), w_1) \quad (8)$$

where the sign (*) means the secular part, w_1 is the generating function of a first order, H_{2p} is the periodic part of a second order and the bracket refers to poisson bracket. The the generating function of a second order can be determine by integrate the periodic part of this order

$$w_2 = \int (H_{2p} + \frac{1}{2}(H_{1per}; w_1) + (H_1^*; w_1)) dt$$

$$w_2 = w_{2A} + w_{2B} + w_{2C} \quad (9)$$

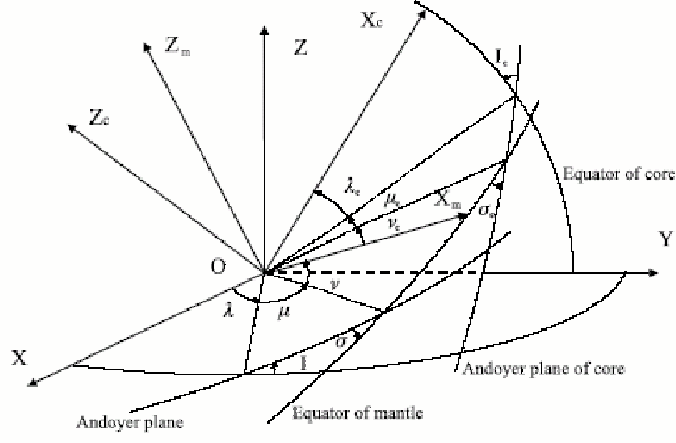


Fig. 1.— Canonical Variables

where

$$w_{2A} = \int (H_2 - H_2^*) dt \quad (10)$$

$$w_{2B} = \frac{1}{2} \int (H_{1per}; w_1) dt \quad (11)$$

$$w_{2C} = \int (H_1^*; w_1) dt \quad (12)$$

The Hamiltonian (eqn.(1)) is formulated canonically by using the set of canonical variables

$\lambda, \mu, \nu, \Lambda, M, N \rightarrow$ for the total Earth

$\lambda_c, \mu_c, \nu_c, \Lambda_c, M_c, N_c \rightarrow$ for the core

(a) Evaluation of w_{2A}

We can write eq. (5) in the form

$$\begin{aligned} H_2 = & -Q_t \left(9 \sum_j \sum_{\rho=\pm 1} \sum_i \sum_{\tau=\pm 1} B_j (B_i \cos(\theta_i - \rho\theta_j) - \sin \sigma C_i(\tau) \cos(h_i - \nu - \rho\theta_j)) \right. \\ & + 6 \sum_j \sum_{\rho=\pm 1} \sum_i \sum_{\tau=\pm 1} C_j(\rho) \left(C_i(\tau) \cos(h_i - \tilde{h}_j) - \frac{1}{2} \sin \sigma D_i(\tau) \cos(h + \mu - \tilde{h}_j) \right) \\ & \left. + 3 \sum_j \sum_{\rho=\pm 1} \sum_i \sum_{\tau=\pm 1} D_i(\rho) \left(\sin \rho C_i(\tau) \cos(h + \nu - \tilde{h} - \tilde{\mu} - \tilde{\nu}) + \frac{1}{2} D_i(\tau) \cos(h + \mu + \nu - \tilde{h} - \tilde{\mu} - \tilde{\nu}) \right) \right) \end{aligned} \quad (13)$$

where:

$$Q_t = \frac{3Gm^*}{a^{*3}} D_t$$

and the suffix i, j are refer to Moon and Sun respectively, G is the gravitational constant, m^* and a^* being respectively the mass and semi-major axis concerning the perturbing body(Moon, Sun),

$$B_i = -\frac{1}{6}(3 \cos^2 I - 1)A_i^0 - \frac{1}{2} \sin 2IA_i^1 - \frac{1}{4} \sin^2 IA_i^2$$

$$C_i(\tau) = -\frac{1}{4} \sin 2IA_i^0 + \frac{\tau}{4} \sin I(1 + \tau \cos I)A_i^2 + \frac{1}{2}(1 + \tau \cos I)(-1 + 2\tau \cos I)A_i^1$$

$$D_i(\tau) = -\frac{1}{2} \sin^2 IA_i^0 + \tau \sin I(1 + \tau \cos I)A_i^1 - \frac{1}{4}(1 + \tau \cos I)^2 A_i^2$$

the numerical values of the coefficients A_i^j are given in Kinoshita (1977) and updated in Kinoshita & Souchay (1990),

$$t_{11} = 2P_2(\sin \tilde{\delta}) - P_2^2(\sin \tilde{\delta}) \cos 2\tilde{\alpha} \qquad t_{12} = -P_2^2(\sin \tilde{\delta}) \sin 2\tilde{\alpha}$$

$$t_{22} = 2P_2(\sin \tilde{\delta}) + P_2^2(\sin \tilde{\delta}) \cos 2\tilde{\alpha} \qquad t_{13} = -P_2^1(\sin \tilde{\delta}) \cos \tilde{\alpha}$$

$$t_{33} = -4P_2(\sin \tilde{\delta}) \qquad t_{23} = -2P_2^1(\sin \tilde{\delta}) \sin \tilde{\alpha}$$

and the spherical function with the simplification of neglecting the second order terms

$$\left(\frac{a^*}{r^*}\right)^3 P_2(\sin \delta) \simeq 3 \sum_i B_i \cos \theta_i - 3 \sin \sigma \sum_i \sum_{\tau=\pm 1} C_i(\tau) \cos(\mu - \tau\theta_i)$$

$$\left(\frac{a^*}{r^*}\right)^3 P_2^1(\sin \delta) \begin{pmatrix} \cos \\ \sin \end{pmatrix} \alpha \simeq 3 \sum_i \sum_{\tau=\pm 1} C_i(\tau) \begin{pmatrix} \sin \\ \cos \end{pmatrix} (\mu + \nu - \tau\theta_i)$$

$$\left(\frac{a^*}{r^*}\right)^3 P_2^2(\sin \delta) \begin{pmatrix} \cos \\ \sin \end{pmatrix} 2\alpha \simeq \begin{pmatrix} + \\ - \end{pmatrix} 3 \sum_i \sum_{\tau=\pm 1} D_i(\tau) \begin{pmatrix} \cos \\ \sin \end{pmatrix} (2\mu + 2\nu - \tau\theta_i)$$

where $\tilde{\alpha}$, $\tilde{\delta}$ being longitude and latitude of the perturbing body referred to the principal axis of the Earth, a^* the semi-major axis of its orbit, taking the approximation $\cos \sigma \simeq 1$, since the angle σ between the angular momentum and the figure axis of the Earth is about 10^{-6} rad, and the coefficient D_t is computed by Takeuchi's Model2(Takeuchi, 1951, which consider as an old but classical work than that recent considerably more rigorous and advanced works), depending on the Earth's model used

$$D_t = \begin{cases} 6.953379 \times 10^{36} \text{cgs} & \text{for Moon} \\ 3.185508 \times 10^{36} \text{cgs} & \text{for Sun} \end{cases}$$

$$\theta_i = (m_1 + m_3 + m_4)\ell_M + (m_3 + m_4)g_M \\ + (m_4 + m_5)h_M + (m_2 - m_4)\ell_S - (g_S + h_S)m_4 - m_5\lambda$$

$$\Omega = h_M - \lambda$$

$$\theta_i = 0 \text{ at } i = (00000) = 0$$

where ℓ , g , h are the Delaunay variables for Moon(M) and Sun(S).

The generating function (10) is obtained by integrating the periodic part of equation (13) which can be related easily to

$$w_{2A} = Q_t \left[\int V_{t0} dt + \int \sin \sigma V_{t1} dt \right] \qquad (14)$$

Now first of all, to carry out the integration of eqn.(13), we follow the following technique

$$\begin{aligned}
 I_1 &= \int M \sin \sigma \cos(k(\mu + \nu) - \gamma\nu - \Phi_{ij}) dt \\
 &= M \sin \sigma [F_1^a \sin(h - \gamma\nu) + F_1^b \cos(h - \gamma\nu)] \\
 &+ M_c \sin \sigma_c [F_2^a \sin(h + \gamma\nu_c) + F_2^b \cos(h + \gamma\nu_c)], \\
 I_2 &= \int M_c \sin \sigma_c \cos(k(\mu + \nu) - \alpha\nu_c - \Phi_{ij}) dt \\
 &= M \sin \sigma [G_1^a \sin(h + \alpha\nu) + G_1^b \cos(h + \alpha\nu)] \\
 &+ M_c \sin \sigma_c [G_2^a \sin(h - \alpha\nu_c) + G_2^b \cos(h - \alpha\nu_c)]
 \end{aligned} \tag{15}$$

where

$$\begin{aligned}
 \Phi_{ij} &= \tau\theta_i - \rho\theta_j; & \alpha, \gamma &= \pm 1, \quad k = 1, 2 \\
 h_i &= \mu + \nu - \tau\theta_i; & \tilde{h}_j &= \tilde{\mu} + \tilde{\nu} - \rho\theta_j \\
 n_{h_{ij}} &= \frac{dh_{ij}}{dt}; & F_1^a &= \frac{\gamma r_4 + n h_{ij}}{f_1 f_2} \\
 F_1^b &= \frac{\gamma \Omega \Gamma \frac{A}{A_c} (f_2 - n h - r_4)}{f_1 f_2^2}; & F_2^a &= \frac{\gamma r_2}{f_1 f_2} \\
 F_2^b &= \gamma \Omega \Gamma \frac{A}{A_c} \frac{r_2}{f_1 f_2^2}; & G_1^a &= -\frac{\alpha r_3}{f_1 f_2} \\
 G_1^b &= \alpha \Omega \Gamma \left(\frac{f_2 - \alpha \frac{A}{A_c} r_3}{f_1 f_2^2} \right); & G_2^a &= \frac{n h_{ij} - \alpha r_1}{f_1 f_2} \\
 G_2^b &= \alpha \Omega \Gamma \frac{A}{A_c} \frac{n h_{ij} - \alpha r_1}{f_1 f_2^2}; & f_1 &= m_1 + n h_{ij} \\
 f_2 &= m_2 + n h_{ij}; & r_1 &= \Omega \left(\frac{A_c}{A_m} + e \frac{A}{A_m} \right) \\
 r_2 &= -\Omega \frac{A}{A_m} (1 + e); & r_3 &= \Omega \frac{A_c}{A_m} (1 + e_c) \\
 r_4 &= -\Omega \frac{A}{A_m} (1 + e_c); & A_m &= A - A_c \\
 e_c &= \frac{C_c - A_c}{A_c}; & m_1 &= \Omega \frac{C - A}{A_m} \\
 m_2 &= -\Omega \left[1 + \frac{A}{A_m} \left(e_c - \frac{R}{\Omega A_c} \right) \right].
 \end{aligned}$$

Then the generating function (10) will be after calculus

$$w_{2A} = w_{2A}^1 + w_{2A}^2 + w_{2A}^3 \tag{16}$$

where

$$\begin{aligned}
 w_{2A}^1 &= -Q_t \left\{ 9 \sum_{i \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} B_j \frac{B_i}{n_j - \rho n_j} \sin(\theta_i - \rho\theta_j) \right. \\
 &+ 6 \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} C_j(\rho) \frac{C_i(\tau)}{\tau n_i - \rho n_j} \sin(h_i - \tilde{h}_j) \\
 &+ \left. \frac{3}{2} \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} D_j(\rho) \frac{D_i(\tau)}{\tau n_i - \rho n_j} \sin(h_i + \mu + \nu - \tilde{h}_j - \tilde{\mu} - \tilde{\nu}) \right\} \\
 w_{2A}^2 &= -Q_t \sin \sigma \left\{ -9 \sum_{i \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} B_j C_i(\tau) [F_1^a \sin(h - \nu) + F_1^b \cos(h - \nu)] \right. \\
 &- 3 \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} C_j(\rho) D_i(\tau) [F_1^a \sin(h - \nu) + F_1^b \cos(h - \nu)] \\
 &+ \left. 3 \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} D_j(\rho) C_i(\tau) [F_1^a \sin(h + \nu) + F_1^b \cos(h + \nu)] \right\}
 \end{aligned}$$

$$\begin{aligned}
w_{2A}^3 &= -Q_t \frac{M_c}{M} \sin \sigma_c \left\{ -9 \sum_{i \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} C_i(\tau) \frac{M_c}{M} \sin(\sigma_c) [F_2^a \sin(h + \nu_c) + F_2^b(h + \nu_c)] \right. \\
&\quad - 3 \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} C_j(\rho) D_i(\tau) [F_2^a \sin(h + \nu_c) + F_2^b \cos(h + \nu_c)] \\
&\quad \left. + 3 \sum_{j \neq 0} \sum_{\rho = \pm 1} \sum_{i \neq 0} \sum_{\tau = \pm 1} D_j(\rho) C_i(\tau) [F_2^a(h - \nu_c) + F_2^b \cos(h - \nu_c)] \right\}
\end{aligned}$$

(b) **Evaluation of w_{2B}**

To evaluate eqn.(11) we use a periodic part of the potential energy in eqn.(4)

$$V_{oper.} = k'_o \sum_{i \neq 0} B_i \cos \theta_i - k'_o \sum_{i \neq 0} \sum_{\tau = \pm 1} C_i(\tau) \sin \sigma \cos(\mu - \tau \theta_i)$$

where

$$k'_o = \frac{3Gm^*}{a^{*3}}(C - A)$$

and C , A are the principal moments of inertia, and the generating function which produced through the first order solution by Getino and Ferrandiz (2001), after some calculations, we can deduce w_{2B} as follows

$$w_{2B} = w_{2B}^1 + w_{2B}^2 + w_{2B}^3 \quad (17)$$

where

$$\begin{aligned}
w_{2B}^1 &= -\frac{1}{4} \sum_{i,j \neq 0} \sum_{\rho = \pm 1} k'_{0i} k'_{0j} \frac{m_{5j}}{\sin I} \frac{1}{n_i - \rho n_j} \left(\frac{B_i}{n_j} \frac{\partial B_j}{\partial I} + \frac{B_j}{n_j} \frac{\partial B_i}{\partial I} \right) \sin(\theta_i - \rho \theta_j) \\
w_{2B}^2 &= \frac{1}{4} \sum_{i,j \neq 0} \sum_{\rho = \pm 1} k'_{0i} k'_{0j} \frac{\sin \sigma}{\sin I} \left(\cos IC_j(\rho) \frac{\partial B_i}{\partial I} - B_i \frac{\partial C_j(\rho)}{\partial I} + \frac{\partial B_i}{\partial I} C_j(\rho) m_{5j} \right) \\
&\quad \times (F_{1j}^a (F_1^a \sin(h - \nu) + F_1^b \cos(h - \nu)) + F_{1j}^b (F_1^a \cos(h - \nu) - F_1^b \sin(h - \nu))) \\
w_{2B}^3 &= \frac{1}{4} \sum_{i,j \neq 0} \sum_{\rho = \pm 1} k'_{0i} k'_{0j} \frac{M_c \sin \sigma_c}{M \sin I} \left(\cos IC_j(\rho) \frac{\partial B_i}{\partial I} - B_i \frac{\partial C_j(\rho)}{\partial I} + \frac{\partial B_i}{\partial I} C_j(\rho) m_{5j} \right) \\
&\quad \times (F_{1j}^a (F_2^a \sin(h + \nu_c) + F_2^b \cos(h + \nu_c)) + F_{1j}^b (F_2^a \sin(h + \nu_c) - F_2^b \cos(h + \nu_c)))
\end{aligned}$$

(c) **Evaluation of w_{2C}**

To evaluate eqn. (12) we use the secular term of the potential energy in eqn. (4)

$$V_{osec.} = k'_o B_o \quad (18)$$

after some calculations, through $\frac{\partial}{\partial M} = \cot I \frac{\partial}{\partial I}$; $\frac{\partial}{\partial \Lambda} = -\frac{1}{\sin I} \frac{\partial}{\partial I}$
we can deduce

$$w_{2c} = w_{2c}^1 + w_{2c}^2 + w_{2c}^3 \quad (19)$$

where

$$w_{2c}^1 = \sum_{i,j \neq 0} \sum_{\rho = \pm 1} k'_{0i} k'_{0j} \frac{m_{5j}}{\sin I} \frac{\partial B_o}{\partial I} \frac{B_j}{n_j^2} \sin \theta_j$$

$$\begin{aligned}
 w_{2c}^2 &= \sum_{i,j \neq 0} \sum_{\rho=\pm 1} k'_{0i} k'_{0j} C_j(\rho) \frac{\partial B_o}{\partial I} \sin \sigma \left(\frac{m_{5j}}{\sin I} - \frac{\cos I}{\sin I} \right) \\
 &\times \left((F_1^a)^2 + G_1^a F_2^a \right) \sin(h - \nu) + \left((F_1^b)^2 + F_2^b G_1^a \right) \cos(h - \nu) \\
 w_{2c}^3 &= \sum_{i,j \neq 0} \sum_{\rho=\pm 1} k'_{0i} k'_{0j} C_j(\rho) \frac{\partial B_o}{\partial I} \sin \sigma_c \frac{M_c}{M} \left(\frac{m_{5j}}{\sin I} - \frac{\cos I}{\sin I} \right) \\
 &\times \left((F_1^a F_2^a + G_2^a F_2^a) \sin(h + \nu_c) + (F_1^b F_2^a + F_2^b G_2^a) \cos(h + \nu_c) \right).
 \end{aligned}$$

IV. FORCED NUTATION

The periodic perturbations, forced nutation are obtained through the generating function by means of the well-known relations

$$\Delta(\mu, \nu, \lambda) = \frac{\partial w}{\partial(M, N, \Lambda)}, \quad \Delta(M, N, \Lambda) = -\frac{\partial w}{\partial(\mu, \nu, \lambda)}.$$

We compute the periodic perturbation of the fundamental planes, that is to say, the plane perpendicular to the angular momentum vector (or Andoyer plane) and the plane perpendicular to the figure axis of the Earth (figure plane or equatorial plane) once the derivatives have been performed, we could identify \tilde{h} which points out that this argument corresponds to the coordinates of the perturbing body responsible of the tidal deformation, with h which points out that this argument corresponding to the coordinates of the perturbed body.

(a) Nutation of Andoyer Plane

The longitude of the node and inclination of this plane are given respectively by λ and $I = \cos^{-1}(\frac{\Lambda}{M})$, the nutation corresponding to these variables, known as poisson terms, are obtained through the equations (Kinoshita, 1977)

$$\Delta\lambda = -\frac{1}{M \sin I} \frac{\partial w_2}{\partial I}, \quad \Delta I = \frac{1}{M \sin I} \left(\frac{\partial w_2}{\partial \lambda} - \cos I \frac{\partial w_2}{\partial \mu} \right). \quad (20)$$

the poisson terms will be obtained by neglecting second order terms in small parameters through the simplification form of generating function (10) as follows:-

Contribution of w_{2A}

$$\Delta\lambda_A = \frac{3Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\tau, \rho=\pm 1} \left\{ \frac{3B_j B_{i,I}}{n_i - \rho n_j} \sin(\theta_i - \rho\theta_j) + \left(\frac{2C_j(\rho) C_{i,I}(\tau)}{\tau n_i - \rho n_j} + \frac{1}{2} \frac{D_j(\rho) D_{i,I}(\tau)}{\tau n_i - \rho n_j} \right) \sin(\tau\theta_i - \rho\theta_j) \right\} \quad (21)$$

$$\begin{aligned}
 \Delta I_A &= \frac{3Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\tau, \rho=\pm 1} \left\{ 3m_{5i} \frac{B_j B_{i,I}}{n_i - \rho n_j} \cos(\theta_i - \rho\theta_j) \right. \\
 &+ \left. \left(\frac{2C_j(\rho) C_i(\tau)}{\tau n_i - \rho n_j} (\tau m_{5i} - \cos I) + \frac{1}{2} \frac{D_j(\rho) D_i(\tau)}{\tau n_i - \rho n_j} (\tau m_{5i} - 2 \cos I) \right) \cos(\tau\theta_i - \rho\theta_j) \right\} \quad (22)
 \end{aligned}$$

Contribution of w_{2B}

$$\Delta\lambda_B = \sum_{i,j \neq 0} \sum_{\tau, \rho=\pm 1} \frac{k'_{oi} k'_{oj}}{4M \sin^2 I} \frac{m_{5j}}{n_j (n_i - \rho n_j)} \left(\frac{\partial B_i}{\partial I} \frac{\partial B_j}{\partial I} + \frac{\partial^2 B_i}{\partial I^2} B_j \right) \sin(\tau\theta_i - \rho\theta_j) \quad (23)$$

$$\Delta I_B = \sum_{i,j \neq 0} \sum_{\tau, \rho=\pm 1} \frac{k'_{oi} k'_{oj}}{4M \sin^2 I} \frac{m_{5i} m_{5j}}{n_j (n_i - \rho n_j)} \left(B_i \frac{\partial B_j}{\partial I} + B_j \frac{\partial B_i}{\partial I} \right). \quad (24)$$

Contribution of w_{2C}

$$\Delta\lambda_C = \sum_{i,j \neq 0} \sum_{\tau, \rho = \pm 1} \frac{-k'_{oi}k'_{oj}}{4M \sin^2 I} m_{5j} \frac{B_j}{n_j^2} \frac{\partial^2 B_{oi}}{\partial I^2} \sin \theta_j, \quad (25)$$

$$\Delta I_C = 0. \quad (26)$$

(b) Nutation of The Figure Plane

The longitude of the node, $\psi_f = -\lambda_f$, and the inclination $\epsilon_f = -I_f$, of this plane are given by Kinoshita(1977)

$$\lambda_f = \lambda + \frac{\sigma}{\sin I} \sin \mu; \quad I_f = I + \sigma \cos \mu$$

following Kinoshita, the periodic perturbation of the increments $\lambda_f - \lambda$, $I_f - I$ called oppolzer terms, are given up to second order by

$$\begin{aligned} \Delta(\lambda_f - \lambda) &= \frac{1}{\sin I} \frac{1}{M \sin \sigma} \left[\sin \mu \left(\frac{\partial w_2}{\partial \nu} - \frac{\partial w_2}{\partial \mu} \right) + \sigma \cos \mu \frac{\partial w_2}{\partial \sigma} \right], \\ \Delta(I_f - I) &= \frac{1}{M \sin \sigma} \left[\cos \mu \left(\frac{\partial w_2}{\partial \nu} - \frac{\partial w_2}{\partial \mu} \right) - \sigma \sin \mu \frac{\partial w_2}{\partial \sigma} \right]. \end{aligned} \quad (27)$$

Oppolzer terms will be obtained by neglecting third order terms through the simplification form of generating function (9) as follows

Contribution of w_{2A}

$$\begin{aligned} \Delta(\lambda_f - \lambda) &= \frac{Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\tau, \rho = \pm 1} \{ (9B_j C_i(\tau) + 3C_j(\rho) D_i(\tau) \\ &\quad - 3D_j(\rho) C_i(\tau)) (F_1^a \sin \theta_i + F_1^b \cos \theta_i) \} \end{aligned} \quad (28)$$

$$\begin{aligned} \Delta(I_f - I) &= -\frac{Q_t}{M} \sum_{i,j \neq 0} \sum_{\tau, \rho = \pm 1} \{ (9B_j C_i(\tau) + 3C_j(\rho) D_i(\tau) \\ &\quad - 3D_j(\rho) C_i(\tau)) (F_1^a \cos \theta_i + F_1^b \sin \theta_i) \}. \end{aligned} \quad (29)$$

Contribution of w_{2B}

$$\begin{aligned} \Delta(\lambda_f - \lambda) &= \frac{k'_{oi}k'_{oj}}{4M \sin^2 I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left(\frac{\partial B_i}{\partial I} C_j(\rho) [\cos I + m_{5i}] - B_i \frac{\partial C_j(\rho)}{\partial I} \right) \\ &\quad \times (F_{1j}^a F_{1i}^a \sin(\tau \theta_i - \rho \theta_j) - F_{1j}^b F_{1i}^a \cos(\tau \theta_i - \rho \theta_j)) \end{aligned} \quad (30)$$

$$\begin{aligned} \Delta(I_f - I) &= \frac{-k'_{oi}k'_{oj}}{4M \sin I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left(\frac{\partial B_i}{\partial I} C_j(\rho) [\cos I + m_{5i}] - B_i \frac{\partial C_j(\rho)}{\partial I} \right) \\ &\quad \times (F_{1j}^a F_{1i}^a \cos(\tau \theta_i - \rho \theta_j) - F_{1j}^b F_{1i}^a \sin(\tau \theta_i - \rho \theta_j)) \end{aligned} \quad (31)$$

Contribution of w_{2C}

$$\Delta(\lambda_f - \lambda) = \frac{k'_{oi}k'_{oj}C_j(\rho)}{M \sin^2 I} \frac{\partial B_o}{\partial I} (\cos I - m_{5i})(E_1 \sin \tau\theta_i - E_2 \cos \tau\theta_i) \quad (32)$$

$$\Delta(I_f - I) = \frac{k'_{oi}k'_{oj}C_j(\rho)}{M \sin I} \frac{\partial B_o}{\partial I} (\cos I - m_{5i})(E_1 \cos \tau\theta_i - E_2 \sin \tau\theta_i) \quad (33)$$

where

$$E_1 = F_1^a F_1^a + G_1^a F_2^a ; \quad E_2 = F_1^b F_1^a + F_2^b G_1^a.$$

V. FINAL EXPRESSION

By using eqns. (20) and (27), the nutation series in longitude $\Delta\psi_f$ and obliquity $\Delta\epsilon_f$ of the figure plane can be collected as follows.

(a) Contribution of w_{2A}

$$\begin{aligned} \Delta\psi_f &= -\frac{Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left\{ \frac{9B_j B_{i,I}}{n_i - \rho n_j} \sin(\theta_i - \rho\theta_j) + \left(\frac{6C_j(\rho)C_{i,I}(\tau)}{\tau n_i - \rho n_j} + \frac{3}{2} \frac{D_j(\rho)D_{i,I}(\tau)}{\tau n_i - \rho n_j} \right) \sin(\tau\theta_i - \rho\theta_j) \right. \\ &\quad \left. + (9B_j C_i(\tau) + 3C_j(\rho)D_i(\tau) - 3D_j(\rho)C_i(\tau))(F_1^a \sin \theta_i + F_1^b \cos \theta_i) \right\} \end{aligned} \quad (34)$$

where the amplitude for longitude in phase and out-of-phase terms respectively are:

$$L_i^{in} = -\frac{Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left\{ \frac{9B_j B_{i,I}}{n_i - \rho n_j} + F_1^a (9B_j C_i(\tau) + 3C_j(\rho)D_i(\tau) - 3D_j(\rho)C_i(\tau)) \right\}$$

$$L_{i,\tau}^{in} = -\frac{Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left(\frac{6C_j(\rho)C_{i,I}(\tau)}{\tau n_i - \rho n_j} + \frac{3}{2} \frac{D_j(\rho)D_{i,I}(\tau)}{\tau n_i - \rho n_j} \right)$$

$$L_i^{out} = \frac{-Q_t}{M \sin I} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} F_1^b (9B_j C_i(\tau) + 3C_j(\rho)D_i(\tau) - 3D_j(\rho)C_i(\tau))$$

and

$$\begin{aligned} \Delta\epsilon_f &= -\frac{Q_t}{M} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left\{ 3m_{5i} \frac{B_i B_j}{n_i - \rho n_j} \cos(\theta_i - \rho\theta_j) \right. \\ &\quad \left. + \left[\frac{2C_i C_j}{\tau n_i - \rho n_j} (\tau m_{5i} - \cos I) + \frac{1}{2} \frac{D_i D_j(\rho)}{\tau n_i - \rho n_j} (\tau m_{5i} - 2 \cos I) \right] \cos(\tau\theta_i - \rho\theta_j) \right. \\ &\quad \left. + (3B_j(\rho)C_i(\tau) + C_j(\rho)D_i(\tau) - D_j(\rho)C_i(\tau))(F_1^a \cos \theta_i + F_1^b \sin \theta_i) \right\} \end{aligned} \quad (35)$$

So the amplitude for obliquity in-phase and out-of-phase terms respectively are

$$O_i^{in} = -\frac{3Q_t}{M} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left\{ 3m_{5i} \frac{B_i(\tau)B_j(\rho)}{n_j - \rho n_j} - F_1^a (3B_j(\rho)C_i(\tau) + C_j(\rho)D_i(\tau) - D_j(\rho)C_i(\tau)) \right\}$$

$$O_{i\tau}^{in} = -\frac{3Q_t}{M} \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \left[\frac{3C_i(\tau)C_j(\rho)}{\tau n_i - \rho n_j} (\tau m_{5i} - \cos I) + \frac{1}{2} \frac{D_i(\tau)D_j(\rho)}{\tau n_i - \rho n_j} (\tau m_{5i} - 2 \cos I) \right]$$

$$O_i^{out} = -\frac{3Q_t}{M} F_1^b \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} (3B_j(\rho)C_i(\tau) + C_j(\rho)D_i(\tau) - D_j(\rho)C_i(\tau))$$

(b) Contribution of w_{2B}

$$\begin{aligned} \Delta\psi_f &= \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} -\frac{k'_{0i} k'_{0j}}{4M \sin^2 I} \left\{ \frac{m_{5i}}{n_j} \left(\frac{\partial B_i \partial B_j}{\partial^2 I} + \frac{\partial^2 B_i}{\partial I^2} B_j \right) \frac{1}{n_i - \rho n_j} \sin(\theta_i - \rho \theta_j) \right. \\ &\quad \left. - \left(\frac{\partial B_i}{\partial I} C_j(\rho) [\cos I + m_{5i}] - B_i \frac{\partial C_j(\rho)}{\partial I} \right) (F_{1j}^a F_{1i}^a \sin(\tau \theta_i - \rho \theta_j) - F_{1j}^b F_{1i}^a \cos(\tau \theta_i - \rho \theta_j)) \right\} \quad (36) \end{aligned}$$

the amplitude for longitude in-phase and out-of-phase terms respectively

$$\begin{aligned} L_i^{in} &= \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \frac{-k'_{0i} k'_{0j}}{4M \sin^2 I} \frac{m_{5j}}{n_j} \left(\frac{\partial B_i \partial B_j}{\partial^2 I} + \frac{\partial^2 B_i}{\partial I^2} B_j \right) \frac{1}{n_i - \rho n_j} \\ L_{i\tau}^{in} &= \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \frac{-k'_{0i} k'_{0j} F_{1i}^a F_{1j}^a}{4M \sin^2 I} \left(\frac{\partial B_i}{\partial I} C_j(\rho) (\cos I + m_{5j}) - B_i \frac{\partial C_j(\rho)}{\partial I} \right) \\ L_{i\tau}^{out} &= \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \frac{k'_{0i} k'_{0j} F_{1i}^a F_{1j}^a}{4M \sin^2 I} \left(\frac{\partial B_i}{\partial I} C_j(\rho) (\cos I + m_{5j}) - B_i \frac{\partial C_j(\rho)}{\partial I} \right) \end{aligned}$$

and

$$\begin{aligned} \Delta\epsilon_f &= \sum_{i,j \neq 0} \sum_{\rho, \tau = \pm 1} \frac{k'_{0i} k'_{0j}}{4M \sin I} \left\{ \left(-\frac{m_{5i} m_{5j}}{\sin I} \left(\frac{B_i}{n_j} \frac{\partial B_j}{\partial I} + \frac{\partial B_i}{\partial I} \frac{B_j}{n_j} \right) \frac{1}{n_i - \rho \theta_j} \cos(\theta_i - \rho \theta_j) \right. \right. \\ &\quad \left. \left. + \left(\frac{\partial B_i}{\partial I} C_j(\rho) (\cos I + m_{5j}) - B_i \frac{\partial C_j(\rho)}{\partial I} \right) (F_{1j}^a F_{1i}^a \cos(\tau \theta_i - \rho \theta_j) - F_{1j}^b F_{1i}^a \sin(\tau \theta_i - \rho \theta_j)) \right\}. \quad (37) \end{aligned}$$

So the amplitude for obliquity in-phase, out-of-phase terms respectively are:

$$\begin{aligned} O_i^{in} &= -\frac{k'_{0i} k'_{0j}}{4M \sin I} \frac{m_{5i} m_{5j}}{\sin I (n_j - \rho n_j)} \left(\frac{B_i}{n_j} \frac{\partial B_j}{\partial I} + \frac{\partial B_i}{\partial I} \frac{B_j}{n_j} \right) \\ O_{i\tau}^{in} &= \frac{k'_{0i} k'_{0j} F_{1j}^a F_{1i}^a}{4M \sin I} \left(\frac{\partial B_i}{\partial I} C_i(\rho) (\cos I + m_{5i}) - B_i \frac{\partial C_j(\rho)}{\partial I} \right) \\ O_{i\tau}^{out} &= -\frac{k'_{0i} k'_{0j} F_{1j}^b F_{1i}^a}{4M \sin I} \left(\frac{\partial B_i}{\partial I} C_j(\rho) (\cos I + m_{5i}) - B_i \frac{\partial C_j(\rho)}{\partial I} \right). \end{aligned}$$

(c) Contribution of w_{2C}

$$\Delta\psi_f = \frac{k'_{oi} k'_{oj}}{M \sin^2 I} m_{5j} \frac{\partial^2 B_{oi}}{\partial I^2} \frac{B_j}{n_j^2} \sin \theta_j - \frac{k'_{oi} k'_{oj} C_j(\rho)}{M \sin^2 I} \frac{\partial B_o}{\partial I} (\cos I - m_{5j}) (E_1 \sin \tau \theta_i - E_2 \cos \tau \theta_i) \quad (38)$$

The amplitude for longitude in-phase and out-of-phase terms respectively are:

$$\begin{aligned} L_i^{in} &= \frac{k'_{oi} k'_{oj}}{M \sin^2 I} m_{5j} \frac{\partial^2 B_{oi}}{\partial I^2} \frac{B_j}{n_j^2} \\ L_{i\tau}^{in} &= -\frac{k'_{oi} k'_{oj}}{M \sin^2 I} E_1 \frac{\partial B_o}{\partial I} (\cos I - m_{5j}) \end{aligned}$$

TABLE 1
 NUMERICAL COEFFICIENTS AND SOME BASIC EARTH'S PARAMETERS

Parameter	Value	Parameter	Values	Sources
Ω	15".04118108	ϵ	84381".412	Allen (2000)
A	$0.80094 \times 10^{45} \text{ gm.cm}^2$	C	$0.80368 \times 10^{45} \text{ gm.cm}^2$	Allen (2000)
a_{\odot}	$0.1495979 \times 14 \text{ cm}$	a_{\subset}	$0384401 \times 10^{11} \text{ cm}$	Allen (2000)
m_{\subset}	$7.353 \times 10^{25} \text{ gm}$	m_{\odot}	$1.989 \times 10^{33} \text{ gm}$	Allen (2000)
R	$6.378136 \times 10^8 \text{ cm}$			Allen (2000)
C_m	$7.040 \times 10^{44} \text{ gm.cm}^2$	A_m	$7.0165 \times 10^{44} \text{ gm.cm}^2$	Barnes et al. (1983)
$k_{o\odot}$	$3475".18832 \text{ arcsec./Jcy}$	$k_{o\subset}$	$7568.01772 \text{ arcsec./Jcy}$	Computed
$k_{t\subset}^c$	$8865".751634 \text{ arcsec./Jcy}$	$k_{t\subset}^m$	$1928".013810 \text{ arcsec./Jcy}$	Computed
$k_{t\odot}^c$	$3832".549573 \text{ arcsec./Jcy}$	$k_{t\odot}^m$	$883".268967 \text{ arcsec./Jcy}$	Computed
$D_{t\subset}^c$	$3.1650834 \times 10^{36} \text{ c.g.s}$	$D_{t\subset}^m$	$4.7499997 \times 10^{36} \text{ c.g.s}$	Computed
$D_{t\odot}^c$	$1.449999 \times 10^{36} \text{ c.g.s}$	$D_{t\odot}^m$	$2.1760878 \times 10^{36} \text{ c.g.s}$	Computed
$Q_{t\subset}^c$	-3.53044×10^{-2}	$Q_{t\subset}^m$	3.897048×10^{-3}	Computed
$Q_{t\odot}^c$	-7.40959×10^{-3}	$Q_{t\odot}^m$	8.17902×10^{-4}	Computed
Γ	4×10^{-6}			Computed

$$L_i^{out} = \frac{k'_{oi}k'_{oj}}{M \sin^2 I} E_2 \frac{\partial B_o}{\partial I}$$

and

$$\Delta \epsilon_f = -\frac{k'_{oi}k'_{oj}}{4M \sin I} \frac{\partial B_o}{\partial I} (\cos I - m_{5i})(E_1 \cos \tau \theta_i + E_2 \sin \tau \theta_i) \quad (39)$$

So the amplitude for obliquity in-phase, out-of-phase terms respectively are:

$$O_{i\tau}^{in} = -\frac{k'_{oi}k'_{oj}E_1}{4M \sin I} \frac{\partial B_o}{\partial I} (\cos I - m_{5j})$$

$$O_{i\tau}^{out} = \frac{-k'_{oi}k'_{oj}E_2}{4M \sin I} \frac{\partial B_o}{\partial I} (\cos I - m_{5j}).$$

VI. NUMERICAL COMPUTATION OF NUTATION SERIES

In this section, we can compute numerically the periodic perturbation of the fundamental plane (eqns.20 and 27), but firstly we need to evaluate numerical coefficients taking into account some of the basic Earth parameters.

Table 1 contains list of numerical coefficients and some basic Earth's parameters (Barnes et al. 1983 ; Allen 2000, and author's computations).

By using the set parameters (Table 1), we can evaluate the nutation series tabulated in Table 2.

VII. CONCLUSION

This work deals with an analytical and numerical computation of a second order Hamiltonian to deduce a nutation series for a two-layer Earth model. In table 1, author computed important coefficients which can be used in the computation of numerical nutation series. In table 2, author calculated a longitude and obliquity (in-phase and out-of-phase) for each mantle and core for 10 main periods in a nutation series in a milliarseconds per Julian century. As shown in Table 2, we find the longitude does not exceed 0.002(mas) and obliquity does not exceed 0.0007 (mas) in-phase through mantle for a period 182.62 days.

ACKNOWLEDGEMENTS

The author is grateful to the anonymous referee for helpful comments.

TABLE 2.
NUTATION SERIES (LONGITUDE AND OBLIQUITY), FOR EACH MANTLE AND CORE (UNIT=MAS).

Period (day)	Longitude				Obliquity			
	in- mantle	-phase core	out-of- mantle	-phase core	in- mantel	-phase core	out-of- mantle	-phase core
-6798.36	-0.15784	0.6791	-0.01390	0.01166	-0.20382	0.26792	0.01365	-0.01145
-3399.18	-0.00109	0.00131	-0.00005	0.00004	-0.00058	0.00049	0.00005	-0.00004
365.26	0.00005	-0.00006	-0.00000	0.00000	-0.00000	0.00000	0.000000	-0.00000
365.22	-0.00003	0.00004	0.00000	-0.00000	-0.00001	0.00001	-0.00000	0.00000
182.62	0.00193	-0.00229	-0.00000	0.00000	0.00067	-0.00056	0.00000	-0.00000
121.75	0.00008	-0.00009	-0.00000	0.00000	0.00003	-0.00002	0.00000	-0.00000
27.55	0.00006	-0.00007	-0.00000	0.00000	-0.00000	0.00000	0.00000	-0.00000
13.66	0.00052	-0.00062	-0.00000	0.000000	-0.00010	0.00009	-0.00000	0.00000
13.63	0.00009	-0.00011	0.00000	-0.00000	-0.00012	0.00010	0.00000	-0.00000
9.13	0.00007	-0.00008	-0.00000	0.00000	-0.00001	0.00001	-0.00000	0.00000

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