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TANGENT DIRECTION OF QUADRATIC RATIONAL Bézier CURVE

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Abstract. In this paper we find the point at which the rational Bézier curve has the given tangent direction. We also analyze the geometric properties of the point of quadratic rational Bézier curve.

1. Preliminaries

Conic section or quadratic rational Bézier curves are widely used in CAD/CAM or Computer Graphics[1, 4, 7, 8, 9, 16]. Any conic section curve can be represented in the quadratic rational Bézier form[14]

$$\mathbf{r}(t) = \frac{B_0^2(t)\mathbf{b}_0 + B_1^2(t)w\mathbf{b}_1 + B_2^2(t)\mathbf{b}_2}{B_0^2(t) + B_1^2(t)w + B_2^2(t)} \qquad t \in [0,1]$$

where $\mathbf{b}_i$, $i = 0, 1, 2$, is the control points, $0 < w < \infty$ is the weight associated to $\mathbf{b}_1$, and $B_i^n(t)$ is the Bernstein polynomial of degree n, i.e.,

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

A lot of properties of quadratic rational Bézier curve have been developed in the recent twenty years[2, 3, 5, 6, 12, 13, 15]. In particular, Farin[10] solved the problem.

Problem 1. Let the control points $\mathbf{b}_i$, $i = 0, 1, 2$, be given. For any two points $\mathbf{q}_i = (1 - r_i)\mathbf{b}_{2i} + r_i\mathbf{b}_1$, $0 < r_i < 1$, $i = 0, 1$, find the weight w satisfying $\mathbf{r}(t)$ is tangent to the line $\overline{\mathbf{q}_0\mathbf{q}_1}$.

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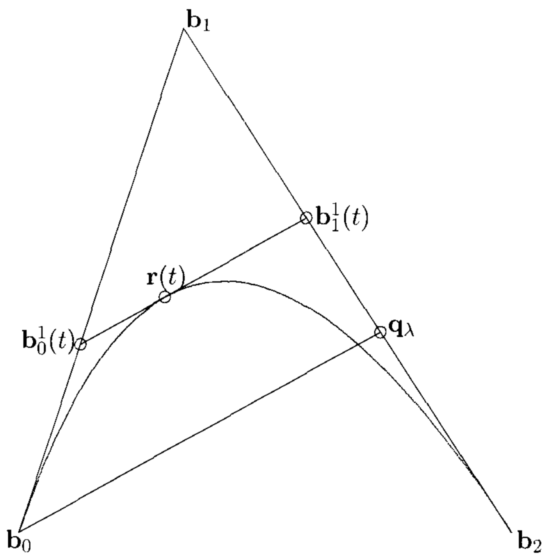

Thus the author is interesting in the following problem

Problem 2. Let the control points $\mathbf{b}_i$, $i = 0, 1, 2$, and weight w be given. For any two points $\mathbf{q}_i = (1 - r_i)\mathbf{b}_{2i} + r_i\mathbf{b}_1$, $0 < r_i < 1$, $i = 0, 1$, find the point $\mathbf{r}(t)$ satisfying $\mathbf{r}'(t)$ is parallel to the line $\overline{\mathbf{q}_0\mathbf{q}_1}$.

It is equivalent to the following problem

Problem 3. Let the control points $\mathbf{b}_i$, $i = 0, 1, 2$, and weight w be given. For any $0 \le \lambda \le 1$, find the point $\mathbf{r}(t)$ satisfying $\mathbf{r}'(t)$ is parallel to the line $\overline{\mathbf{b}_0\mathbf{q}_\lambda}$, where $\mathbf{q}_\lambda = (1 - \lambda)\mathbf{b}_2 + \lambda\mathbf{b}_1$.

In this paper we solve the problem above and analyze the geometric properties of the point $\mathbf{r}(t)$.

2. Tangent direction of quadratic rational Bézier curve

In the following proposition, we give the solution of Problem 3.

Proposition 2.1. *The tangent line of the quadratic rational Bézier* $\mathbf{r}(t)$ *is parallel to the line* $\overline{\mathbf{b}_0\mathbf{q}_\lambda}$ *if and only if*

$$t(w,\lambda) = \begin{cases} 1/2 & (\lambda = 0) \\ \frac{1-\lambda}{2-\lambda} & (\lambda \neq 0,\ w = 1) \\ \frac{-(2w(1-\lambda)+\lambda))+\sqrt{\lambda^2+4w^2(1-\lambda)}}{2\lambda(w-1)} & (\lambda \neq 0,\ w \neq 1) \end{cases}$$

Proof. The tangent line of the quadratic rational Bézier curve $\mathbf{r}(t)$ is parallel to the line $\overline{\mathbf{b}_0\mathbf{q}_\lambda}$ if and only if

$$\overline{\mathbf{b}_0^1(t)\mathbf{b}_1^1(t)} \parallel \overline{\mathbf{b}_0\mathbf{q}_\lambda}.$$

It is equivalent to $ratio(\mathbf{b}_1, \mathbf{b}_0^1(t), \mathbf{b}_0) = ratio(\mathbf{b}_1, \mathbf{b}_1^1(t), \mathbf{q})$, where the ratio is defined[11] by

$$ratio(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \frac{||\mathbf{p}_1 - \mathbf{p}_2||}{||\mathbf{p}_2 - \mathbf{p}_3||}$$

for collinear three points $\mathbf{p}_i$, $i = 1, 2, 3$. Since

$$\mathbf{b}_0^1(t) = \frac{(1-t)\mathbf{b}_0 + tw\mathbf{b}_1}{(1-t)+wt}, \quad \mathbf{b}_1^1(t) = \frac{(1-t)w\mathbf{b}_1 + t\mathbf{b}_2}{(1-t)w+t},$$

we have

$$\begin{aligned} ratio(\mathbf{b}_1, \mathbf{b}_0^1(t), \mathbf{b}_0) &= \frac{1-t}{wt} \\ ratio(\mathbf{b}_1, \mathbf{b}_1^1(t), \mathbf{q}) &= \frac{\frac{t}{(1-t)w+t}}{(1-\lambda) - \frac{t}{(1-t)w+t}} = \frac{t}{(1-\lambda)((1-t)w+t)-t} \end{aligned}$$

The equation $ratio(\mathbf{b}_1, \mathbf{b}_0^1(t), \mathbf{b}_0) = ratio(\mathbf{b}_1, \mathbf{b}_1^1(t), \mathbf{q})$ yields

$$f_1(t) = \frac{1-t}{wt} - \frac{t}{(1-\lambda)((1-t)w+t)-t} = 0$$

which is equivalent

$$f_2(t) = \lambda(w-1)t^2 + (2w - 2w\lambda + \lambda)t - w(1-\lambda) = 0.$$

If $\lambda = 0$, then $f_2(t)$ has unique solution $t = 1/2$, and if $\lambda \neq 0$ and $w = 1$, then $f_2(t)$ has unique solution $t = (1-\lambda)/(2-\lambda)$. If $\lambda \neq 0$ and $w \neq 1$, since $f_2(0) = -w(1-\lambda) < 0$ and $f_2(1) = w > 0$, the quadratic

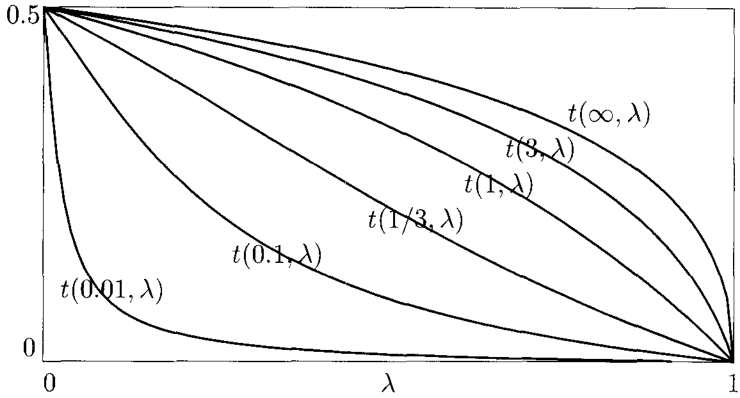

FIGURE 1. $t(w, \lambda)$ for $\lambda \in [0, 1]$.

polynomial $f_2(t)$ has an unique solution in [0,1]. Since $2w(1-\lambda)+\lambda > 0$, the unique solution is equal to

$$t(w, \lambda) = \frac{-(2w(1-\lambda)+\lambda)) + \sqrt{\lambda^2 + 4w^2(1-\lambda)}}{2\lambda(w-1)}.$$

□

In following proposition we obtain some geometric properties of $t(w, \lambda)$.

Proposition 2.2. *$t(w, \lambda)$ is differentiable in $(0, \infty) \times [0, 1]$, $t(w, \lambda)$ decreases as λ increases, and*

$$\lim_{\lambda \to 0} t(w, \lambda) = 1/2, \qquad \lim_{\lambda \to 1} t(w, \lambda) = 0, \tag{2.1}$$

$$\lim_{w \to 1} t(w, \lambda) = \frac{1-\lambda}{2-\lambda}, \qquad \lim_{w \to \infty} t(w, \lambda) = \frac{\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}. \tag{2.2}$$

Proof. Since

$$t(w, \lambda) = \frac{-(2w(1-\lambda)+\lambda)^2 - (\lambda^2 + 4w^2(1-\lambda))}{2\lambda(w-1)(-(2w(1-\lambda)+\lambda)) - \sqrt{\lambda^2 + 4w^2(1-\lambda)})}$$

$$= \frac{2w(1-\lambda)}{2w(1-\lambda)+\lambda+\sqrt{\lambda^2 + 4w^2(1-\lambda)})} \tag{2.3}$$

and its nominator is positive for all $(w, \lambda) \in (0, \infty) \times [0, 1]$, $t(w, \lambda)$ is differentiable in the region. Equation (2.3) yields

$$\frac{dt(w,\lambda)}{d\lambda} = -2w\frac{1 + (w^2(1-\lambda)+\lambda)/\sqrt{\lambda^2 + 4w^2(1-\lambda)}}{(2w(1-\lambda)+\lambda+\sqrt{\lambda^2+4w^2(1-\lambda)})^2} < 0$$

so that $t(w, \lambda)$ decreases as λ increases. Also, by Equation (2.3), we obtain Equations (2.1)-(2.2). □

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