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STRONG DEDUCTIVE SYSTEMS OF BL-ALGEBRAS

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Abstract. The notion of strong deductive system of a BL-algebra is introduced, and a characterization of a strong deductive system is given. A relation between a strong deductive system and a deductive system is given. It will be seen that every strong deductive system can be expressed as the union of special sets.

1. Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [3], arising from the continuous triangular norms (t-norms), familiar in the frameworks of fuzzy set theory. The main example of a BL-algebra is the interval $[0,1]$ endowed with the structure induced by a continuous t-norm. Generally, BL-algebras rise as Lindenbaum algebras from certain logical axioms in a similar manner as MV-algebras do from the axioms of Lukasiewicz logic. In fact, MV-algebras are BL-algebras. The converse, however, is not true. A basic tool in the study of a BL-algebra is a *deductive system* (called *filter* in [3]). From logical point of view, deductive systems correspond sets of provable formulas. Buşneag et al. [1] and Jun et al. [4, 5, 6] studied deductive systems of a BL-algebra. In this article, we introduce the notion of a strong deductive system of a BL-algebra, and give a characterization of a strong deductive system. We provide a relation between a strong deductive system and a

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deductive system. We show that every strong deductive system can be expressed as the union of special sets.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type τ. A *generalized BL-algebra* (cf. [2]) is a system $\mathbf{A} = (A, \wedge, \vee, *, \multimap, \top) \in K(\tau)$, where $\tau = (2, 2, 2, 2, 0)$, such that

(A1) $(A, *, \top)$ is an abelian monoid, that is, $*$ is associative, commutative and $x * \top = x$,

(A2) $\mathbf{L}(\mathbf{A}) := (A, \vee, \wedge, \top)$ is a lattice with greatest element $\top$, that is, the operations $\vee$ and $\wedge$ are both associative, commutative and idempotent; $x \wedge (x \vee y) = x = x \vee (x \wedge y)$, $x \wedge \top = x$,

(A3) $x \multimap x = \top$,

(A4) $(x * y) \multimap z = x \multimap (y \multimap z)$,

(A5) $x \wedge y = x * (x \multimap y)$,

(A6) $(x \multimap y) \vee (y \multimap x) = \top$.

A *BL-algebra* is a bounded generalized BL-algebra, that is, it is a system $\mathbf{A} = (A, \wedge, \vee, * \multimap, \bot, \top) \in K(\tau)$, where $\tau = (2, 2, 2, 2, 0, 0)$, such that

- $(A, \wedge, \vee, *, \multimap, \top)$ is a generalized BL-algebra,
- $\bot$ is the lower bound of $\mathbf{L}(\mathbf{A})$.

In a BL-algebra $\mathbf{A} = (A, \wedge, \vee, * \multimap, \bot, \top)$, the following properties hold for every $x, y, z \in A$ (cf. [1], [3], [7]):

(b1) $x \leq y$ if and only if $x \multimap y = \top$.

(b2) $\top \multimap x = x$, $x \multimap \top = \top$.

(b3) $x \multimap y \leq (z \multimap x) \multimap (z \multimap y)$.

(b4) $(x \multimap y) \multimap (x \multimap z) \leq x \multimap (y \multimap z)$.

(b5) $x \multimap (y \multimap z) = y \multimap (x \multimap z)$.

A BL-algebra $\mathbf{A}$ is said to be *implicative* (see [4]) if it satisfies the following inequality:

$$(2.1) \qquad (\forall x, y, z \in A)\, (x \multimap (y \multimap z) \leq (x \multimap y) \multimap (x \multimap z)).$$

Recall that the power BL-algebra $\mathcal{P}(X)$ of a set X is an implicative BL-algebra (see [4]).

3. Strong deductive systems

In what follows, let $\mathbf{A}$ denote a BL-algebra $\mathbf{A} = (A, \wedge, \vee, * \multimap, \bot, \top)$ unless otherwise specified.

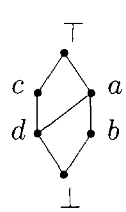
Definition 3.1. A subset P of A is called a *strong deductive system* of $\mathbf{A}$ if it satisfies the following conditions:

(z1) $x \multimap a \in P$ for all $x \in A$ and $a \in P$.

(z2) $(a \multimap (b \multimap x)) \multimap x \in P$ for all $x \in A$ and $a, b \in P$.

Using (A3) and (z1), we know that every strong deductive system contains $\top$.

Example 3.1. Let $A = \{\bot, a, b, c, d, \top\}$ be a set with Hasse diagram and Cayley tables as follows:



$*$	$\top$	a	b	c	d	$\bot$
$\top$	$\top$	a	b	c	d	$\bot$
a	a	b	b	d	$\bot$	$\bot$
b	b	b	b	$\bot$	$\bot$	$\bot$
c	c	d	$\bot$	c	d	$\bot$
d	d	$\bot$	$\bot$	d	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\multimap$	$\top$	a	b	c	d	$\bot$
$\top$	$\top$	a	b	c	d	$\bot$
a	$\top$	$\top$	a	c	c	d
b	$\top$	$\top$	$\top$	c	c	c
c	$\top$	a	b	$\top$	a	b
d	$\top$	$\top$	a	$\top$	$\top$	a
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

For every $x, y \in A$, define $x \wedge y = x * (x \multimap y)$ and

$$x \vee y = ((x \multimap y) \multimap y) * (((x \multimap y) \multimap y) \multimap ((y \multimap x) \multimap x)).$$

Then $\mathbf{A} = (A, \wedge, \vee, * \multimap, \bot, \top)$ is a BL-algebra (cf. [5]). We know that $P_1 := \{a, b, \top\}$ is a strong deductive system of $\mathbf{A}$, but $P_2 := \{a, \top\}$ is not a strong deductive system of $\mathbf{A}$ because there exists $b \in A$ such that $(a \multimap (a \multimap b)) \multimap b = b \notin P_2$.

Proposition 3.2. *If P is a strong deductive system of $\mathbf{A}$, then*

$$(\forall a \in P)\,(\forall x \in A)\,((a \multimap x) \multimap x \in P). \tag{3.1}$$

Proof. The proof follows by taking $b = a$ and $a = \top$ in (z2) and using (b2). □

Corollary 3.3. *Let P be a strong deductive system of $\mathbf{A}$. If $a \in P$ and $a \leq x$, then $x \in P$.*

Proof. Let $a \in P$ and $x \in A$ be such that $a \leq x$. Using (b1), (b2) and Proposition 3.2, we have $x = \top \multimap x = (a \multimap x) \multimap x \in P$. This completes the proof. □

Proposition 3.4. *Let P be a subset of A such that*

(z3) $\top \in P$,

(z4) *whenever $x \multimap (a \multimap y) \in P$ and $a \in P$ then $x \multimap y \in P$.*

If $b \in P$ and $b \leq x$, then $x \in P$.

Proof. Let $b \in P$ and $x \in A$ be such that $b \leq x$. Then

$$\top \multimap (b \multimap x) = \top \multimap \top = \top \in P,$$

and so $x = \top \multimap x \in P$ by (z4) and (b2). □

Theorem 3.5. *Let P be a subset of A. Then P is a strong deductive system of $\mathbf{A}$ if and only if it satisfies the conditions* (z3) *and* (z4).

Proof. Let P be a strong deductive system of $\mathbf{A}$. Obviously, $\top \in P$. Let $x, y, z \in A$ be such that $x \multimap (y \multimap z) \in P$ and $y \in P$. Then $(y \multimap z) \multimap z \in P$ by Proposition 3.2. Since

$$((y \multimap z) \multimap z) \multimap ((x \multimap (y \multimap z)) \multimap (x \multimap z)) = \top$$

by (b1) and (b3), it follows from (b2) and (z2) that

$$\begin{aligned} x \multimap z &= \top \multimap (x \multimap z) \\ &= (((y \multimap z) \multimap z) \multimap ((x \multimap (y \multimap z)) \multimap (x \multimap z))) \multimap (x \multimap z) \in P. \end{aligned}$$

Conversely, suppose that P satisfies conditions (z3) and (z4). Let $x \in A$ and $a \in P$. Since $x \multimap (a \multimap a) = x \multimap \top = \top \in P$ by (z3), we have $x \multimap a \in P$ by (z4), i.e., (z1) is valid. Note that $(a \multimap x) \multimap (a \multimap x) = \top \in P$ so from (z4) that $(a \multimap x) \multimap x \in P$. Since $(a \multimap x) \multimap x \leq (b \multimap (a \multimap x)) \multimap (b \multimap x)$ for all $b \in P$, we have $(b \multimap (a \multimap x)) \multimap (b \multimap x) \in P$ by Proposition 3.4. It follows from (z4) that $(b \multimap (a \multimap x)) \multimap x \in P$ which proves (z2). Hence P is a strong deductive system of $\mathbf{A}$. □

Definition 3.6. [7] A subset D of A is called a *deductive system* of $\mathbf{A}$ if it satisfies:

(i) $\top \in D$,

(ii) $(\forall a \in D)\ (\forall b \in A)\ (a \multimap b \in D \Rightarrow b \in D)$.

Theorem 3.7. *Every strong deductive system is a deductive system.*

Proof. Let D be a strong deductive system of $\mathbf{A}$. Then $\top \in D$ by Theorem 3.5. Let $a \in D$ and $y \in A$ be such that $a \multimap y \in D$. Taking $x = \top$ in (z4) and using (b2), we have $\top \multimap (a \multimap y) = a \multimap y \in D$. It follows from (z4) and (b2) that $y = \top \multimap y \in D$. Hence D is a deductive system of $\mathbf{A}$. □

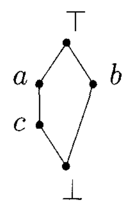
In Example 3.1 the set $P_2 = \{a, \top\}$ is a deductive system of $\mathbf{A}$ which is not a strong deductive system of $\mathbf{A}$. This shows that the converse of Theorem 3.7 is not valid in general.

For any $a, b \in A$, we consider a set $\mathbf{A}_a^b := \{x \in A \mid a \multimap (b \multimap x) = \top\}$. Note that $\mathbf{A}_a^b = \mathbf{A}_b^a$ for all $a, b \in A$. In Example 3.1, the set $\mathbf{A}_a^\top = \{a, \top\}$ is not a strong deductive system of $\mathbf{A}$. Hence we know that $\mathbf{A}_a^b$ may not be a strong deductive system of $\mathbf{A}$ in general.

Proposition 3.8. *Let $y \in A$ be such that $y \multimap z = \top$ for all $z \in A$. Then $\mathbf{A}_x^y = A$ for all $x \in A$.*

Proof. Straightforward. □

Example 3.2. Let $A = \{\bot, a, b, c, \top\}$ be a set with Hasse diagram and Cayley tables as follows:



$*$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	a	a	$\bot$	c	$\bot$
b	b	$\bot$	b	$\bot$	$\bot$
c	c	c	$\bot$	c	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\multimap$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	$\top$	$\top$	b	c	$\bot$
b	$\top$	a	$\top$	c	c
c	$\top$	$\top$	b	$\top$	b
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

For every $x, y \in A$, define $x \wedge y = x * (x \multimap y)$ and

$$x \vee y = ((x \multimap y) \multimap y) * (((x \multimap y) \multimap y) \multimap ((y \multimap x) \multimap x)).$$

Then $\mathbf{A} = (A, \wedge, \vee, * \multimap, \bot, \top)$ is a BL-algebra. Using Proposition 3.8, we have $\mathbf{A}_x^{\bot} = A$ for all $x \in A$. Furthermore, we know that $\mathbf{A}_{\top}^{\top} = \{\top\}$, $\mathbf{A}_a^{\top} = \mathbf{A}_a^a = \mathbf{A}_a^b = \{a, \top\}$, $\mathbf{A}_b^{\top} = \mathbf{A}_b^b = \{b, \top\}$, $\mathbf{A}_c^{\top} = \mathbf{A}_a^c = \mathbf{A}_c^c = \{a, c, \top\}$, and $\mathbf{A}_b^c = A$ are strong deductive systems of $\mathbf{A}$.

Theorem 3.9. *Assume that $\mathbf{A}$ is implicative. For any $a, b \in A$, the set $\mathbf{A}_a^b$ is a strong deductive system of $\mathbf{A}$.*

Proof. Let $x \in A$ and $u, v \in \mathbf{A}_a^b$. Then

$$\begin{aligned} a \multimap (b \multimap (x \multimap u)) &= a \multimap ((b \multimap x) \multimap (b \multimap u)) \\ &= (a \multimap (b \multimap x)) \multimap (a \multimap (b \multimap u)) \\ &= (a \multimap (b \multimap x)) \multimap \top = \top, \end{aligned}$$

and

$$\begin{aligned} & a \multimap (b \multimap ((u \multimap (v \multimap x)) \multimap x)) \\ &= (a \multimap (b \multimap (u \multimap (v \multimap x)))) \multimap (a \multimap (b \multimap x)) \\ &= ((a \multimap (b \multimap u)) \multimap (a \multimap (b \multimap (v \multimap x)))) \multimap (a \multimap (b \multimap x)) \\ &= (\top \multimap ((a \multimap (b \multimap v)) \multimap (a \multimap (b \multimap x)))) \multimap (a \multimap (b \multimap x)) \\ &= (a \multimap (b \multimap x)) \multimap (a \multimap (b \multimap x)) = \top. \end{aligned}$$

Hence $x \multimap u \in \mathbf{A}_a^b$ and $(u \multimap (v \multimap x)) \multimap x \in \mathbf{A}_a^b$, which shows that $\mathbf{A}_a^b$ is a strong deductive system of $\mathbf{A}$. □

Theorem 3.10. *A nonempty subset P of A is a strong deductive system of $\mathbf{A}$ if and only if $\mathbf{A}_a^b \subseteq P$ for all $a, b \in P$.*

Proof. Assume that P is a strong deductive system of $\mathbf{A}$. For every $a, b \in P$, let $w \in \mathbf{A}_a^b$. Then $a \multimap (b \multimap w) = \top \in P$, and so

$$w = \top \multimap w = (a \multimap (b \multimap w)) \multimap w \in P$$

by (z2). Hence $\mathbf{A}_a^b \subseteq P$. Conversely suppose that $\mathbf{A}_a^b \subseteq P$ for all $a, b \in P$. Observe that $\top \in \mathbf{A}_a^b \subseteq P$. Let $x, y, z \in A$ be such that $x \multimap (y \multimap z) \in P$ and $y \in P$. Since

$$\begin{aligned}&(x \multimap (y \multimap z)) \multimap (y \multimap (x \multimap z))\\&= (y \multimap (x \multimap z)) \multimap (y \multimap (x \multimap z)) = \top\end{aligned}$$

by (b5) and (A3), we have $x \multimap z \in \mathbf{A}_y^{x \multimap (y \multimap z)} \subseteq P$. It follows from Theorem 3.5 that P is a strong deductive system of $\mathbf{A}$. □

Corollary 3.11. *Every strong deductive system is closed under the operation $*$.*

Proof. Let D be a strong deductive system of L and let $a, b \in D$. Then $a \multimap (b \multimap (a * b)) = (a * b) \multimap (a * b) = \top$ by (A4) and (A3), and so $a * b \in \mathbf{A}_a^b \subseteq D$ by Theorem 3.10. □

Theorem 3.12. *Every strong deductive system P of $\mathbf{A}$ can be expressed as the union of sets of the form $\mathbf{A}_a^b$ for every $a, b \in P$.*

Proof. Let P be a strong deductive system of $\mathbf{A}$ and let $x \in P$. Observe that $x \in \mathbf{A}_x^\top$, and so $P \subseteq \bigcup_{x \in P} \mathbf{A}_x^\top \subseteq \bigcup_{a,b \in P} \mathbf{A}_a^b$. Now let $y \in \bigcup_{a,b \in P} \mathbf{A}_a^b$. Then $y \in \mathbf{A}_u^v$ for some $u, v \in P$. It follows from Theorem 3.10 that $y \in P$ so that $\bigcup_{a,b \in P} \mathbf{A}_a^b \subseteq P$. This completes the proof. □

Corollary 3.13. *Every strong deductive system P of $\mathbf{A}$ can be expressed as the union of sets of the form $\mathbf{A}_a^\top$ for every $a \in P$.*

4. Concluding remarks

In this paper, we introduced the notion of a strong deductive system of a BL-algebra, and gave a characterization of a strong deductive system. We showed that every strong deductive system can be expressed as the union of special sets. The results of this paper will be devoted to study of MV-algebras, lattice implication algebras, Łukasiewicz' logic, Gödel's logic and the product logic, which are different extensions of basic logic. Moreover, it will be devoted to the problem to reveal the logical content of various methods from fuzzy logic which play a specific role in fuzzy control and expert systems, e.g. Zadeh's compositional rule of inference, generalized modus ponens, min-composition, generalized quantification, etc. Some important issues for future work are: (i) developing the properties of a strong deductive system, (ii) defining new deductive systems which are related to given strong deductive systems, and (iii) finding useful results on the new structures.

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