Honam Mathematical J. **29** (2007), No. 3, pp. 427–443

SPANNING COLUMN RANK PRESERVERS OF INTEGER MATRICES

Kyung-Tae Kang and Seok-Zun Song*

Abstract. The spanning column rank of an $m \times n$ integer matrix A is the minimum number of the columns of A that span its column space. We compare the spanning column rank with column rank of matrices over the ring of integers. We also characterize the linear operators that preserve the spanning column rank of integer matrices.

1. Introduction

There are many papers on the study of linear operators that preserve certain matrix functions such as rank, determinant and so on over fields. We can find them in [4]. Also the (nonnegative) integer matrices are important topics of many researchers([1],[2],[6],[7]).

Frobenius(1897), Marcus and Moyles(1959), Marcus and May(1976), Marcus and Purves(1959), Beasley(1970), Minc(1976) and Kovacs(1977) characterized those linear operators on the matrices over field that preserve: determinant and characteristic polynomial, rank, permanent, the rth symmetric function($r \geq 4$), and so on(see [4]).

In 1983, McDonald [5] found that the characterizations of determinant and rank functions were valid over more general rings.

In 1985, analogues of Marcus and Moyles's work on rank were obtained by Beasley, Gregory and Pullman [1] for nonnegative integers.

Received June 4, 2007. Accepted July 6, 2007.
2000 Mathematics Subject Classification: 15A03, 15A04, 15A36.
Key words and phrases: spanning column rank, linear operator, (P,Q)-operator.
* Corresponding author.

Recently, Song([6]) researched on the spanning column rank of matrices over semirings and Song et al.([3], [8]) characterized linear operators that preserve (spanning) column rank of matrices over semirings of nonnegative reals.

In this paper, we obtain characterizations of those linear operators on $m \times n$ matrices over ring of integers that preserve spanning column rank.

2. Preliminaries and Basic Results

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{M}_{m\times n}(\mathbb{Z})$ denote the set of all $m \times n$ matrices with entries in the ring of integers. Addition, multiplication by scalars, and the product of matrices are defined similarly as over a field. Let $\mathbb{E}_{m,n} = \{E_{ij} \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$, where E_{ij} is the $m \times n$ matrix whose $(i, j)^{\text{th}}$ entry is 1 and whose other entries are 0. We call each member of $\mathbb{E}_{m,n}$ a *cell*. The zero matrix is denoted by O. The identity matrix of order k is denoted by I_k. The transpose of a matrix A, denoted by A^t, is defined in the usual way.

The *factor rank*, $r(A)$, of a nonzero matrix A in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is defined as the least integer k for which there exist matrices $B \in \mathbb{M}_{m\times k}(\mathbb{Z})$ and $C \in \mathbb{M}_{k\times n}(\mathbb{Z})$ such that $A = BC$. The factor rank of a zero matrix is zero. Also we can easily obtain that $0 \le r(A) \le \min\{m, n\}$. The *real rank* of A will be denoted by $\rho(A)$.

If A is a matrix in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $\rho(A) = k$, then $A = BC$ for some $m \times k$ and $k \times n$ matrices B and C over the real field $\mathbb{R}$: for, $\rho(A) = k$ implies that there exist nonsingular $m \times m$ and $n \times n$ real matrices U and V, respectively such that $A = U(I_k \oplus O)V$, where O is the $(m-k)\times(n-k)$ zero matrix. Let $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$ denote the i^{th} column and the j^{th} row of U and V, respectively for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. It follows from $A = U(I_k \oplus O)V$ that $A = \boldsymbol{u}_1\boldsymbol{v}_1 + \boldsymbol{u}_2\boldsymbol{v}_2 + \cdots + \boldsymbol{u}_k\boldsymbol{v}_k = BC$, where $B = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_k]$ and $C = [\boldsymbol{v}_1^t \ \cdots \ \boldsymbol{v}_k^t]^t$. Then B and C are $m \times k$ and

$k \times n$ real matrices, respectively. This shows that $\rho(X) \leq r(X)$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$. In fact, we obtain that $\rho(X) = r(X)$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$ (see Theorem 2.2).

An $n \times n$ integer matrix A is called *nonsingular* if for any vector $\boldsymbol{x}$ in $\mathbb{Z}^n$, $A\boldsymbol{x} = \mathbf{0}$ implies that $\boldsymbol{x} = \mathbf{0}$.

Lemma 2.1. *The factor rank of a matrix in* $\mathbb{M}_{m\times n}(\mathbb{Z})$ *is unchanged by pre- or post-multiplication by a nonsingular matrix.*

Proof. The proof is straightforward. □

Theorem 2.2. *For any matrix* A *in* $\mathbb{M}_{m\times n}(\mathbb{Z})$*, we have* $\rho(A) = r(A)$.

Proof. Since $\rho(A) \leq r(A)$ for all A in $\mathbb{M}_{m\times n}(\mathbb{Z})$, it suffices to show that $r(A) \leq \rho(A)$ for all A in $\mathbb{M}_{m\times n}(\mathbb{Z})$. If $\rho(A) = k$, then there exist $m \times k$ and $k \times n$ real matrices B and C, respectively such that $A = BC$. Since entries of A are integers, we may assume that B and C are matrices over the rational field $\mathbb{Q}$. Let α and β are the least common multiples in $\mathbb{Z}$ of denominators of nonzero entries of B and C, respectively. Then we have that αB and βC are $m \times k$ and $k \times n$ integer matrices, respectively. Thus we have $r(\alpha\beta A) \leq k$ since $\alpha\beta A = (\alpha B)(\beta C)$. Notice that $\alpha\beta A = DA$, where D is the diagonal matrix in $\mathbb{M}_{m\times m}(\mathbb{Z})$ whose main diagonal entries are $\alpha\beta$. By Lemma 2.1, the factor ranks of A and $\alpha\beta A$ are equal since D is nonsingular. Consequently, we obtain that $r(A) = r(\alpha\beta A) \leq \rho(A)$. □

If $\mathbb{V}$ is a nonempty subset of $\mathbb{Z}^n \equiv \mathbb{M}_{n\times 1}(\mathbb{Z})$ which is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbb{Z}$. A nonzero vector $\boldsymbol{p} = [p_1, p_2, \cdots, p_n]^t$ in $\mathbb{Z}^n$ is *irreducible* if the greatest common divisor of p_i's is 1 (that is, $\gcd\{p_1, \ldots, p_n\} = 1$). A subset $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_d\}$ of $\mathbb{Z}^n$ is called *linearly dependent* if there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ in $\mathbb{Z}$, not all zeros, such that $\sum_{i=1}^{d} \alpha_i \boldsymbol{s}_i = \mathbf{0}$; S is called *linearly independent* if it is not linearly dependent.

Proposition 2.3. *Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ be linearly independent vectors in $\mathbb{Z}^n$. Then for any nonzero vector $\boldsymbol{b}$ in $\mathbb{Z}^n$, there exist nonzero integer β and integers α_i, not all zero, such that $\beta \boldsymbol{b} = \alpha_1 \boldsymbol{p}_1 + \alpha_2 \boldsymbol{p}_2 + \cdots + \alpha_n \boldsymbol{p}_n$.*

Proof. Let A be the $n \times n$ matrix whose columns are $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$. Then A is nonsingular, and hence $\det(A)$ is a nonzero integer. Consider a system $A\boldsymbol{x} = \boldsymbol{b}$ of n linear equations in n unknowns. By Cramer's rule, this system has a unique solution $x_i = \frac{\det(A_i)}{\det(A)}$ in the rational numbers for all $i = 1, 2, \ldots, n$, where A_i is the matrix obtained by replacing the entries in the i^{th} column of A by the entries in $\boldsymbol{b}$. Then we have

$$\boldsymbol{b} = \frac{\det(A_1)}{\det(A)} \boldsymbol{p}_1 + \frac{\det(A_2)}{\det(A)} \boldsymbol{p}_2 + \cdots + \frac{\det(A_n)}{\det(A)} \boldsymbol{p}_n.$$

If we take $\beta = \det(A)$ and $\alpha_i = \det(A_i)$, then the result follows. □

For nonzero $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^n$, we say $\boldsymbol{x}$ is similar to $\boldsymbol{y}$, which denoted by $\boldsymbol{x} \simeq \boldsymbol{y}$ if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent. Then we can easily show that $\simeq$ is an equivalence relation in $\mathbb{Z}^n$.

Lemma 2.4. *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonzero vectors in $\mathbb{Z}^n$ with $\boldsymbol{x} \simeq \boldsymbol{y}$, then there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{x} = \alpha \boldsymbol{p}$ and $\boldsymbol{y} = \beta \boldsymbol{p}$ for some nonzero integers α and β.*

Proof. Since $\boldsymbol{x} \simeq \boldsymbol{y}$, $a\boldsymbol{x} = b\boldsymbol{y}$ for some nonzero integers a and b. Let $\boldsymbol{x} = [x_1, \cdots, x_n]^t$, $\boldsymbol{y} = [y_1, \cdots, y_n]^t$ and $\alpha = \gcd(x_1, \ldots, x_n)$. Then there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{x} = \alpha \boldsymbol{p}$. Thus $a\boldsymbol{x} = b\boldsymbol{y}$ becomes

$$a\alpha \boldsymbol{p} = b\boldsymbol{y}. \tag{2.1}$$

Let $r = \gcd(a\alpha, b)$, $r_1 = \frac{a\alpha}{r}$ and $r_2 = \frac{b}{r}$. Then r_1 and r_2 are nonzero in $\mathbb{Z}$ with $\gcd(r_1, r_2) = 1$, and (2.1) becomes

$$r_1\boldsymbol{p} = r_2\boldsymbol{y}. \tag{2.2}$$

Therefore we have r_1 divides every r_2y_i for all $i = 1, \ldots, n$. Since r_1 is relatively prime to r_2, it follows that r_1 divides every entry in $\boldsymbol{y}$. Thus we have $\boldsymbol{y} = r_1\boldsymbol{c}$ for some nonzero vector $\boldsymbol{c}$ in $\mathbb{Z}^n$. By the cancellation, (2.2) becomes $\boldsymbol{p} = r_2\boldsymbol{c}$. Then r_2 is a unit in $\mathbb{Z}$ because r_2 divides every entry in the irreducible vector $\boldsymbol{p}$. That is, $r_2 = \pm 1$ so that $\boldsymbol{y} = \pm r_1\boldsymbol{p}$ by (2.2). If we take $\beta = \pm r_1$, the result follows. □

Corollary 2.5. *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are irreducible vectors in $\mathbb{Z}^n$ with $\boldsymbol{x} \simeq \boldsymbol{y}$, then $\boldsymbol{y} = \pm\boldsymbol{x}$.*

Proof. It is an immediate consequence of Lemma 2.4. □

As with fields, a *basis* for a vector space $\mathbb{V}$ is a spanning subset of least cardinality. That cardinality is the *dimension*, $\dim(\mathbb{V})$, of $\mathbb{V}$.

The *column space* of a matrix A in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is the vector space that is spanned by its columns. The *column rank*, $c(A)$ of A in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is the dimension of the column space of A. Then Beasley and Pullman [2] showed that

$$r(X) = c(X) \tag{2.3}$$

for all matrices X in $\mathbb{M}_{m\times n}(\mathbb{Z})$.

The *spanning column rank*, $sc(A)$ of A in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is the minimum number of the columns of A which span its column space ([8]). As with factor rank, the zero matrix is assigned column rank and spanning column rank 0.

It follows that

$$0 \le c(X) \le sc(X) \le n \tag{2.4}$$

for all matrices X in $\mathbb{M}_{m\times n}(\mathbb{Z})$.

Over a field $\mathbb{F}$ we have $c(X) = sc(X)$ for all X in $\mathbb{M}_{m\times n}(\mathbb{F})$. For, if $c(X) = k$, then the column space of X has dimension k. So any r columns of X are linearly dependent for r which is greater than k. Hence $sc(X) \leq k$. Therefore the column rank and spanning column rank of any matrix X in $\mathbb{M}_{m\times n}(\mathbb{F})$ are equal by (2.4). But the following Example shows that they may differ over the ring of integers.

Example 2.6. Consider a matrix

$$A = \begin{bmatrix} p & q \\ p & q \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{Z}),$$

where p and q are distinct positive prime integers. Then there exist nonzero integers α and β such that $\alpha p + \beta q = 1$. Thus $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis of the column space of A. Thus $c(A) = 1$. But we have $sc(A) = 2$ since $\begin{bmatrix} p \\ p \end{bmatrix} \neq a \begin{bmatrix} q \\ q \end{bmatrix}$ and $\begin{bmatrix} q \\ q \end{bmatrix} \neq b \begin{bmatrix} p \\ p \end{bmatrix}$ for any a and b in $\mathbb{Z}$.

Lemma 2.7. *For any nonzero $1\times n$ matrix A over $\mathbb{Z}$, we have $sc(A) = 1$ or 2.*

Proof. If $n = 1$ or 2, there is nothing to prove. So we may assume $n \geq 3$. Let $A = [a_1\ a_2 \cdots\ a_n]$ be any $1 \times n$ matrix over $\mathbb{Z}$. Let $\alpha = \gcd(a_1, \ldots, a_n)$. Then we have $a_i = \alpha a_i'$ for some integers a_i' for all $i = 1, \ldots, n$ with $\gcd(a_1', \ldots, a_n') = 1$. If $a_i' = \pm 1$ for some i, then $\{a_i\}$ spans the column space of A. Thus $sc(A) = 1$. If $a_i' \neq \pm 1$ for all i, it follows from $\gcd(a_1', \ldots, a_n') = 1$ that there exist two distinct indices k and l in $\{1, \ldots, n\}$ such that $\gcd(a_k', a_l') = 1$, equivalently $xa_k' + ya_l' = 1$ for some nonzero integers x and y. Then $x\alpha a_k' + y\alpha a_l' = \alpha$, equivalently $xa_k + ya_l = \alpha$. This implies that α is in the column space of A and thus $\{a_k, a_l\}$ spans the column space of A. Hence $sc(A) = 2$. □

The following is an immediate consequence of Lemma 2.7.

Corollary 2.8. *For a nonzero $1 \times n$ matrix A over $\mathbb{Z}$, $sc(A) = 1$ if and only if there exists an entry a_k in A such that a_k is a common divisor of all entries in A.*

Lemma 2.9. *For a matrix X in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $sc(X) = 1$, any two columns of X are linearly dependent.*

Proof. Since $sc(X) = 1$, it follows that there exists a nonzero k^{th} column $\boldsymbol{x}_k$ of X such that $\{\boldsymbol{x}_k\}$ spans the column space of X. Let $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ be any two nonzero columns of X. Then $\boldsymbol{x}_i = \alpha\boldsymbol{x}_k$ and $\boldsymbol{x}_j = \beta\boldsymbol{x}_k$ for some nonzero integers α and β. From these two equalities, we obtain that $\beta\boldsymbol{x}_i - \alpha\boldsymbol{x}_j = \mathbf{0}$. Thus the Lemma follows. □

In general, the converse of Lemma 2.9 is not true. For example, consider a matrix A in Example 2.6. Clearly, two columns of A are linearly dependent, while $sc(A) = 2$.

For any two matrices A in $\mathbb{M}_{m\times n}(\mathbb{Z})$ and B in $\mathbb{M}_{n\times t}(\mathbb{Z})$, we have

$$sc(AB) \leq sc(B). \tag{2.5}$$

For, if $sc(B) = k$, then there exist k columns $\boldsymbol{b}_{i_1}, \boldsymbol{b}_{i_2}, \ldots, \boldsymbol{b}_{i_k}$ of B of minimum cardinality that spans its column space. Thus we have that any j^{th} column of B is of the form $\boldsymbol{b}_j = \sum_{l=1}^{k} \alpha_l \boldsymbol{b}_{i_l}$ for some integers $\alpha_1, \ldots, \alpha_k$. This shows that for any j^{th} column of AB is of the form $A\boldsymbol{b}_j = \sum_{l=1}^{k} \alpha_l A\boldsymbol{b}_{i_l}$. Thus, any column of AB can be written as a linear combination of $A\boldsymbol{b}_{i_1}, A\boldsymbol{b}_{i_2}, \ldots, A\boldsymbol{b}_{i_k}$. Therefore we have $sc(AB) \leq k = sc(B)$.

Lemma 2.10. *For any matrix A in $\mathbb{M}_{m\times n}(\mathbb{Z})$, $sc(A) = 1$ if and only if A can be factored as $A = \boldsymbol{x}\boldsymbol{a}^t$ for some nonzero vectors $\boldsymbol{x}$ in $\mathbb{Z}^m$ and $\boldsymbol{a}$ in $\mathbb{Z}^n$ with $sc(\boldsymbol{a}^t) = 1$.*

Proof. It is straightforward to see that $sc(\boldsymbol{x}\boldsymbol{a}^t) = 1$ for nonzero vectors $\boldsymbol{x}$ in $\mathbb{Z}^m$ and $\boldsymbol{a}$ in $\mathbb{Z}^n$ with $sc(\boldsymbol{a}^t) = 1$. Conversely, assume that $sc(A) = 1$. Then there exists one nonzero column $\boldsymbol{a}_k$ of A such that all columns $\boldsymbol{a}_i$ of A can be expressed as a scalar multiple of $\boldsymbol{a}_k$ so that $\boldsymbol{a}_i = \alpha_i \boldsymbol{a}_k$ for some integers α_i and for all $i = 1, \ldots, n$. Therefore $A = \boldsymbol{a}_k[\alpha_1, \cdots, \alpha_n]$ with $\alpha_k = 1$. If we let $\boldsymbol{x} = \boldsymbol{a}_k$ and $\boldsymbol{a} = [\alpha_1, \cdots, \alpha_n]^t$, then $A = \boldsymbol{x}\boldsymbol{a}^t$ and $\boldsymbol{a}^t$ has spanning column rank 1 since $\alpha_k = 1$. Thus the Lemma follows. □

3. Spanning column rank preservers

In this section we have characterizations of the linear operators that preserve the spanning column rank of matrices over $\mathbb{Z}$.

Suppose that T is an operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Say that T is a

(i) *linear operator* if $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $\alpha, \beta \in \mathbb{Z}$ and for all $X, Y \in \mathbb{M}_{m\times n}(\mathbb{Z})$.

(ii) *spanning column rank preserver* if $sc(T(X)) = sc(X)$ for all $X \in \mathbb{M}_{m\times n}(\mathbb{Z})$.

(iii) *spanning column rank-r preserver* if $sc(T(X)) = r$ whenever $sc(X) = r$ for all $X \in \mathbb{M}_{m\times n}(\mathbb{Z})$.

Lemma 3.1. *For a given matrix Q in $\mathbb{M}_{m\times m}(\mathbb{Z})$, define a linear operator T on $\mathbb{M}_{m\times n}(\mathbb{Z})$ by $T(X) = QX$ for all $X \in \mathbb{M}_{m\times n}(\mathbb{Z})$. Then T is a spanning column rank preserver if and only if Q is nonsingular.*

Proof. Assume that Q is nonsingular. It follows from (2.5) that $sc(T(X)) \leq sc(X)$. Thus it is sufficient to show that $sc(X) \leq sc(T(X))$. For any matrix $X = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_n]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, we have $QX = [Q\boldsymbol{x}_1\ Q\boldsymbol{x}_2\ \cdots\ Q\boldsymbol{x}_n]$. If $sc(T(X)) = sc(QX) = k$, there exist k columns $Q\boldsymbol{x}_{i_1}, \ldots, Q\boldsymbol{x}_{i_k}$ of QX, of minimum cardinality that spans its column space. Let $\boldsymbol{x}_j$ be any j^{th} column of X. Then $Q\boldsymbol{x}_j$ is the j^{th} column of

QX, and hence $Q\boldsymbol{x}_j = \sum_{t=1}^{k} \alpha_t Q\boldsymbol{x}_{i_t} = Q\sum_{t=1}^{k} \alpha_t \boldsymbol{x}_{i_t}$. Since Q is nonsingular, it follows that $\boldsymbol{x}_j = \sum_{t=1}^{k} \alpha_t \boldsymbol{x}_{i_t}$. This shows that $\{\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_k}\}$ spans the column space of X. Thus, $sc(X) \leq k = sc(QX)$. Therefore we have $sc(X) = sc(QX)$, and hence T is a spanning column rank preserver on $\mathbb{M}_{m\times n}(\mathbb{Z})$.

Conversely, assume that Q is singular in $\mathbb{M}_{m\times m}(\mathbb{Z})$. Then $Q\boldsymbol{x}= \mathbf{0}$ for some nonzero vector $\boldsymbol{x}$ in $\mathbb{Z}^m$. Consider the matrix $X = [\boldsymbol{x}\ \boldsymbol{x}\ \cdots\ \boldsymbol{x}]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then we have $sc(X) = 1$, but $sc(T(X)) = 0$ since $T(X) = QX = [Q\boldsymbol{x}\ Q\boldsymbol{x}\ \cdots\ Q\boldsymbol{x}] = O$. Therefore T does not preserve spanning column rank 1. This contradiction shows that Q must be nonsingular. □

Let P be a given matrix in $\mathbb{M}_{n\times n}(\mathbb{Z})$. Define a linear operator T on $\mathbb{M}_{m\times n}(\mathbb{Z})$ by $T(X) = XP$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Even if P is nonsingular, T may not be a spanning column rank preserver on $\mathbb{M}_{m\times n}(\mathbb{Z})$. For example, let $P = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$ be a matrix in $\mathbb{M}_{2\times 2}(\mathbb{Z})$, where p and q are distinct prime integers. Consider a matrix $A = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$ in Example 2.6 with $sc(A) = 2$. But the spanning column rank of $AP = \begin{bmatrix} pq & pq \\ pq & pq \end{bmatrix}$ is 1.

Let $P = [\,p_{ij}\,]$ be an $n\times n$ matrix over $\mathbb{Z}$. We say that P is an *absolute permutation matrix* if $[\,|p_{ij}|\,]$ is a permutation matrix in $\mathbb{M}_{n\times n}(\mathbb{Z})$, where $|p_{ij}|$ is the absolute value of p_{ij}. Thus all nonzero entries of P are either 1 or -1. Furthermore, we can easily show that if P and P' are absolute permutation matrices in $\mathbb{M}_{n\times n}(\mathbb{Z})$, then their product, PP', is also an absolute permutation matrix.

We say that a linear operator T on $\mathbb{M}_{m\times n}(\mathbb{Z})$ is a *nontransposing (Q, P)-operator* if there exist a nonsingular matrix Q in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and an absolute permutation matrix P in $\mathbb{M}_{n\times n}(\mathbb{Z})$ such that $T(X) = QXP$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$.

For any index i in $\{1, \ldots, n\}$, $\boldsymbol{e}_i^{(n)}$ denotes a vector in $\mathbb{Z}^n$ with "1" in the i^{th} position and zero elsewhere.

Lemma 3.2. *Let T be a linear operator defined by $T(E_{ij}) = b_{ij}\, \boldsymbol{q}_i\, \boldsymbol{p}_j^t$ for all cells E_{ij} in $\mathbb{M}_{m\times n}(\mathbb{Z})$, where $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m$ and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are linearly independent vectors in $\mathbb{Z}^m$ and $\mathbb{Z}^n$, respectively and all b_{ij} are nonzero integers. If T is a spanning column rank-1 preserver on $\mathbb{M}_{m\times n}(\mathbb{Z})$, then T is a nontranposing (Q, P)-operator.*

Proof. Assume that T is a spanning column rank-1 preserver on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Let

$$Q' = [\boldsymbol{q}_1 \ \cdots \ \boldsymbol{q}_m], \quad P' = [\boldsymbol{p}_1 \ \cdots \ \boldsymbol{p}_n]^t \ \text{ and } \ B = [\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_n],$$

where each $\boldsymbol{b}_i = [b_{1i}\ b_{2i}\ \cdots\ b_{mi}]^t$ is a vector in $\mathbb{Z}^m$. Then Q' and P' are nonsingular matrices in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and $\mathbb{M}_{n\times n}(\mathbb{Z})$, respectively. Thus, we can write for any matrix $X = [x_{ij}]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$,

$$T(X) = Q'X'P', \tag{3.1}$$

where $X' = [x_{ij}b_{ij}]$ is a matrix in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Let $\boldsymbol{p}_j = [a_{j1}, a_{j2}, \cdots, a_{jn}]^t$ for all $j = 1, \ldots, n$. Since the columns of $(P')^t$ is linearly independent, without loss of generality, we may assume that $a_{jj} \neq 0$ for all $j = 1, \ldots, n$ (if necessary, we permute the columns of $(P')^t$).

First, we show that for all $i, j = 1, \ldots, n$,

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ \pm a_{11} & \text{if } i = j. \end{cases}$$

Consider a matrix $X_1 = E_{11} + \sum\limits_{k=2}^{n} d_k E_{1k} \in \mathbb{M}_{m\times n}(\mathbb{Z})$, where each d_k is in $\mathbb{Z}$. Then $sc(X_1) = 1$ since $\{\boldsymbol{e}_1^{(m)}\}$ spans the column space of X_1. Thus, the spanning column rank of

$$T(X_1) = \boldsymbol{q}_1 \left[b_{11}a_{11} + \sum_{k=2}^{n} d_k b_{1k} a_{k1} \ \middle| \ \cdots \ \middle| \ b_{11}a_{1n} + \sum_{k=2}^{n} d_k b_{1k} a_{kn} \right]$$

must have 1, equivalently $sc(C) = sc([c_1 \ \cdots \ c_n]) = 1$, where

$$c_i = b_{11}a_{1i} + \sum_{k=2}^{n} d_k b_{1k} a_{ki}.$$

Assume that $a_{k1} \neq 0$ for some $k = 2, 3, \ldots, n$. Notice that $\mathbb{Z}$ has infinitely many prime integers. Thus, we can select prime integers $d_2, \ldots, d_n$ such that $c_i \neq \pm 1$ for all $i = 1, \ldots, n$ and $\gcd(c_1, \ldots, c_n) = 1$ since the number of a_{ij}'s and b_{ij}'s is finite. By Corollary 2.8, $sc(C) = 2$, a contradiction. Therefore $a_{k1} = 0$ for all $k = 2, \ldots, n$. Similarly, if we consider a matrix $X_j = E_{1j} + \sum_{k=1,\ k\neq j}^{n} d_k E_{1k}$ with $sc(X_j) = 1$, we can easily obtain $a_{kj} = 0$ for all k in $\{1, \ldots, n\} \setminus \{j\}$. Therefore $a_{ij} = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$.

Suppose that $a_{jj} \neq \pm a_{11}$ for some $j = 1, \ldots, n$. Then either $|a_{jj}| > |a_{11}|$ or $|a_{jj}| < |a_{11}|$, say $|a_{jj}| > |a_{11}|$. Consider a spanning column rank 1 matrix $Y = pE_{11} + E_{1j}$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, where p is a prime integer with $\gcd(p, b_{1j}a_{jj}) = 1$. It follows from Corollary 2.8 that the spanning column rank of

$$T(Y) = \boldsymbol{q}_1[p\, b_{11},\ 0,\ \cdots,\ 0,\ b_{1j}a_{jj},\ 0,\ \cdots,\ 0]$$

must have 2, a contradiction. Thus, we have that $a_{jj} = \pm a_{11}$ for all $j = 1, \ldots, n$.

Next, we claim that $\boldsymbol{b}_j = \pm \boldsymbol{b}_1$ for all $j = 1, \ldots, n$. Suppose that $\boldsymbol{b}_j$ is neither $\boldsymbol{b}_1$ nor $-\boldsymbol{b}_1$ for some j. Consider a matrix $Z = q\sum_{k=1}^{m} E_{k1} + \sum_{k=1}^{m} E_{kj}$ with $sc(Z) = 1$, where q is a prime integer such that $\gcd(q, b_{1j}, b_{2j}, \ldots, b_{mj}) = 1$. Then

$$T(Z) = Q'[q\, a_{11}\boldsymbol{b}_1 \ \ \boldsymbol{0} \ \ \cdots \ \ \boldsymbol{0} \ \ \pm a_{11}\boldsymbol{b}_j \ \ \boldsymbol{0} \ \ \cdots \ \ \boldsymbol{0}]$$

must have spanning column rank 1. By Lemma 3.1, the spanning column rank of $G = [q\,\boldsymbol{b}_1 \ \ \boldsymbol{0} \ \ \cdots \ \ \boldsymbol{0} \ \ \pm \boldsymbol{b}_j \ \ \boldsymbol{0} \ \ \cdots \ \ \boldsymbol{0}]$ must also have 1. But $q\,\boldsymbol{b}_1 \neq \alpha\,\boldsymbol{b}_j$ and $\boldsymbol{b}_j \neq \beta(q\,\boldsymbol{b}_1)$ for any α, β in $\mathbb{Z}$. Thus $sc(G) = 2$, a contradiction. Therefore we have that $\boldsymbol{b}_j$ is either $\boldsymbol{b}_1$ or $-\boldsymbol{b}_1$ for all

$j = 1, \ldots, n$. It follows that

$$B = [\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_n] = \boldsymbol{b}_1[\alpha_1, \alpha_2, \cdots, \alpha_n],$$

where $\alpha_i = 1$ or -1, equivalently $b_{ij} = b_{i1}\alpha_j$.

Let $D = diag(b_{11}, \ldots, b_{m1})$ and $F = diag(\alpha_1, \ldots, \alpha_n)$ be diagonal matrices in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and $\mathbb{M}_{n\times n}(\mathbb{Z})$, respectively. Then for any matrix $X = [x_{ij}]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, the $(i,j)^{\text{th}}$ entry of DXF is $b_{i1}x_{ij}\alpha_j = x_{ij}b_{ij}$. Therefore (3.1) becomes

$$T(X) = Q'DXFP' \tag{3.2}$$

for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$. If we let $Q \equiv a_{11}Q'D$ and $P \equiv \frac{1}{a_{11}}FP'$, then (3.2) becomes $T(X) = QXP$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Here, Q is nonsingular in $\mathbb{M}_{m\times m}(\mathbb{Z})$ since D is nonsingular, and P is an absolute permutation matrix since $\frac{1}{a_{11}}P'$ and F are absolute permutation matrices. Thus the Lemma follows. □

Corollary 3.3. *Any nontransposing (Q,P)-operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$ is a spanning column rank preserver.*

Proof. The proof follows from Lemma 3.1 and definitions of a nontransposing (Q,P)-operator and an absolute permutation matrix. □

Lemma 3.4. *Let T be a linear operator defined by $T(X) = X^t$ for all X in $\mathbb{M}_{n\times n}(\mathbb{Z})$ with $n \geq 2$. Then T does not preserve spanning column rank r for $r \geq 1$.*

Proof. Let A be the matrix in Example 2.6. Then we have $sc(A) = 2$, while $sc(A^t) = 1$. Consider $B = A^t \oplus 0_{n-2} \in \mathbb{M}_{n\times n}(\mathbb{Z})$. Then $sc(B) = 1$ but $T(B) = B^t$ has spanning column rank 2. Let

$$C = B \oplus I_k \oplus 0_{n-k-2} \in \mathbb{M}_{n\times n}(\mathbb{Z}).$$

Then $sc(C) = 1 + k$ but $T(C) = C^t$ has spanning column rank $2 + k$. Therefore T does not preserve spanning column rank r for $r \geq 1$. □

Proposition 3.5. *Let A and B be matrices in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $sc(A) = sc(B) = 1$ and let $A = \boldsymbol{x}\boldsymbol{a}^t$ and $B = \boldsymbol{y}\boldsymbol{b}^t$ be any factorizations. If $sc(A+B) = 1$, then we have $\boldsymbol{x} \simeq \boldsymbol{y}$ or $\boldsymbol{a} \simeq \boldsymbol{b}$.*

Proof. Suppose that $\boldsymbol{x} \not\simeq \boldsymbol{y}$ and $\boldsymbol{a} \not\simeq \boldsymbol{b}$. It follows from the definition of the similar relation "$\simeq$" that $\boldsymbol{x}$, $\boldsymbol{y}$ are linearly independent in $\mathbb{Z}^m$, and $\boldsymbol{a}$, $\boldsymbol{b}$ are also linearly independent in $\mathbb{Z}^n$. Then we can easily show that $r(A+B) = 2$, and thus $c(A+B) = 2$ by (2.3). It follows from (2.4) that $sc(A+B) \geq 2$, a contradiction. Therefore $\boldsymbol{x} \simeq \boldsymbol{y}$ or $\boldsymbol{a} \simeq \boldsymbol{b}$. □

In general, the converse of Proposition 3.5 is not true. For example, consider two matrices $A = \boldsymbol{x}[1 \;\; 1]$ and $B = \boldsymbol{x}[1 \;\; 2]$ in $\mathbb{M}_{m\times 2}(\mathbb{Z})$, where $\boldsymbol{x}$ is a nonzero vector in $\mathbb{Z}^m$. Then $sc(A) = sc(B) = 1$, while $sc(A+B) = 2$ since $A + B = \boldsymbol{x}[2 \;\; 3]$.

Theorem 3.6. *Let T be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then T is an injective spanning column rank-1 preserver if and only if T is a nontransposing (Q,P)-operator.*

Proof. Assume that $T(X) = QXP$ for all X in $\mathbb{M}_{m\times n}(\mathbb{Z})$, where Q is nonsingular and P is an absolute permutation matrix. Clearly, T is injective. It follows from Corollary 3.3 that T is a spanning column rank-1 preserver.

Conversely, assume that T is an injective spanning column rank-1 preserver on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Note that every cell has spanning column rank 1. Thus by Lemma 2.10, for any cell E_{ij} in $\mathbb{E}_{m,n}$, we can write

$$T(E_{ij}) = \boldsymbol{u}^{ij}{\boldsymbol{v}_{ij}}^t$$

for some nonzero vectors $\boldsymbol{u}^{ij}$ in $\mathbb{Z}^m$ and $\boldsymbol{v}_{ij}$ in $\mathbb{Z}^n$ with $sc({\boldsymbol{v}_{ij}}^t) = 1$. Let j and k be arbitrary in $\{1, \ldots, n\}$. Since $E_{ij} + E_{ik}$ has spanning column rank 1, the spanning column rank of $T(E_{ij} + E_{ik}) = \boldsymbol{u}^{ij}{\boldsymbol{v}_{ij}}^t + \boldsymbol{u}^{ik}{\boldsymbol{v}_{ik}}^t$ must be 1. It follows from Proposition 3.5 that $\boldsymbol{u}^{ij} \simeq \boldsymbol{u}^{ik}$ or $\boldsymbol{v}_{ij} \simeq \boldsymbol{v}_{ik}$.

Now, we will show that for a fixed i in $\{1, \ldots, m\}$, either

$$\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{i2} \simeq \cdots \simeq \boldsymbol{u}^{in} \quad \text{or} \quad \boldsymbol{v}_{i1} \simeq \boldsymbol{v}_{i2} \simeq \cdots \simeq \boldsymbol{v}_{in}. \tag{3.3}$$

Suppose that $\boldsymbol{v}_{i1} \not\simeq \boldsymbol{v}_{ij}$ for some j in $\{2, \ldots, n\}$. By Proposition 3.5, we have $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ij}$ because $sc(T(E_{i1} + E_{ij})) = 1$. If $\boldsymbol{u}^{i1} \not\simeq \boldsymbol{u}^{ik}$ for some k in $\{2, \ldots, m\}$, then we have $\boldsymbol{v}_{i1} \simeq \boldsymbol{v}_{ik}$ by Proposition 3.5. Therefore $\boldsymbol{v}_{ij} \not\simeq \boldsymbol{v}_{ik}$ because $\simeq$ is an equivalence relation. Thus $\boldsymbol{u}^{ij} \simeq \boldsymbol{u}^{ik}$ and this would imply $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ik}$ because $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ij}$. This contradicts to $\boldsymbol{u}^{i1} \not\simeq \boldsymbol{u}^{ik}$, and thus (3.3) is established.

Similarly, we can show that for a fixed j in $\{1, \ldots, n\}$, either

$$\boldsymbol{u}^{1j} \simeq \boldsymbol{u}^{2j} \simeq \cdots \simeq \boldsymbol{u}^{mj} \quad \text{or} \tag{3.4}$$

$$\boldsymbol{v}_{1j} \simeq \boldsymbol{v}_{2j} \simeq \cdots \simeq \boldsymbol{v}_{mj}. \tag{3.5}$$

If $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{i2} \simeq \cdots \simeq \boldsymbol{u}^{in}$, then there exist an irreducible vector $\boldsymbol{q}_i$ in $\mathbb{Z}^m$ and nonzero integers α_j such that $\boldsymbol{u}^{ij} = \alpha_j \boldsymbol{q}_i$ for all $j = 1, \ldots, n$. Thus we can write $T(E_{ij}) = \boldsymbol{u}^{ij} {\boldsymbol{v}_{ij}}^t$ as $T(E_{ij}) = \boldsymbol{q}_i (\alpha_j \boldsymbol{v}_{ij})^t$ for all $j = 1, \ldots, n$. Therefore we can restate (3.3) as follows. For a fixed i in $\{1, \ldots, m\}$, either

$$\boldsymbol{u}^{i1} = \boldsymbol{u}^{i2} = \cdots = \boldsymbol{u}^{in} = \boldsymbol{q}_i \quad \text{or} \tag{3.6}$$

$$\boldsymbol{v}_{i1} = \boldsymbol{v}_{i2} = \cdots = \boldsymbol{v}_{in} = \boldsymbol{p}_i, \tag{3.7}$$

where $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$ are irreducible vectors.

Assume that (3.6) holds for some $i = 1, \ldots, m$. If $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{in}$ are linearly dependent, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{Z}$, not all zeros, such that $\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_{ij} = \mathbf{0}$. Consider a nonzero matrix $X = \sum_{j=1}^{n} \alpha_j E_{ij}$. Then we have

$$T(X) = T\left(\sum_{j=1}^{n} \alpha_j E_{ij} \right) = \boldsymbol{u}^{i1} \left(\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_{ij} \right)^t = O = T(O), \tag{3.8}$$

a contradiction to the fact that T is injective. Thus $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{in}$ are linearly independent. Analogous statements are satisfied in case (3.4), (3.5) or (3.7).

Next, we will show that if (3.6) holds for a fixed $i = 1, \ldots, m$, then (3.5) must hold for all $j = 1, \ldots, n$, and consequently (3.6) must hold for all i. Suppose that (3.4) holds for some $j = 1, \ldots, n$. Then $\boldsymbol{u}^{ij} (= \boldsymbol{q}_i)$ appears both in (3.6) and (3.4). It follows from (3.4) and $\boldsymbol{u}^{ij} = \boldsymbol{q}_i$ that there exist nonzero integers α_s such that $\boldsymbol{u}^{sj} = \alpha_s \boldsymbol{q}_i$ for all $s = 1, \ldots, m$. Note that $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{in}$ are linearly independent since (3.6) is satisfied. By Proposition 2.3, there exist nonzero integer β_s and integers β_{sk}, not all zero, such that $\beta_s \boldsymbol{v}_{sj} = \sum_{k=1}^{n} \beta_{sk} \boldsymbol{v}_{ik}$ for all $s = 1, \ldots, m$. Then we have

$$\beta_s \boldsymbol{u}^{sj} {\boldsymbol{v}_{sj}}^t = \sum_{k=1}^{n} \beta_{sk} \boldsymbol{u}^{sj} {\boldsymbol{v}_{ik}}^t = \sum_{k=1}^{n} \beta_{sk} \alpha_s \boldsymbol{q}_i {\boldsymbol{v}_{ik}}^t = \sum_{k=1}^{n} \beta_{sk} \alpha_s \boldsymbol{u}^{ik} {\boldsymbol{v}_{ik}}^t,$$

equivalently

$$T(\beta_s E_{sj}) = T\Big(\sum_{k=1}^{n} \beta_{sk} \alpha_s E_{ik} \Big) \tag{3.9}$$

for all $s \in \{1, \ldots, m\} \backslash \{i\}$. This contradicts to the fact that T is injective. Thus we have established that either

$$\boldsymbol{u}^{ij} = \boldsymbol{q}_i \quad \text{and} \quad \boldsymbol{v}_{ij} = b_{ij} \boldsymbol{p}_j \tag{3.10}$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m$ and $\boldsymbol{p}_1 \ldots, \boldsymbol{p}_n$ are linearly independent irreducible vectors and b_{ij} are nonzero integers, or

$$\boldsymbol{v}_{ij} = b_{ij} \boldsymbol{p}_i \quad \text{and} \quad \boldsymbol{u}^{ij} = \boldsymbol{q}_j \tag{3.11}$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ and $\boldsymbol{q}_1 \ldots, \boldsymbol{q}_n$ are linearly independent irreducible vectors and b_{ij} are nonzero integers.

Now, we assume that (3.10) holds. Then we have that for any cell E_{ij} in $\mathbb{M}_{m \times n}(\mathbb{Z})$, $T(E_{ij}) = b_{ij} \boldsymbol{q}_i \boldsymbol{p}_j^t$, where $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m$ and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are linearly independent vectors in $\mathbb{Z}^m$ and $\mathbb{Z}^n$, respectively and each b_{ij} is nonzero. It follows from Lemma 3.2 that $T(X) = QXP$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z})$, where Q is a nonsingular matrix in $\mathbb{M}_{m \times m}(\mathbb{Z})$ and P is an absolute permutation matrix in $\mathbb{M}_{n \times n}(\mathbb{Z})$. Thus T is a nontransposing (Q, P)-operator.

Next, assume that (3.11) holds. Then $m = n$, and by the similar argument of the proof of Lemma 3.2, we have that $T(X) = QX^tP$ for all X in $\mathbb{M}_{n\times n}(\mathbb{Z})$, where Q is a nonsingular matrix and P is an absolute permutation matrix in $\mathbb{M}_{n\times n}(\mathbb{Z})$. But, if we consider the matrix C in Lemma 3.4, we can easily obtain that $sc(C) = 1 + k$, while $sc(T(C)) = 2 + k$ for all $k \geq 0$. Thus T does not preserve spanning column rank r for all $r \geq 1$. Hence (3.11) is impossible. □

Corollary 3.7. *Let T be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $n \geq 2$. Then the following statements are equivalent:*

(i) *T is an injective spanning column rank-1 preserver;*
(ii) *T is a nontransposing (Q, P)-operator;*
(iii) *T is a spanning column rank preserver;*
(iv) *T preserves spanning column ranks 1, 2 and 3.*

Proof. The equivalence of (i) and (ii) follows from Theorem 3.6. By Corollary 3.3, (ii) implies (iii). Clearly, (iii) implies (iv). Thus it suffices to show that (iv) implies (ii). Assume that T preserves spanning column ranks 1, 2 and 3. In the proof of Theorem 3.6, we use the injectivity only in identities (3.8) and (3.9). Consider the identity (3.8) with $X = \sum_{j=1}^{n} \alpha_j E_{ij}$. It follows from Lemma 2.7 that $sc(X) = 1$ or 2. But $sc(T(X)) = sc(O) = 0$, a contradiction to the fact that T preserves spanning column ranks 1 and 2. Consider the identity (3.9) with $Y = \beta_s E_{sj} - \sum_{k=1}^{n} \beta_{sk}\alpha_s E_{ik}$ for all $s \in \{1, \ldots, m\}\setminus\{i\}$. Then we can easily show that $sc(Y) = 2$ or 3 by Lemma 2.7, while $sc(T(Y)) = sc(O) = 0$. This contradicts to the fact that T preserves spanning column ranks 2 and 3. Therefore (ii) is satisfied. □

Thus we have characterized the linear operators that preserve the spanning column rank of matrices over the ring of integers.

References

[1] L. B. Beasley, D. A. Gregory and N. J. Pullman, *Nonnegative rank-preserving operators*, Linear Algebra Appl., **65**(1985), 207-223.

[2] L. B. Beasley and N. J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl., **101**(1988), 33-48.

[3] L. B. Beasley, S. Z. Song, K. T. Kang and B. K. Sarma, *Column ranks and their preservers over nonnegative real matrices*, Linear Algebra Appl., **399**(2005), 3-16.

[4] C. K. Li et al., *Linear preserver problems references*, Linear and Multilinear Algebra, **33**(1992), 121-129.

[5] B. R. McDonald, *R-linear endomorphisms of $(R)^n$ preserving invariants*, Mem. Amer. Math. Soc., **46**(1983) No. 287.

[6] S. Z. Song, *On spanning column rank of matrices over semirings*, Bull. Korean Math. Soc., **32**(1995), 337-342.

[7] S. Z. Song, *Linear operators that preserve maximal column ranks of nonnegative integer matrices*, Proc. Amer. Math. Soc., **126**(1998), 2205-2211.

[8] S. Z. Song and S. G. Hwang, *Spanning column ranks and their presevers of nonnegative matrices*, Linear Algebra Appl., **254**(1997), 485-495.

Kyung-Tae Kang
Department of Mathematics
Cheju National University
Jeju 690-756, Republic of Korea
E-mail: `kangkt@cheju.ac.kr`

Seok-Zun Song
Department of Mathematics
Cheju National University
Jeju 690-756, Republic of Korea
E-mail: `szsong@cheju.ac.kr`