Honam Mathematical J. **29** (2007), No. 3, pp. 403–413

NORMED HILBERT ALGEBRAS

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Abstract. In this paper, we introduce the notion of a norm in Hilbert algebras, and discuss some properties of Cauchy sequences.

1. Introduction

The origin of Hilbert algebras is in Mathematical Logic. In 1923, D. Hilbert was the first who remarked the possibility of studying a very interesting part of the classical propositional calculus taking as axioms only the one verified by logical implication (this field is known as *positive implicative propositional calculus*). So, the equation class H of all Hilbert algebras constitutes a natural algebraic semantics for the pure-implicational fragment of intuitionistic logic, and therefore the algebras from H are sometimes called *positive implication algebras* ([6]). D. Buşneag ([2]) defined a pseudo-valuation on a Hilbert algebra and investigated several properties. Also, he ([3]) provided several theorems on extensions of pseudo-valuations(valuations). In this paper, we introduce the notion of a norm in Hilbert algebras, and discuss some properties of Cauchy sequences.

Received May 29, 2007. Accepted June 30, 2007.
2000 Mathematics Subject Classification: 03G25, 06A06, 06F35.
Key words and phrases: (normed) Hilbert algebra, subalgebra, open/cloesd sphere, Cauchy sequence.

2. Preliminaries

By a *Hilbert algebra* ([4]) we mean an algebra $(A, \to, 1)$ of type (2,0) satisfying the following identities:

(a_1) $x \to (y \to x) = 1$,
(a_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(a_3) If $x \to y = y \to x = 1$, then $x = y$

for any $x, y, z \in A$.

In [4] it was proved that (a_1)-(a_3) are equivalent with (a_4)-(a_7):

(a_4) $x \to x = 1$,
(a_5) $1 \to x = x$,
(a_6) $x \to (y \to z) = (x \to y) \to (x \to z)$,
(a_7) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$

for any $x, y, z \in A$.

If A is a Hilbert algebra, then the relation $x \leq y$ if and only if $x \to y = 1$ is a partial order on A (which will be called the *natural ordering on* A); with respect to this ordering 1 is the greatest element of A. A subset S of a Hilbert algebra A is called a *subalgebra* of A if $x \to y \in S$ for all $x, y \in S$. A subset D of a Hilbert algebra A is called a *deductive system* of A if (i) $1 \in D$, (ii) $x, x \to y \in D$ imply $y \in D$. A *bounded Hilbert algebra* is a Hilbert algebra with least element 0 relative to the natural order.

In a Hilbert algebra A, the following hold for all $x, y, z \in A$:

(c_1) $x \to 1 = 1$,
(c_2) $x \leq y \to x$,
(c_3) $x \to (y \to z) = y \to (x \to z)$,
(c_4) $x \leq (x \to y) \to y$,
(c_5) $((x \to y) \to y) \to y = x \to y$,
(c_6) $x \to y \leq (y \to z) \to (x \to z)$,
(c_7) If $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$,

(c_8) $x \to (x \to y) = x \to y$.

Example 2.1. ([2, 3]) (1) Classical propositional calculus, where $\to$ denotes the logical implication $\Leftrightarrow$, is a Hilbert algebra.
(2) If $(A, \leq)$ is a poset with the greatest element 1, then $(A, \to, 1)$ is a Hilbert algebra, where for $x, y \in A$, $x \to y = 1$ if $x \leq y$ and y otherwise.
(3) If $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean lattice, then $(A, \to, 1)$ is a bounded Hilbert algebra, where $\to$ is defined by $x \to y := (\neg x) \vee y$ for any $x, y \in A$.
(4) Consider a set $A := \{a, b, c, d, 1\}$ and the operation " $\to$ " defined by the table:

$\to$	a	b	c	d	1
a	1	d	1	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

Then $(A, \to, 1)$ is a Hilbert algebra and $\{1, a, c\}$ is a subalgebra of A.

W. H. Wu([3]) introduced the notion of fuzzy implication algebras, and Y. B. Jun ([6]) discussed Cauchy sequences in a normed fuzzy implicative algebra. We show that the Hilbert algebra is a different algebraic structure from a fuzzy implication algebra by simply providing examples.

A nonempty subset X together with a binary operation $\to$ and a zero element 0 is called a *fuzzy implication algebra* (*FI-algebra*, for short) ([3]) if the following axioms are satisfied for all $x, y, z \in X$:

(I1) $x \to (y \to z) = y \to (x \to z)$,

(I2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(I3) $x \to x = 1$,

(I4) $x \to y = y \to x = 1 \implies x = y$,

(I5) $0 \to x = 1$,

where $1 = 0 \to 0$.

Example 2.2. ([6]) Let $X := \{0, a, b, c, 1\}$ be a set with the following Cayley table:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(X, \rightarrow, 0)$ is an FI-algebra. Since $(b \rightarrow (b \rightarrow a)) \rightarrow ((b \rightarrow b) \rightarrow (b \rightarrow a)) = c$, it cannot be a Hilbert algebra.

On the while, in Example 2.1-(4), the Hilbert algebra $(A, \rightarrow, 1)$ has no element α satisfying $\alpha \rightarrow \alpha = 1$ and $\alpha \rightarrow x = 1, \forall x \in A$, which means that $(A, \rightarrow, 1)$ is not a fuzzy implication algebra.

3. Normed Hilbert Algebras

Definition 3.1. Let A be a Hilbert algebra. A function which assigns to each element $x \in A$ the real number $||x||$ is called a *norm* on A if it satisfies, for all $x, y, z \in A$, the following axioms:

(N1) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 1$.

(N2) $||x \rightarrow y|| \leq ||x \rightarrow z|| + ||z \rightarrow y||$.

A Hilbert algebra A together with a norm is called a *normed Hilbert algebra.* The real number $||x||$ is called the *norm* of x.

Example 3.2. (1) The norm $|| \; || : A \rightarrow \mathbb{R}$, defined by $||x|| := 0$ for any $x \in A$, is a norm on A (called a *trivial norm*).

(2) If M is a finite set with n elements and $A = \mathcal{P}(M)$ is the Boolean lattice of the power set of M, then a map $|| \; || : \mathcal{P}(M) \rightarrow \mathbb{R}$ defined by $||X|| = n - |X|$ is a norm on X, where $|X|$ is the number of elements of X.

Example 3.3. We denote by $Ds(A)$ the set of all deductive systems of a Hilbert algebra. If $X \subseteq A$, we denote by $(X)= \cap\{D \in Ds(A)|X \subseteq D\}$ ((X) is called the *deductive system generated by* X). If $X = \{x_1, x_2, \cdots, x_n\}$ we denote by $(x_1, x_2, \cdots, x_n)=(\{x_1, x_2, \cdots, x_n\})$; the deductive system generated by one element $a \in A$, will denote by $[a)$ and easily to verify that $[a) =\{x \in A|a \leq x\}$ ($[a)$ is said to be *principal*). For $D_1, D_2 \in Ds(A)$, we define

$$D_1 \wedge D_2 := D_1 \cap D_2,\ D_1 \vee D_2 := D_1 \cup D_2,\ D_1 \rightarrow D_2 := \{a \in A|[a) \cap D_1 \subseteq D_2\}.$$

Then $(Ds(A), \vee, \wedge)$ is a lattice with 0 and 1 where 0 is $\{1\}$ and 1 is A ([1]). If $D \in Ds(A)$, then it is easy to show that the map $||\ ||_D : A \rightarrow \mathbb{R}$ defined by

$$||x||_D := \begin{cases} 0 & \text{if } x \in D \\ 1 & \text{if } x \notin D \end{cases}$$

is a norm on A.

Proposition 3.4. *Let A be a normed Hilbert algebra. Define a function*

$$d : A \times A \rightarrow \mathbb{R} \text{ by } d(x, y) := ||y \rightarrow x||$$

for all $(x, y) \in A \times A$. Then, for every $x, y, z \in A$,

(i) $d(x, 1) \geq 0$ *and* $d(x, 1) = 0$ *if and only if* $x = 1$,
(ii) $d(x, y) \leq d(x, z) + d(z, y)$,
(iii) $d(x, x) = 0$,
(iv) $d(1, x) = 0$,
(v) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,
(vi) $x \leq y \Rightarrow ||y|| \leq ||x||$,
(vii) $d(x, y) \leq ||x||$,
(viii) $d(x \rightarrow y, z) = d(z \rightarrow y, x)$,
(ix) $x \leq y \Rightarrow d(y, z) \leq d(x, z)$ *and* $d(z, x) \leq d(z, y)$.

Proof. (i) $d(x, 1) = ||1 \rightarrow x|| = ||x|| \geq 0$, and obviously $d(x, 1) = 0$ if and only if $x = 1$.
(ii) by (N2).

(iii) $d(x,x) = ||x \to x|| = ||1|| = 0$ by (N1).

(iv) $d(1,x) = ||x \to 1|| = ||1|| = 0$ by (c_1) and (N1).

(v) If $d(x,y) = d(y,x) = 0$, then $||y \to x|| = ||x \to y|| = ||0||$ and so $y \to x = 1 = x \to y$. Hence, by (a_3), we have $x = y$.

(vi) If $x \leq y$, then $x \to y = 1$ and thus

$$||y|| = ||1 \to y|| \leq ||1 \to x|| + ||x \to y|| = ||x|| + ||1|| = ||x||.$$

(vii) Since $x \leq y \to x$, it follows from (vi) that $d(x,y) \leq ||x||$.

(viii) Since $x \to (y \to z) = y \to (x \to z)$, we get

$$d(x \to y, z) = ||z \to (x \to y)|| = ||x \to (z \to y)|| = d(z \to y, x).$$

(ix) Assume that $x \leq y$. Then $z \to x \leq z \to y$ and $y \to z \leq x \to z$. It follows from (vi) that $d(y,z) \leq d(x,z)$ and $d(z,x) \leq d(z,y)$. □

Proposition 3.5. *Let A be a normed Hilbert algebra. Then the set $J := \{x \in A \mid ||x|| = 0\}$ is a deductive system of A.*

Proof. By (N1), $1 \in J$. Let $x, x \to y \in J$, for any $y \in A$. Then $||x|| = 0$ and $||x \to y|| = 0$.

$$||y|| = ||1 \to y|| \leq ||1 \to x|| + ||x \to y|| = ||x|| + ||x \to y|| = 0.$$

Since $||x|| \geq 0 \ \forall x \in A$, we have $||y|| = 0$ and so $y \in J$. Thus J is a deductive system of A. □

Let A be a normed Hilbert algebra. For any $x \in A$ and a positive real number δ, we let $S(x,\delta)$ denote the set of points within a distance of δ from x:

$$S(x,\delta) := \{y \in A | d(y,x) < \delta\}.$$

We call $S(x,\delta)$ the *open sphere* with center x and radius δ. The set

$$\bar{S}(x,\delta) := \{y \in A | d(y,x) \leq \delta\}$$

is called the *closed sphere* with center x and radius δ.

Proposition 3.6. *Let A be a normed Hilbert algebra. For any $x \in A$ and a positive real number δ, the open* (resp. *closed*) *sphere $S(x, \delta)$* (resp. *$\bar{S}(x, \delta)$*) *is a subalgebra of A.*

Proof. Let $y, z \in S(x, \delta)$. Then $d(y, x) < \delta$. Since $y \leq z \to y$, it follows from Proposition 3.4-(ix) that $d(z \to y, x) \leq d(y, x) < \delta$ so that $z \to y \in S(x, \delta)$. Hence $S(x, \delta)$ is a subalgebra of A. Similarly, $\bar{S}(x, \delta)$ is a subalgebra of A. □

Definition 3.7. Let A be a normed Hilbert algebra. The sequence $\{x_n\}$ of members of A is said to *converges to* $p \in A$, denoted by $\lim_{n\to\infty} x_n = p$, if for every $\epsilon > 0$ there exists a positive integer n_0 such that $n > n_0$ implies $d(x_n, p) < \epsilon$ and $d(p, x_n) < \epsilon$.

Proposition 3.8. *Let A be a normed Hilbert algebra. Every convergent sequence in A has a unique limit.*

Proof. Let $\{x_n\}$ be a sequence in A which converges to p and q. Let $\epsilon > 0$. Then there exists a positive integer n_0 such that $n > n_0$ implies

$$d(x_n, p) < \frac{\epsilon}{2},\ d(p, x_n) < \frac{\epsilon}{2},\ d(x_n, q) < \frac{\epsilon}{2},\ \text{and}\ d(q, x_n) < \frac{\epsilon}{2}.$$

Using Proposition 3.4-(ii), we have

$$d(p, q) \leq d(p, x_n) + d(x_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and

$$d(q, p) \leq d(q, x_n) + d(x_n, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n > n_0$. Since ϵ is arbitrary, it follows that $d(p, q) = 0 = d(q, p)$ so from Proposition 3.4-(v) that $p = q$. This completes the proof. □

In general, sequences of real numbers may or may not have convergent subsequence, and every bounded sequence of real numbers contain a convergent subsequence.

Theorem 3.9. *Let $\{x_n\}$ be a sequence in a normed Hilbert algebra A and let $\{x_{i_n}\}$ be a subsequence of $\{x_n\}$. If $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} x_{i_n} = p$.*

Proof. Assume that $\lim_{n\to\infty} x_n = p$. Then for every $\epsilon > 0$, there exists a positive integer n_0 such that

$$n > n_0 \text{ implies } d(x_n, p) < \epsilon \text{ and } d(p, x_n) < \epsilon.$$

Note that there exists a positive integer k such that if $n > k$ then $i_n > n_0$. It follows that $d(x_{i_n}, p) < \epsilon$ and $d(p, x_{i_n}) < \epsilon$. Hence $\{x_{i_n}\}$ converges to p. This completes the proof. □

Theorem 3.10. *Let $\{x_n\}$ be a sequence in a normed Hilbert algebra A which converges to $p \in A$. Then for every $y \in A$, the sets $\{d(x_n, y)\}$ and $\{d(y, x_n)\}$ are bounded.*

Proof. Since $\lim_{n\to\infty} x_n = p$, the sets $\{d(x_n, p)\}$ and $\{d(p, x_n)\}$ are bounded. Hence there exist $M_1 > 0$ and $M_2 > 0$ such that $d(x_n, p) \leq M_1$ and $d(x_n, p) \leq M_2$. Using Proposition 3.4-(ii), we have

$$d(x_n, y) \leq d(x_n, p) + d(p, y) \leq M_1 + d(p, y) \leq M + N$$

and

$$d(y, x_n) \leq d(y, p) + d(p, x_n) \leq d(y, p) + M_2 \leq M + N$$

where $M := \max\{M_1, M_2\}$ and $N := \max\{d(p, y), d(y, p)\}$. Consequently, $\{d(x_n, y)\}$ and $\{d(y, x_n)\}$ are bounded. □

Definition 3.11. Let $\{x_{i_n}\}$ be a subsequence of a sequence $\{x_n\}$ in a normed Hilbert algebra A. If $\{x_{i_n}\}$ converges, its limits is called a *subsequential limit* of $\{x_n\}$.

Theorem 3.12. *The subsequential limits of a sequence $\{x_n\}$ in a normed Hilbert algebra A form a closed subset of X.*

Proof. Let E^* be the set of all subsequential limits of $\{x_n\}$ and let q be a limit point of E^*. We have to show that $q \in E^*$.

Choose n_1 so that $x_{n_1} \neq q$. (If no such n_1 exists, then E^* has only one point, and there is nothing to prove.) Put $\delta := d(q, x_{n_1})$. Suppose $n_1, \cdots, n_{i-1}$ are chosen. Since q is a limit point of E^*, there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$ and $d(q, x) < 2^{-i}\delta$. Since $x \in E^*$, there exists an integer $n_i > n_{i-1}$ such that $d(x, x_{n_i}) < 2^{-i}\delta$ and $d(x_{n_i}, x) < 2^{-i}\delta$. Thus

$$d(q, x_{n_i}) \leq d(q, x) + d(x, x_{n_i}) < 2^{-i}\delta + 2^{-i}\delta$$

and

$$d(x_{n_i}, q) \leq d(x_{n_i}, x) + d(x, q) < 2^{-i}\delta + 2^{-i}\delta$$

for $i = 1, 2, 3, \cdots$. This says that $\{x_{n_i}\}$ converges to q. Hence $q \in E^*$. □

Definition 3.13. A sequence $\{x_n\}$ in a normed Hilbert algebra A is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists a positive integer n_0 such that $n, m > n_0$ implies $d(x_n, x_m) < \epsilon$ and $d(x_m, x_n) < \epsilon$.

Theorem 3.14. *Every convergent sequence in a normed Hilbert algebra is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a sequence in a normed Hilbert algebra A which converges to $p \in A$. Let $\epsilon > 0$. Then there exists a positive integer n_0 such that

$$n > n_0 \text{ implies } d(x_n, p) < \frac{\epsilon}{2} \text{ and } d(p, x_n) < \frac{\epsilon}{2}$$

and

$$m > n_0 \text{ implies } d(x_m, p) < \frac{\epsilon}{2} \text{ and } d(p, x_m) < \frac{\epsilon}{2}.$$

Consequently, $n, m > n_0$ implies

$$d(x_n, x_m) \leq d(x_n, p) + d((p, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and

$$d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.15. *Let $\{x_n\}$ be a Cauchy sequence in a normed Hilbert algebra A. If a subsequence $\{x_{i_n}\}$ of $\{x_n\}$ converges to a point $p \in A$, then the Cauchy sequence itself converges to p.*

Proof. Let $\epsilon > 0$. We need to find a positive integer n_0 such that

$$n > n_0 \text{ implies } d(x_n, p) < \epsilon \text{ and } d(p, x_n) < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists a positive integer n_0 such that $n, m > n_0$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$ and $d(x_m, x_n) < \frac{\epsilon}{2}$. Also, since the subsequence $\{x_{i_n}\}$ converges to p, there exists a positive integer i_m such that $d(x_{i_m}, p) < \frac{\epsilon}{2}$ and $d(p, x_{i_m}) < \frac{\epsilon}{2}$. Observe that we can choose i_m so that $i_m > n_0$. Accordingly, $n > n_0$ implies

$$d(x_n, p) \leq d(x_n, x_{i_m}) + d(x_{i_m}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and

$$d(p, x_n) \leq d(p, x_{i_m}) + d(x_{i_m}, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ converges to p. This completes the proof. □

References

[1] D. Buşneag, *A note on deductive systems of Hilbert algebra*, Kobe J. Math. **2** (1985), 29-35.

[2] D. Buşneag, *Hilbert algebras with valuation*, Math. Japonica **44(2)** (1996), 285-289.

[3] D. Buşneag, *On extensions of pseudo-valuations on Hilbert algebras*, Discrete Mathematics **263** (2003), 11-24.

[4] A. Diego, Sur les algèbras de Hilbert, Hermann, Paris, Collection de Logique Math. **21**, 1966.

[5] Y. B. Jun, *Deductive system of Hilbert algebras*, Math. Japonica **43(1)** (1996), 51-54.

[6] Y. B. Jun, *Normed Fuzzy Implication Algebras*, Advances in Algebras (to appear).

[7] H. Rasiowa, An Algebraic Approach to Non-classical Logic, North-Holland, Amsterdam, 1974.

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