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EXPONENTIAL INEQUALITY AND ALMOST SURE CONVERGENCE FOR THE NEGATIVELY ASSOCIATED SEQUENCE

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Abstract. For bounded negatively associated random variables we derive almost sure convergence and specify the associated rate of convergence by establishing exponential inequality.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$. We start with definition. A finite family $\{X_1, X_2, \cdots, X_n\}$ is said to be negatively associated(NA) if for every disjoint subsets A, $B \subset \{1, \cdots, n\}$ and for any real coordinatewise non-decreasing functions f on R^A and, g on R^B, $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

This concept was introduced by Joag-Dev and Proschan[3]. As pointed out and proved by Joag-Dev and Proschan[3], a number of well-known multivariate distributions possess the negative association property such as multinominal distribution, multivariate hypergeometric distribution, negatively correlated normal distribution, permutation distribution, and joint distribution of ranks.

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Because of their wide applications in multivariate statistical analysis and reliability theory, the concept of negative associated random variables has received extensive attention recently. We refer to Joag-Dev and Proschan[3] for fundamental properties, Newman[8] for the central limit theorem, Matula[7] for the maximal inequalities, Roussas[9] for the Hoeffding[2] inequality, Shao and Su[11] for law of the iterated logarithm, Kim and Ko[5] for complete convergence of weighted sum and Shao[10] for weak convergence.

Set $S_n = \sum_{i=1}^{n} X_i$ and $\overline{S}_n$ for S_n/n. The problem of proving exponential bounds for the probabilities $P(|\overline{S}_n| \geq \epsilon)$ $(\epsilon > 0)$ is of paramount importance, both in probability and statistics. From a statistical viewpoint such inequalities can be used, among other things, for the purpose of providing rates of convergence(both in probability sense and almost surely) for estimates of various quantities. Especially so in a nonparametric setting, where the advantages of structure are not available to the investigator. Exponential inequalities for various kinds of random variables were studied extensively. Some of these exponential inequalities were studied by Devroye[1], Roussas[9] and Shao[10].

In this paper, we derive the almost sure convergence of negatively associated sequence, and specify the associated rate of convergence by establishing exponential inequality.

2. Notations, assumptions and preliminary results

For the formulation of the result to be made in this paper, introduction of some notation is required. The notation is closed related to the way the proofs are carried out. Namely, for positive integer $1 \leq p = p(n) < n$ and $p \to \infty$, divide the set $\{1, 2, \cdots, n\}$ into successive groups each containing p elements. Let r be the largest integer with,

$$0 < r < n, \ r \to \infty, \ \textit{and } 2pr \leq n, \tag{1}$$

which implies that $n/2pr \to 1$. Thus, the set $\{1, 2, \cdots, n\}$ is into $2r$ groups, each consisting of p elements, the remaining $n - 2rp(< p)$ elements constitute a set which may be empty.

With p and r as in above define the random variables U_i, V_i, $i = 1, 2, \cdots, r$ and W_n by

$$U_i = X_{2(i-1)p+1} + \cdots + X_{(2i-1)p}, \tag{2}$$

$$V_i = X_{(2i-1)p+1} + \cdots + X_{2ip}, \tag{3}$$

$$W_n = X_{2pr+1} + \cdots + X_n, \tag{4}$$

$$\overline{U}_n = \frac{1}{n}\sum_{i=1}^{n} U_i, \quad \overline{V}_n = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \overline{W}_n = \frac{W_n}{n}, \tag{5}$$

so that

$$\overline{S}_n = \overline{U}_n + \overline{V}_n + \overline{W}_n. \tag{6}$$

The basic assumption is that the negatively associated random variables X_i's are mean zero and bounded, i.e, $|X_i| \le M$, $i \ge 1$.

Define ϵ_n by

$$\epsilon_n = (\frac{\alpha M^2}{2})^{\frac{1}{2}} (\frac{\log n}{r})^{\frac{1}{2}}, \tag{7}$$

where M is an assumption, r is as in (1), and α is an arbitrary constant > 1.

Lemma 2.1(Devroye[1]) Let X be a centered random variable. If there exist $a, b \in R$ such that $P(a \le X \le b) = 1$, then for every $\lambda > 0$,

$$E(\exp(\lambda X)) \le exp(\frac{\lambda^2 (b-a)^2}{8}). \tag{8}$$

Lemma 2.2(Newman[8]) Let $\{X_i, i \ge 1\}$ be a sequence of negatively associated random variables. Then

$$E(exp\sum_{i=1}^{n} X_i) \le \prod_{i=1}^{n} [E(expX_i)]. \tag{9}$$

3. Main results

Lemma 3.1 Let $\overline{U}_n$ be defined in (5) and let $\epsilon_n > 0$. Suppose X_i's are negatively associated random variables with $EX_i = 0$ and bounded, i.e. $|X_i| \leq M$ for all $i \geq 1$. Then,

$$P(\overline{U}_n \geq \epsilon_n) \leq exp(-\frac{2r\epsilon_n^2}{M^2}). \tag{10}$$

Proof. Note that $U_1, \cdots, U_r$ are negatively associated by a consequence of property P_6 in Joag-Dev and Proschan[3] and $|U_i| \leq pM$ for all $i \geq 1$ by definition of U_i in (2). For some $\lambda > 0$, we have

$$E(\exp^{\lambda \overline{U}_n}) \leq \prod_{i=1}^{n} [E(exp(\frac{\lambda}{n})U_i)]. \tag{11}$$

by Lemma 2.2. At this point, we apply Lemma 2.1, which states that, if $EX = 0$ and $a \leq X \leq b$ then for every $\lambda > 0$, $E(exp(\lambda X)) \leq exp[\lambda^2(b-a)^2/8]$. Take $X = U_i$, so that $|U_i| \leq pM$ and $b - a = 2pM$. Then we obtain

$$E(exp((\frac{\lambda}{n})U_i) \leq exp(\frac{\lambda^2 p^2 M^2}{2n^2}), \tag{12}$$

which yields

$$\begin{aligned} \prod_{i=1}^{n} E(exp((\frac{\lambda}{n})U_i) &\leq exp(\frac{\lambda^2 M^2 p^2 r}{2n^2}) \\ &\leq exp(\frac{\lambda^2 M^2}{8r}), \end{aligned}$$

because

$$\frac{\lambda^2 M^2 p^2 r}{2n^2} = \frac{\lambda^2 M^2}{8r}(\frac{2pr}{n})^2 \leq \frac{\lambda^2 M^2}{8r}$$

by (1). From (11) and (12) it follows that

$$E(exp(\lambda \overline{U}_n)) \leq exp(\frac{\lambda^2 M^2}{8r}). \tag{13}$$

Therefore, for $\epsilon_n > 0$

$$P(\overline{U}_n \geq \epsilon_n) \leq exp(-\lambda\epsilon_n + \frac{\lambda^2 M^2}{8r}). \tag{14}$$

Minimizing, with respect to λ, the right-hand in (14), we obtain

$$P(\overline{U}_n \geq \epsilon_n) \leq exp(-\frac{2r{\epsilon_n}^2}{M^2}), \ for\ \lambda = \frac{4r\epsilon_n}{M^2}. \tag{15}$$

This completes the proof of the lemma.

From Lemma 3.1, we have the following result.

Lemma 3.2 Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, for all $i \geq 1$ and let ϵ_n specified by (7), it holds

$$P(\overline{U}_n \geq \epsilon_n) \leq n^{-\alpha}. \tag{16}$$

Similarly, we obtain the following results as in $\overline{U}_n$ in Lemmas 3.1 and 3.2.

Lemma 3.3 Let $\overline{V}_n$ be defined in (5). Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, for all $i \geq 1$ and let ϵ_n specified by (7). Then,

$$P(\overline{V}_n \geq \epsilon_n) \leq exp(-\frac{2r{\epsilon_n}^2}{M^2}), \ for\ \lambda = \frac{4r\epsilon_n}{M^2}.$$

Lemma 3.4 Let $\overline{V}_n$ be defined in (5). Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, for all $i \geq 1$ and let ϵ_n specified by (7). Then,

$$P(\overline{V}_n \geq \epsilon_n) \leq n^{-\alpha}. \tag{17}$$

The following observation is meant to explain that may dispense with $\overline{W}_n$ as defined in (5).

Lemma 3.5 Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, for all $i \geq 1$ and let ϵ_n specified by (7). Then, $P(\overline{W}_n \geq \epsilon_n) = 0$ for all sufficiently large n.

Proof. W_n consists of $(n-2pr)$ terms and $n-2pr<p$. Then $|\overline{W}_n| \le pM/n$ so that $P(\overline{W}_n \ge \epsilon_n) \le P(M \ge n\epsilon_n/p)$. The last expression $P(M \ge n\epsilon_n/p)$, however, is 0, for all sufficiently large n, because

$$\begin{aligned} \frac{n\epsilon_n}{p} &= (\frac{\alpha M^2}{2})^{\frac{1}{2}}(\frac{n^2\log n}{p^2 r})^{\frac{1}{2}} \\ &= (2\alpha M^2)^{\frac{1}{2}}(\frac{n}{2pr})(r\log n)^{\frac{1}{2}} \to \infty, \end{aligned}$$

this is so because $n/2pr \to 1$.

Lemma 3.6 Let $\{X_i, i \ge 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \le M$ and $EX_i = 0$, $i \ge 1$ and let $\overline{S}_n$ and ϵ_n be defined by (6) and (7), respectively. Then

$$(18) \qquad P(|\overline{S}_n| \ge \epsilon_n) \le 4\ exp(-cr\epsilon_n^2),\ c = \frac{2}{9M^2}.$$

Proof. It consists, essentially, in combining Lemmas 3.1 - 3.5. The random variables $-X_i$, $i = 1, \cdots, n$, have the same properties as the random variables X_i, $i = 1, \cdots, n$. Thus, we always have by (15)

$$(19) \qquad \begin{aligned} P(|\overline{U}_n| \ge \epsilon_n) &= P(\overline{U}_n \ge \epsilon_n) + P(-\overline{U}_n \ge \epsilon_n) \\ &\le 2\ exp(-2r\epsilon_n^2/M^2), \end{aligned}$$

and similarly for $P(\overline{V}_n \ge \epsilon_n)$, that is, we have

$$(20) \qquad \begin{aligned} P(|\overline{V}_n| \ge \epsilon_n) &= P(\overline{V}_n \ge \epsilon_n) + P(-\overline{V}_n \ge \epsilon_n) \\ &\le 2\ exp(-2r\epsilon_n^2/M^2). \end{aligned}$$

Therefore

$$\begin{aligned} P(|\overline{S}_n| \ge 3\epsilon_n) &\le P(|\overline{U}_n| \ge \epsilon_n) + P(|\overline{V}_n| \ge \epsilon_n) + P(|\overline{W}_n| \ge \epsilon_n) \\ &\le P(|\overline{U}_n| \ge \epsilon_n) + P(|\overline{V}_n| \ge \epsilon_n) \\ &\le 4\ exp(-2r\epsilon_n^2/M^2)\ for\ every\ n \ge n_0. \end{aligned}$$

Since $P(|\overline{W}_n| \ge \epsilon_n) \to 0$, by Lemma 3.5. Replacing ϵ_n by $\epsilon_n/3$ we obtain finally,

$$(21) \qquad P(|\overline{S}_n| \ge \epsilon_n) \le 4\ exp(-2cr\epsilon_n^2),\ c = \frac{2}{9M^2},\ n \ge n_0.$$

Theorem 3.7 Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, for all $i \geq 1$ and let ϵ_n be defined by (7). Assume

$$\frac{2r\epsilon_n^2}{M^2} = \log n^\alpha$$

for $\alpha > 1$. Then

$$\frac{S_n}{n} \to 0, \ a.s., \tag{22}$$

at the rate of $1/\epsilon_n$, where

$$\frac{1}{\epsilon_n} = (\frac{2}{\alpha M^2})^{\frac{1}{2}} (\frac{r}{\log n})^{\frac{1}{2}}.$$

Proof. The specification of ϵ_n by (7) and (21) lead to the convergence $\overline{S}_n \to 0$ a.s. at the rate of $1/\epsilon_n$:

From (21) we have

$$\sum_{h=1}^{\infty} P(|\overline{S}_n| \geq \epsilon_n) \leq 4 \sum_{h=1}^{\infty} exp(-\frac{4r{\epsilon_n}^2}{9M^2}) \leq c \sum_{h=1}^{\infty} n^{-\alpha} < \infty, \ \alpha > 1.$$

Thus $\overline{S}_n \to 0$ a.s..

Remark. Regarding the latter part of lemma 3.6, proceed as follows. For value of ϵ_n specified in (7), the rate of convergence is given by

$$\frac{1}{\epsilon_n} = (\frac{2}{\alpha M^2})^{\frac{1}{2}} (\frac{r}{\log n})^{\frac{1}{2}}.$$

For this choice of ϵ_n, λ becomes

$$\lambda = (\frac{8\alpha}{M^2})^{\frac{1}{2}} (r \log n)^{\frac{1}{2}}.$$

Finally, we derive an extension of Hoeffding inequality to the negative associated case.

Theorem 3.8 Let $\{X_i, i \geq 1\}$ be a sequence of negatively associated random variables such that $a_i \leq X_i \leq b_i$ and $EX_i = 0$, for all $i \geq 1$. Then, for every $t > 0$,

$$P(\overline{S}_n \geq t) \leq 2\ exp[-\frac{2n^2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}]. \tag{23}$$

If $|X_i| \leq M,\ i = 1, 2, \cdots, n$, then, for every $t > 0$,

$$P(\overline{S}_n \geq t) \leq 2\ exp[-\frac{nt^2}{2M^2}]. \tag{24}$$

Proof. By Lemma 2.1, we have

$$E\ exp(\lambda X_i) \leq exp(\frac{\lambda^2(b_i - a_i)^2}{8}), for\ \lambda > 0,$$

and

$$\begin{aligned} P(\overline{S}_n \geq t) &= P(exp(\lambda S_n) \geq \lambda nt) \\ &\leq exp(-\lambda nt)E\prod_{i=1}^{n} exp(\lambda X_i) \\ &\leq exp(-\lambda nt)\prod_{i=1}^{n} E[exp(\lambda X_i)] \\ &\leq exp[\frac{\lambda^2}{8}\sum_{i=1}^{n}(b_i - a_i)^2 - \lambda nt]. \end{aligned} \tag{25}$$

By minimizing(with respect to λ) the right-hand side in (25), the desired (23) follows. Replacing $b_i - a_i$ by $2M$ in (25) the desired result (24) also follows.

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