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ON THE PRIME SPECTRUM OF A MODULE OVER A COMMUTATIVE NOETHERIAN RING

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Abstract. Let R be a commutative ring and let M be an R-module. Let $X = Spec(M)$ be the prime spectrum of M with Zariski topology. Our main purpose in this paper is to specify the topological dimensions of X, where X is a Noetherian topological space, and compare them with those of topological dimensions of $Supp_R(M)$. Also we will give a characterization for the irreducibility of X and we obtain some related results.

1. Introduction

Throughout this paper R will denote a commutative ring with an identity, all modules are unitary and the notation "$\subset$" denotes the strict inclusion. Further $\mathbb{Z}$ will denote the ring of integers and Q the field of quotients of $\mathbb{Z}$. A proper submodule P of M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$ (see [6]). The prime spectrum of M is denoted by $Spec(M)$ (or $Spec_R(M)$) and defined as

$$Spec(M) = \{P : P \text{ is a prime submodule of } M\}.$$

Let N be a submodule of M. Then the variety of N is denoted by $V(N)$ and defined (see [5]) as

$$V(N) = \{P \in Spec(M) : (P :_R M) \supseteq (N :_R M)\}.$$

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Further $V^*(N)$ is defined to be

$$V^*(N) = \{P \in Spec(M) : P \supseteq N\}.$$

The elements of the set

$$Z_M = \{V(N) : N \text{ is a submodule of } M\}$$

satisfy the axioms for closed sets in a topological space on $X = Spec(M)$ (see [4] and [5]). The above resulting topology is called the Zariski topology relative to M.

Let X be a topological space. Then X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (or maximal condition).

Also X is said to be irreducible if X is not the union of two proper closed subsets. This is the case if and only if every non-empty open subset of X is connected. For a non-empty Noetherian topological space X, the dimension of X, denoted by $dim(X)$, is defined (see [3, Chap. 19]) as

$$dim(X) = sup\{n : Z_0 \supset Z_1 \supset ... \supset Z_n \text{ is a chain of irreducible closed subset of } X\}.$$

(The dimension of the empty space is defined to be -1.) Further the connectedness dimension of X ,denoted by $c(X)$, is defined as

$$c(X) = min\{dim(Z) : Z \subseteq X, Z \text{ is closed and } X\backslash Z \text{ is disconnected}\}.$$

Also the subdimension of X, denoted by $sdim(X)$, is defined as

$$sdim(X) = min\{dim(T) : T \text{ is an irreducible component of } X\}.$$

The study of the topological connectivity of algebraic sets is a fundamental subject in algebraic geometry. There are some well-known results concerning the connectivity of $Spec(R)$ (with Zariski topology), where R is a commutative Noetherian ring (for example see [2, 19]). Among these results, it is shown that $c(Spec(R))$ can be specified in terms of Krull dimension of some quotient of R.

Let M be an R-module and let $X = Spec(M)$ be the topological space with Zariski topology. Set $\bar{R} = R/Ann_R(M)$. Further we assume that the map $f : X \to Spec(\bar{R})$ defined by $P \mapsto (P :_R M)/Ann_R(M)$, is surjective. This condition may accurse in many cases, for example when M is a finitely generated or a faithfully flat R-module, then f is surjective (see remark 2.6). In this paper we will specify the set of irreducible components of X in terms of $V(pM)$, where $p \in Min_R(Supp_R(M))$ (see 2.7). In 3.1 we will consider the conditions under which X is a Noetherian topological space and we will describe the topological dimensions $dim(X)$, $c(X)$ and $sdim(X)$ in terms of Krull dimensions of some quotient of R and compare them with those of topological dimensions of $Supp_R(M)$ (see theorems, 3.5, 3.6 and 3.7). Further we will give a characterization for the irreducibility of X (see 4.2). Finally we obtain some further information concerning the connectivity of X (see 4.3 and 4.5).

Let M be an R-module. Throughout the remainder of this paper X will denote $Spec_R(M)$ and $f : X \to Spec(\bar{R})$ defined by $P \mapsto (P :_R M)/Ann_R(M)$, is called the natural map of X. Also the Krull dimension of M and $Min(Supp_R(M))$ are denoted respectively by $K.dim(M)$ and $Min(M)$. Further for a Noetherian ring R, $dim(Supp_R(M))$, $c(Supp_R(M))$, and $sdim(Supp_R(M))$ are denoted respectively by $dim(M)$, $c(M)$, and $sdim(M)$.

2. Auxiliary results

Definition 2.1 (see [6]). A proper submodule P of M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$. It follows from definition that $(P :_R M)$ is a prime ideal.

Remark 2.2. Let M be an R-module and let N and L be submodules of M. Then we have the following.

(a) If $(N :_R M) = (L :_R M)$, then $V(N) = V(L)$. The converse is also true if both N and L are prime (see [5, Section 2, Result 1]).

(b) $V(N) = V((N :_R M)M)$ and $V(IM) = V^*(IM)$ for every ideal I of R (see [5, Section 2, Result 3]). Also

$$V(0) = Spec_R(M) \text{ and } V(M) = \emptyset,$$

$$V(N) \cup V(L) = V(N \cap L).$$

Further for any index set Λ (see [5, page 419]),

$$\cap_{\lambda \in \Lambda} V(N_\lambda) = V(\sum_{\lambda \in \Lambda} (N_\lambda : M)M).$$

Also we have

$$\cap_{\lambda \in \Lambda} V^*(N_\lambda) = V^*(\sum_{\lambda \in \Lambda} (N_\lambda)).$$

(c) (see [5, 3.4]). If the natural map of X is surjective, then $V(Ann_R(M)) = Supp_R(M)$.

Remark 2.3. Let M be an R-module such that the natural map of X is surjective. By 2.2 (c) and the fact that $V(I)$ is homeomorphic to $Spec(R/I)$ for every ideal of R, $Supp_R(M)$ is homeomorphic to $Spec(R/Ann(M))$, and $g : X \to Supp_R(M)$ defined by $P \mapsto (P :_R M)$ is a surjective map.

Remark 2.4.

(a) Let M be an R-module. For each $f \in R$, we define $X_f = X \backslash V(fM)$. Then every X_f is an open set of X. Further the set $B = \{X_f : f \in R\}$ forms a base for the Zariski topology on X and for any $f, g \in R$, $X_{fg} = X_f \cap X_g$ (see [5, 4.2 and 4.3]).

(b) Let M be an R-module such that the natural map f of X is surjective. Then for every $f \in R$, X_f is compact (see [5, 4.5]). In particular $X_1 = Spec_R(M)$ is compact.

(c) Let M be an R-module such that the natural map f of X is surjective. Then Y is an irreducible closed subset of $X = Spec_R(M)$ if and only if $Y = V(P)$ for some $P \in X$ (see [5, 5.7]).

Remark 2.5. Let M and M' be R-modules. Set $X = Spec_R(M)$, $X' = Spec_R(M')$, and let $f : M \to M'$ be an epimorphism. Then the map $g : X' \to X$ defined by $P' \mapsto f^{-1}(P')$ is continuous (see [5, 3.9]).

Remark 2.6. For every non-zero R- module M the natural map of X is surjective in each of the following cases (see [5, 3.5]).

(a) M is a finitely generated R-module.

(b) M is a faithfully flat R-module.

(c) M is a ring S containing R as a subring (with the same identity) such that the spectral map $\theta : Spec(S) \to Spec(R)$ defined by $P \mapsto P \cap R$ is surjective. (For example S is an integral extension of R).

Lemma 2.7. Let M be an R-module such that the natural map of X is surjective. Set

$$A = \{V(pM) : p \in Min(M)\}.$$

Then A is equal the set of all irreducible components of X.

Proof. Let Y be an irreducible component of $Spec_R(M)$. Then by 2.4 (c), $Y = V(P)$, for some $P \in Spec_R(M)$. Hence by 2.2 (c)

$$Y = V(P) = V((P :_R M)M)$$

where $(P :_R M) \in Supp(M)$. Set $p := (P :_R M)$. It is enough to show that $p \in Min(M)$. To see this let $q \in Supp(M)$ and $q \subseteq p$. Since the natural map of X is surjective, there exists prime submodule Q of M such that $(Q :_R M) = q$. Hence $Y = V(P) \subseteq V(Q)$. Since Y is irreducible component, it follows that $p = q$ by 2.2 (a). Conversely let $Y \in A$. Then there exists $p \in Min(M)$ such that $Y = V(pM)$. By 2.3, there is $P \in Spec(M)$ such that $(P :_R M) = p$. Hence by 2.2 (b),

$$Y = V(pM) = V((P :_R M)M) = V(P).$$

Thus Y is irreducible by 2.4 (c). Now let $Y = V(P) \subseteq V(Q)$, where Q is a prime submodule of M. Since $P \in V(Q)$ and $p \in Min(M)$, it follows that $(P :_R M) = (Q :_R M)$. Hence by 2.2 (b),

$$Y = V((P :_R M)M) = V((Q :_R M)M) = V(Q).$$

This completes the proof.

Example 2.8. Let M be an R-module such that the natural map of X is surjective. Let the zero submodule of M have a primary decomposition. Then $X = Spec(M)$ has only finitely many irreducible components.

Proof. Let $0 = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition of the zero submodule of M, where Q_i is a P_i-primary ideal for each i, $1 \leq i \leq n$. Then by [7, 1.3],

$$W.Ass_R(M) = \{P_1, P_2, ..., P_n\}.$$

(Here $W.Ass_R(M)$ denotes the set of weakly associated primes of M.) Further by [8, 1.5], $W.Ass_R(M)$ and $V(Ann_R(M))$ have the same minimal members. Now the result follows from 2.2 (c) and 2.7.

Proposition 2.9. Let M be an R-module such that the natural map of X is surjective. Let I be an ideal of R. Then

(a) The natural map of $Spec_R(M/IM)$ is also surjective.

(b) $\sqrt{Ann_R(M/IM)} = \sqrt{(I + Ann_R(M))}$.

(c) $Spec_R(M/IM)$ is homeomorphic to $V(IM)$.

Proof. (a) The proof is straightforward and we omit it.

(b) It is enough to prove that $V(Ann_R(M/IM)) = V(I + Ann_R(M))$. It is clear that $V(Ann_R(M/IM)) \subseteq V(I + Ann_R(M))$. So let $p \in V(I + Ann_R(M))$. Then by 2.2 (c) and [7, 4.1 and 3.2], $p \in Supp_R(M/IM)$.

But $V(Ann_R(M/IM)) = Supp_R(M/IM)$ by part (a) and 2.2 (c). It turns out that $p \in V(Ann_R(M/IM))$ as desired.

(c) Let N be a submodule of M and let $f : M \to M/N$ be the natural homomorphism. We can see easily that

$$Spec_R(M/N) = \{P/N : P \in Spec_R(M) \text{ and } P \supseteq N\}.$$

Let $g : Spec_R(M/N) \to Spec_R(M)$ be the map defined by $P/N \mapsto f^{-1}(P/N) = P$. Then g is continuous by 2.5. Also g is one to one and

$$Im(g) = \{P : P \in Spec_R(M) \text{ and } P \supseteq N\} = V^*(N).$$

Further for any submodule L of M with $L \supseteq N$ we have $f(V(L/N)) = V(L)$. This implies that f^{-1} is a continuous map. Hence $Spec_R(M/N)$ is homeomorphic to $V^*(N)$. Now by 2.2 (b), $V(IM) = V^*(IM)$ for every ideal I of R. Hence $Spec_R(M/IM)$ is homeomorphic to $V(IM)$ and the proof is completed.

3. Topological dimensions of $Spec_R(M)$

Throughout this section R is a commutative Noetherian ring.

We know that for the Noetherian ring R, $Spec(R)$ is a Noetherian topological space. In the following theorem we will generalize this fact.

Theorem 3.1. Let R be a Noetherian ring and let M be an R-module such that the natural map of X is surjective. Then $Spec_R(M)$ is a Noetherian topological space.

Proof. Set $X = Spec_R(M)$. By [1, Chap.6, Exe. 6], it is enough to show that every open subset of X is compact. To see this, let G be an open subset of X. Since $\{X_r = X \backslash V(rM) : r \in R\}$ forms a basis (see

2.4 (a)) for the Zariski topology on X. Hence by 2.2,

$$G = \cup_{i\in I} X_{r_i} = X\backslash \cap_{i\in I} V(r_iM) = X\backslash \cap_{i\in I} V^*(r_iM) = X\backslash V^*(\sum_{i\in I} r_iM)$$

$$= X\backslash V^*((\sum_{i\in I}(r_i))M).$$

Since R is a Noetherian ring, $\sum_{i\in I}(r_i) = \sum_{i=1}^{n} Rx_i$, where $x_i \in R$ for $i = 1, 2, ...n$. Hence we have

$$G = X\backslash V^*(\sum_{i=1}^{n} x_iM) = X\backslash \cap_{i=1}^{n} V^*(x_iM) = X\backslash \cap_{i=1}^{n} V(x_iM) = \cup_{i=1}^{n} X_{x_i}.$$

But each X_{x_i}, $i = 1, 2, ..., n$ is compact by 2.4 (b). Hence G is a compact set and the proof is completed.

Example 3.2. Let M be a Noetherian R-module. Then $Spec_R(M)$ is a Noetherian topological space by 2.6 and 3.1. But the converse is not true in general. For example let $M = Q$ and $R = \mathbb{Z}$. Then $Spec_{\mathbb{Z}}(Q) = \{0\}$ (See [2, P. 36, Exe. 8]) is a Noetherian topological space while $M = Q$ is not a Noetherian R-module.

Lemma 3.3. Let M be an R-module such that the natural map of X is surjective. Then

$$dim(Supp_R(M)) = K.dim(M) = K.dim(R/Ann_R(M)).$$

Proof. Let $\bar{R} = R/Ann_R(M)$. By 2.3,

$$dim(Supp_R(M)) = dim(Spec\bar{R}) = K.dim(\bar{R}).$$

On the other hand by using 2.2 (c), we have

$$K.dim(M) = K.dim(\bar{R}).$$

Hence $dim(Supp_R(M)) = K.dim(M) = K.dim(\bar{R})$. This completes the proof.

Let M be an R-module such that the natural map of X is surjective. As we mentioned in 3.1, $X = Spec_R(M)$ is a Noetherian topological space. Hence the topological dimensions, $dim(Spec_R(M))$, $c(Spec_R(M))$ and $sdim(Spec_R(M))$ are meaningful. Now let R be a commutative Noetherian ring, then $dim(Spec(R)) = K.dim(R)$. The following theorem extends this fact.

Theorem 3.4. Let M be an R-module such that the natural map of X is surjective. Then

$$dim(Spec_R(M)) = K.dim_R(M).$$

Proof. Let $K.dim_R(M) = s$. Then by 3.3, there exists a chain

$$p_0 \subset p_1 \subset ... \subset p_s,$$

of prime ideals of R such that $p_i \supseteq Ann_R(M)$. Since the natural map of X is surjective, for every $i = 0, 1, ..., s$, there exists a prime submodule P_i of M such that $p_i = (P_i : M)$. Hence by 2.2,

$$V(P_0) \supset V(P_1) \supset ... \supset V(P_s).$$

Thus by 2.4 (c), $dim(Spec_R(M)) \geq s$. The reverse inclusion is proved similarly and the proof is completed.

Corollary 3.5. Let M be an R-module such that the natural map of X is surjective. Then

$$dim(Spec_R(M)) = dim(Supp_R(M)) = K.dim_R(M) = K.dim(R/Ann_R(M)).$$

Proof. This follows from 3.3 and 3.4.

Theorem 3.6. Let M be an R-module such that the natural map of X is surjective. Then

$$c(Spec_R(M)) = c(M) =$$

$$min\{K.dim(R/((\cap_{p\in C}\ p) + (\cap_{p\in D}\ p)) : C \cup D = Min(M)\}.$$

Proof. By [3, 19.1.15], for a Noetherian topological space T, we have

$$c(T) = min\{dim((\bigcup_{i\in A} T_i) \cap (\bigcup_{j\in B} T_j)) : (A, B) \in \phi(n)\},$$

where,

$\phi(n) = \{(A, B) : A\, and\, B\, are\, non\, empty\, subset\, of\, \{1, 2, ..., n\}\, and\, A\cup B = \{1, 2, ...,$

and $T_1, T_2, ..., T_n$ are irreducible components of T. Hence by 2.7, we have

$c(Spec_R(M)) = min\{dim((\bigcup_{p\in C} V(pM))\cap(\bigcup_{p\in D} V(pM)) : C\cup D = Min(M)\}.$

Now by 2.2 (b),

$c(Spec_R(M)) = min\{dim(V((\bigcap_{p\in C} pM)\cap V((\bigcap_{p\in D} pM)) : C\cup D = Min(M)\}$

Thus by using 2.2 (b),

$$c(Spec_R(M)) =$$

$min\{dim(V((\bigcap_{P\in C} pM :_R M)M+(\bigcap_{p\in D} pM :_R M)M) : C\cup D = Min(M)\}.$

Let H be a subset of $Min(M)$. Then by using 2.2 (c) and 2.9 (b) we can see that

$$(\bigcap_{p\in H} pM :_R M) = \bigcap_{p\in H} p.$$

By using this and 2.9 (c),

$c(Spec_R(M)) = min\{dim(V(((\cap_{p\in C}\, p)+(\cap_{p\in D}\, p))M)) : C\cup D = Min(M)\}$

$= min\{dimSpec(M/((\cap_{p\in C}\, p) + (\cap_{p\in D}\, p))M) : C \cup D = Min(M)\}.$

Now set

$$I := (\bigcap_{P\in C} p + \bigcap_{p\in D} p).$$

Then by 2.9 (a), the natural map of $Spec_R(M/IM)$ is surjective. Hence by 3.5,

$dim(Spec_R(M/IM)) = K.dim(R/Ann_R(M/IM)) = K.dim(R/\sqrt{Ann_R(M/IM)}).$

But $I \supseteq Ann_R(M)$ by 2.2 (c). Hence by 2.9 (b),

$$dim(Spec(M/IM)) = K.dim(R/I).$$

This implies that,

$$c(Spec_R(M)) = min\{K.dim(R/(\cap_{p\in C} + \cap_{p\in D}) : C \cup D = Min(M)\}.$$

Set $\bar{R} = R/Ann_R(M)$. Then by 2.3 we have

$$c(M) = c(Spec(\bar{R})) = c(\bar{R}).$$

Now by [3, 19.2.5]

$$c(\bar{R}) = min\{K.dim(\bar{R}/((\cap_{\bar{p}\in C} \bar{p}) + (\cap_{\bar{p}\in D} \bar{p})) : C \cup D = Min(\bar{R})\}.$$

Hence by the above arguments and 2.2 (c), we have

$$c(M) = min\{K.dim(R/((\cap_{p\in C} p)+(\cap_{p\in D} p)) : C\cup D = Min(M)\} = c(Spec_R(M)).$$

This completes the proof.

Theorem 3.7. Let M be an R-module such that the natural map of X is surjective. Then

$$sdim(Spec_R(M)) = sdim(M) =$$

$$min\{K.dim(R/p) : p \in Min(M)\}.$$

Proof. By [3, 19.2.2], for a Noetherian topological space T,

$$sdim(T) = min\{dim(T_i) : T_i \text{ is an irreducible component of } X\}.$$

So by 2.7,

$$sdim(Spec_R(M)) = min\{dim(V(pM)) : p \in Min(M)\}.$$

Hence by 2.9 (c), 2.9 (a), and 3.5,

$$sdim(Spec_R(M)) = min\{K.dim(M/pM) : p \in Min(M)\}.$$

Now by using 2.9 (a), 3.5, and 2.9 (b),

$$K.dim(M/pM) = K.dim(R/Ann(M/pM)) = K.dim(R/\sqrt{Ann(M/pM)})$$

$$= K.dim(R/(p + Ann(M))) = K.dim(R/p).$$

Hence

$$sdim(Spec_R(M)) = min\{K.dim(R/p) : p \in Min(M)\}.$$

Set $\bar{R} = R/Ann_R(M)$. Now by [3, 19.2.5] and 2.2 (c),

$$sdim(Spec\bar{R}) = min\{K.dim(\bar{R}/\bar{p}) : \bar{p} \in Min(\bar{R})\}$$
$$= min\{K.dim(R/p) : p \in Min(M)\}.$$

But by 2.3,

$$sdim(Spec\bar{R}) = sdim(Supp(M)) = sdim(M).$$

Hence

$$sdim(Spec_R(M)) = sdim(M) = min\{K.dim(R/p) : p \in Min(M)\}.$$

This completes the proof.

Example 3.8. Suppose that M is a finitely generated or a faithfully flat R-module. Then we have the following.

(a) $dim(Spec(M)) = dim(M) = K.dim(M)$.

(b) $c(Spec(M)) = c(M)$.

(c) $sdim(Spec(M)) = sdim(M)$.

Proof. The results follows from 3.5, 3.6 and 3.7.

The following example shows that the equalities in 3.5, 3.6, and 3.7 are not true in general.

Example 3.9. Let $M = Q$ and $R = \mathbb{Z}$. Then $Spec_R(M) = \{0\} = V(0)$ is an irreducible Noetherian topological space. Since the only irreducible closed subset of $Spec_{\mathbb{Z}}(Q)$ is $V(0) = 0$, it follows that $dim(Spec_R(M)) = dim(Spec_{\mathbb{Z}}(Q)) = 0$. Also

$$dim(M) = sup\{ht(p) : p \in Supp(M)\} = sup\{ht(p) : p \in Spec(\mathbb{Z})\} = 1.$$

Hence $dim(Spec_R(M)) \neq dim(M)$. Further

$$c(M) = c(Supp(M)) = c(Spec(\mathbb{Z})) = c(\mathbb{Z}) = K.dim(\mathbb{Z}) = 1.$$

But by [3, 19.1.10 (iii)],

$$c(Spec_R(M)) = dim(Spec_R(M)) = dim(Spec_{\mathbb{Z}}(Q)) = 0.$$

Hence $c(Spec_R(M)) \neq c(M)$. Similarly, one can show that $sdim(Spec_R(M)) \neq sdim(M)$.

4. Some topological properties of $Spec_R(M)$

Proposition 4.1 Let M be an R-module such that the natural map of X is surjective. Let f and g be two elements of R. Then we have the following.

(a) $f \in \sqrt{Ann(M)}$ if and only if $X_f = \emptyset$.

(b) f is a unit element if and only if $X_f = X$

(c) $\sqrt{((f) + Ann(M))} = \sqrt{((g) + Ann(M))}$ if and only if $X_f = X_g$.

Proof. We just prove (a) and the proof of other parts is similar. Let $X_f = \emptyset$, then $V(fM) = Spec(M)$. Now let $p \in Supp(M)$, then by 2.3 there exists $P \in Spec(M)$ such that $(P :_R M) = p$. But $P \in V(fM)$. It turns out that for every $p \in Supp(M)$, $(fM :_R M) \subseteq p$. Hence by 2.2 (c), $f \in \cap_{p \supseteq Ann(M)} p$ so that $f \in \sqrt{Ann(M)}$. Conversely, if $f \in \sqrt{Ann(M)}$, then there exists $n \in N$ such that $f^n M = 0$. Hence for every $P \in Spec(M)$, $f \in (P :_R M)$. Thus $V(fM) = Spec(M)$ so that $X_f = \emptyset$. This completes the proof.

Theorem 4.2. Let M be an R-module such that the natural map of X is surjective. Then the following are equivalent.

(a) $X = Spec_R(M)$ is an irreducible topological space.

(b) $Supp(M)$ is an irreducible topological space.

(c) $\sqrt{Ann(M)}$ is a prime ideal of R.

(d) $Spec_R(M) = V(pM)$ for some $p \in Supp(M)$.

(e) $Min(M)$ consists of a single prime p.

Proof. (a) $\Longrightarrow$ (b) Let f be the natural map of X. By [5, 3.1], f is a continuous map. Hence $Im(f) = Spec(R/Ann_R(M))$ is also irreducible. This implies that $Supp(M)$ is an irreducible topological space by 2.3.

(b) $\Longrightarrow$ (c) Let $Supp(M)$ be an irreducible topological space. Hence $Spec(R/Ann_R(M))$ is an irreducible topological space by 2.3. This implies that $X_f \cap X_g \neq \emptyset$ for every pair of non-empty open sets X_f and X_g in X. Hence $\sqrt{Ann(M)}$ is a prime ideal of R by 4.1 (a).

(c) $\Longrightarrow$ (d) We can see that $Spec_R(M) = V(\sqrt{(Ann_R(M))}M)$. Hence $Spec_R(M) = V(pM)$, where $p = \sqrt{(Ann_R(M))} \in Supp(M)$ by 2.2 (c).

(d) $\Longrightarrow$ (a) Let $Spec_R(M) = V(pM)$ for some $p \in Supp(M)$. Since the natural map of X is surjective, there exists $P \in Spec_R(M)$ such that $(P :_R M) = p$. Hence $Spec_R(M) = V(pM) = V((P :_R M)M) = V(P)$ by 2.2. (b). This implies that $X = Spec_R(M)$ is an irreducible topological space by 2.4 (c).

(e) $\Longleftrightarrow$ (a) This is an immediate consequence of 2.7.

Corollary 4.3. Let R be a commutative Noetherian ring. Let M be an R-module such that the natural map of X is surjective. Then $c(Spec(M)) = K.dimM$ in each of the following cases.

(a) $Supp(M)$ is an irreducible topological space.
(b) $\sqrt{Ann(M)}$ is a prime ideal of R.
(c) $Spec_R(M) = V(pM)$ for some $p \in Supp(M)$.
(d) $Min(M)$ consists of a single prime p.

Proof. By [3, 19.1.10], for every irreducible Noetherian topological space T, $dim(T) = c(T)$. Hence $c(Spec(M)) = dim(Spec_R(M))$ by 3.1 and 4.2. Thus $c(Spec(M)) = K.dimM$ by 3.5.

The following example shows that $Supp(M)$ and $spec_R(M)$ don't have the same behavior always.

Example 4.4. For a commutative ring S it is well known that $Spec(S)$, is T_0-space; that is, for every $p, q \in Spec(R)$, $p \neq q$ there is either neighborhood of p does not intersect $\{q\}$ or a neighborhood of q that does not intersect $\{p\}$. Now let M be a finitely generated R-module. Since $Supp(M)$ is homeomorphic to $Spec(R/Ann(M))$, then $Supp(M)$ is a T_0-space too. However, $Spec(M)$ is not a T_0-space in general. To see this, take M as a finitely generated R-module which is not a multiplication R-module. Then $Spec_R(M)$ is not a T_0 space by [5, 6.6].

In [5] it is shown that if (R, m) is a commutative quasi local, then $Spec_R(M)$ is a connected topological space. The following extends this result.

Theorem 4.5. Let M be an R-module such that the natural map of X is surjective. Set $\bar{R} = R/Ann_R(M)$. Assume there exists $p \in Spec(\bar{R})$ such that the natural homomorphism $\phi : \bar{R} \to \bar{R}_p$ is one to one. Then $Spec_R(M)$ is a connected topological space.

Proof. Let $\phi^* : Spec(\bar{R}_p) \to Spec(\bar{R})$ given by $Q \longmapsto \phi^{-1}(Q)$, be the induced map. Since ϕ is one to one we have $cl(\phi^*(Spec(\bar{R}_p))) = Spec(\bar{R})$ by [1, 1.21]. (Here for a topological space T, $cl(T)$ denotes the topological closure of T). Since $\bar{R}_p$ is quasi local, $Spec(\bar{R}_p)$ is a connected topological space. This implies that $\phi^*(Spec(\bar{R}_p))$ is a connected space because ϕ^* is a continuous map. It is well- known that if T is a connected space, then so also is $cl(T)$. Hence $Spec(\bar{R})$ is a connected space. But by [5, 3.8], $Spec_R(M)$ is a connected if and only if $Spec(\bar{R})$ is connected. This completes the proof.

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