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FUZZY FOLDNESS OF IMPLICATIVE ORDERED FILTERS IN IMPLICATIVE SEMIGROUPS

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Abstract. The notion of fuzzy n-fold implicative ordered filters in implicative semigroups is introduced, and related properties are investigated. Relations between fuzzy ordered filters and fuzzy n-fold implicative ordered filters are provided. Characterizations of a fuzzy n-fold implicative ordered filter are given, and an extension property for a fuzzy n-fold implicative ordered filter is obtained.

1. Introduction

The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [3]. The first is a generalization of implicative semilattice (see Nemitz [7] and Blyth [2]) and has a close relation with implication in mathematical logic and set theoretic difference (see Birkhoff [1] and Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role which is shown by Nemitz [7]. Motivated by this, Chan and Shum [3] established some elementary properties, and constructed quotient structure of implicative semigroups via ordered filters. Jun et al. [6] discussed ordered filters of implicative semigroups. To deeply study of implicative semigroups it is undoubtedly necessary to establish more complete theory of ordered filters for it. Jun [9] investigated more further properties of implicative semigroups and of their ordered filters which were started

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in general case by Chan and Schum [3]. In particular, he introduced the notion of an n-fold implicative ordered filter in implicative semigroups, and stated some equivalent conditions in order that an ordered filter would be an implicative ordered filter and obtained the so called extension property for implicative ordered filters. He also gave a condition for the ordered filter $\{1\}$ to be n-fold implicative. In this paper, we consider the fuzzification of the notion of n-fold implicative ordered filter in implicative semigroups. We give a relation between fuzzy ordered filters and fuzzy n-fold implicative ordered filters. We provide characterizations of a fuzzy n-fold implicative ordered filter, and obtain an extension property for a fuzzy n-fold implicative ordered filter.

2. Preliminaries

We recall some definitions and results. By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering "$\leq$" and a binary operation "$\cdot$" such that for all $x, y, z \in S$, we have:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(2) $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,

(3) $x \cdot y \leq x$ and $x \cdot y \leq y$.

An n.p.o. semigroup $(S; \leq, \cdot)$ is said to be *implicative* if there is an additional binary operation $* : S \times S \to S$ such that for any elements x, y, z of S,

(4) $z \leq x * y$ if and only if $z \cdot x \leq y$.

The operation $*$ is called *implication*. From now on, an implicative n.p.o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup $(S; \leq, \cdot, *)$ is said to be *commutative* if it satisfies

(5) $x \cdot y = y \cdot x$ for all $x, y \in S$,

that is, $(S, \cdot)$ is a commutative semigroup.

In any implicative semigroup $(S;\leq,\cdot,*)$, $x*x=y*y$ and this element is the greatest in S; it will be denoted by 1.

Some elementary properties of implicative semigroups are summarized by the following.

Proposition 2.1. (Chan and Shum [3; Theorem 1.4]) *Let S be an implicative semigroup. Then for every $x, y, z \in S$, the following hold:*

(6) $x \leq 1$, $x * x = 1$, $x = 1 * x$,

(7) $x \leq y * (x \cdot y)$,

(8) $x \leq x * x^2$,

(9) $x \leq y * x$,

(10) *if* $x \leq y$ *then* $x * z \geq y * z$ *and* $z * x \leq z * y$,

(11) $x \leq y$ *if and only if* $x * y = 1$,

(12) $x * (y * z) = (x \cdot y) * z$,

(13) *if S is commutative then* $x * y \leq (s \cdot x) * (s \cdot y)$ *for all s in S.*

Definition 2.2. (Chan and Shum [3; Definition 2.1]) Let S be an implicative semigroup and let F be a nonempty subset of S. Then F is called an *ordered filter* of S if

(F1) $x \cdot y \in F$ for every $x, y \in F$, that is, F is a subsemigroup of S.

(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

The following result gives an equivalent condition of an ordered filter.

Proposition 2.3. (Jun et al. [6; Proposition 2]) *Suppose S is an implicative semigroup. Then a non-empty subset F of S is an ordered filter if and only if it satisfies the following conditions:*

(F3) $1 \in F$,

(F4) $x * y \in F$ *and* $x \in F$ *imply* $y \in F$.

Now we note important elementary properties of a commutative implicative semigroup, which follows from (5), (6) and (12).

Proposition 2.4. *If S is a commutative implicative semigroup, then for any $x, y, z \in S$,*

(14) $x * (y * z) = y * (x * z)$.

(15) $y * z \leq (x * y) * (x * z)$.

(16) $x \leq (x * y) * y$.

Definition 2.5. (Jun [5]) Let S be an implicative semigroup. A non-empty subset F of S is called an *implicative ordered filter* of S if it satisfies (F3) and

(F5) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$

for all $x, y, z \in S$.

Definition 2.6. [13] A fuzzy set μ in S is called a *fuzzy ordered filter* (briefly, FO-filter) of S if it satisfies:

(c1) $(\forall x \in S)\ (\mu(x) \leq \mu(1))$.

(c2) $(\forall x, y \in S)\ (\mu(y) \geq \min\{\mu(x * y), \mu(x)\})$.

Note that every FO-filter is order preserving.

Definition 2.7. [7] A fuzzy set μ in S is called a *fuzzy implicative ordered filter* (briefly, FIO-filter) of S if it satisfies the conditions (c1) and

(c3) $(\forall x, y, z \in S)\ (\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(x * y)\})$.

Note that every FIO-filter is an FO-filter, but the converse is not true in general (see [7]).

3. n-fold implicative ordered filters

In what follows let S and $\mathbb{N}$ denote an implicative semigroup and the set of all positive integers, respectively, unless otherwise specified. For any elements x and y of S and $n \in \mathbb{N}$, we use the notation $x^n * y$ instead of $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = y$.

Definition 3.1. [9] A non-empty subset F of S is called an *n-fold implicative ordered filter* of S if it satisfies (F3) and

(F6) $x^n * (y * z) \in F$ and $x^n * y \in F$ imply $x^n * z \in F$

for all $x, y, z \in S$.

Note that the 1-fold implicative ordered filter is an implicative ordered filter.

We begin with the fuzzification of Definition 3.1.

Definition 3.2. A fuzzy set μ in S is called a *fuzzy n-fold implicative ordered filter* (briefly, $FnIO$-filter) of S if it satisfies conditions (c1) and

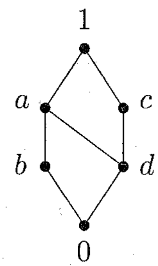
(c4) $(\forall x, y, z \in S)$ $(\mu(x^n * z) \geq \min\{\mu(x^n * (y * z)), \mu(x^n * y)\})$.

Note that every $F1IO$-filter is an FIO-filter.

Example 3.3. Let $S := \{1, a, b, c, d, 0\}$ be a set with Cayley tables and Hasse diagram as follows:

$\cdot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1



Then $(S; \leq, \cdot, *)$ is an implicative semigroup (see [9]). Let μ be a fuzzy set in S defined by

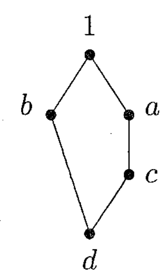
$$\mu(x) := \begin{cases} \alpha & \text{if } x \in \{1, a, b\}, \\ \beta & \text{otherwise}, \end{cases} \tag{3.1}$$

for all $x \in S$ where $\alpha > \beta$ in $[0, 1]$. It is easy to see that μ is an $F2IO$-filter of S.

Example 3.4. Let $S := \{1, a, b, c, d\}$ be an implicative semigroup with Cayley tables and Hasse diagram as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1



Let μ be a fuzzy set in S defined by

$$\mu(x) := \begin{cases} \alpha & \text{if } x \in \{1, a, c\}, \\ \beta & \text{otherwise}, \end{cases} \tag{3.2}$$

for all $x \in S$ where $\alpha > \beta$ in $[0, 1]$. It is easy to see that μ is an $FnIO$-filter of S.

Theorem 3.5. *Every $FnIO$-filter of S is an FO-filter of S.*

Proof. Let μ be an $FnIO$-filter of S. Taking $x = 1$ in (c4) and using (6), we have

$$\mu(z) \geq \min\{\mu(y * z), \mu(y)\}.$$

Hence μ is an FO-filter of S. □

The converse of Theorem 3.5 is not true as seen in the following example.

Example 3.6. Consider the implicative semigroup S in Example 3.3. Let μ be a fuzzy set in S defined by $\mu(1) > \mu(x)$ for all $x(\neq 1) \in S$. Then μ is an FO-filter of S. But

$$\mu(d * 0) = \mu(a) < \mu(1) = \min\{\mu(d * (a * 0)), \mu(d * a)\},$$

and so μ is not an $F1IO$-filter of S.

Proposition 3.7. *Every $FnIO$-filter μ of S satisfies the following inequality.*

$$(\forall x, y \in S)\, (\forall k \in \mathbb{N})\, (\mu(x^n * y) \geq \mu(x^{n+k} * y)). \tag{3.3}$$

Proof. Let μ be an $FnIO$-filter of S and let $x, y \in S$. For $k = 1$, we have

$$\begin{aligned} \mu(x^n * y) &\geq \min\{\mu(x^n * (x * y)), \mu(x^n * x)\} \\ &= \min\{\mu(x^n * (x * y)), \mu(1)\} \\ &= \mu(x^{n+1} * y) \end{aligned}$$

by (c4), (6) and (c1). Assume that the inequality (3.3) is valid for $k = r$, i.e.,

$$(\forall x, y \in S)\, (\mu(x^n * y) \geq \mu(x^{n+r} * y)). \tag{3.4}$$

Then

$$\mu(x^{n+r} * y) \geq \min\{\mu(x^{n+r} * (x * y)), \mu(x^{n+r} * x)\} = \min\{\mu(x^{n+r+1} * y), \mu(1)\} = \mu(x^{n+r+1} * y) \tag{3.5}$$

by (c4), (6) and (c1). Combining (3.4) and (3.5) induces

$$\mu(x^n * y) \geq \mu(x^{n+r+1} * y).$$

This completes the proof. □

We give conditions for an FO-filter to be an $FnIO$-filter.

Theorem 3.8. *Let μ be an FO-filter of a commutative implicative semigroup S. If μ satisfies the condition* (3.3), *then μ is an $FnIO$-filter of S.*

Proof. Assume that μ satisfies the condition (3.3). Using (10), (14) and (15), we have

$$x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z)) = (x^n * y) * (x^{2n} * z).$$

Since μ is order preserving, it follows from (c2) and (3.3) that

$$\begin{aligned} \mu(x^n * z) &\geq \mu(x^{2n} * z) \\ &\geq \min\{\mu((x^n * y) * (x^{2n} * z)), \mu(x^n * y)\} \\ &\geq \min\{\mu(x^n * (y * z)), \mu(x^n * y)\} \end{aligned}$$

so that μ is an $FnIO$-filter of S. □

Theorem 3.9. *Let μ be an FO-filter of a commutative implicative semigroup S. Then the following are equivalent.*

(i) *μ is an $FnIO$-filter of S.*

(ii) $(\forall x, y \in S)\ (\mu(x^{n+1} * y) \leq \mu(x^n * y))$.

(iii) $(\forall x, y, z \in S)\ (\mu(x^n * (y * z)) \leq \mu((x^n * y) * (x^n * z)))$.

Proof. (i) ⇒ (ii) This follows from Proposition 3.7.

(ii) ⇒ (iii) Suppose (ii) holds and let $x, y, z \in S$. Since

$$x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z)),$$

it follows that

$$\begin{aligned}\mu(x^n * (y * z)) &\le \mu(x^n * ((x^n * y) * (x^n * z)))\\ &= \mu(x^n * (x^n * ((x^n * y) * z)))\\ &= \mu(x^{n+1} * (x^{n-1} * ((x^n * y) * z)))\\ &\le \mu(x^n * (x^{n-1} * ((x^n * y) * z)))\\ &= \mu(x^{n+1} * (x^{n-2} * ((x^n * y) * z)))\\ &\le \mu(x^n * (x^{n-2} * ((x^n * y) * z)))\\ &\cdots\\ &\le \mu(x^n * ((x^n * y) * z))\\ &= \mu((x^n * y) * (x^n * z)).\end{aligned}$$

(iii) $\Rightarrow$ (i) Assume that (iii) holds and let $x, y, z \in S$. Then

$$\begin{aligned}\mu(x^n * z) &\ge \min\{\mu((x^n * y) * (x^n * z)), \mu(x^n * y)\}\\ &\ge \min\{\mu(x^n * (y * z)), \mu(x^n * y)\}.\end{aligned}$$

Hence μ is an $FnIO$-filter of S. □

The following is a characterization of an $FnIO$-filter.

Theorem 3.10. *Let μ be a fuzzy set in a commutative implicative semigroup S. Then μ is an $FnIO$-filter of S if and only if it satisfies* (c1) *and*

(c5) $(\forall x, y, z \in S)$ $(\mu(y^n * z) \ge \min\{\mu(x * (y^{n+1} * z)), \mu(x)\})$.

Proof. Suppose that μ is an $FnIO$-filter of S and let $x, y, z \in S$. Then

$$\mu(y^n * z) \ge \mu(y^{n+1} * z) \ge \min\{\mu(x * (y^{n+1} * z)), \mu(x)\}$$

by Theorem 3.9 and (c2). Conversely, assume that μ satisfies the conditions (c1) and (c5). Then

$$\begin{aligned}\mu(y) &= \mu(1^n * y)\\ &\ge \min\{\mu(x * (1^{n+1} * y)), \mu(x)\}\\ &= \min\{\mu(x * y), \mu(x)\},\end{aligned}$$

and so μ is an FO-filter of S. Using (6), (c1) and (c5), we have

$$\begin{aligned}\mu(x^n * y) &\geq \min\{\mu(1 * (x^{n+1} * y)), \mu(1)\}\\ &= \min\{\mu(x^{n+1} * y), \mu(1)\}\\ &= \mu(x^{n+1} * y).\end{aligned}$$

It follows from Theorem 3.9 that μ is an $FnIO$-filter of S. □

We give an extension property for an $FnIO$-filter.

Theorem 3.11. *Let μ and ν be FO-filters of a commutative implicative semigroup S such that $\mu \subset \nu$ and $\mu(1) = \nu(1)$. If μ is an $FnIO$-filter of S, then so is ν.*

Proof. Assume that μ is an $FnIO$-filter of S. For any $x, y \in S$, we have

$$\begin{aligned}\nu((x^{n+1} * y) * (x^n * y)) &\geq \mu((x^{n+1} * y) * (x^n * y))\\ &= \mu(x^n * ((x^{n+1} * y) * y))\\ &\geq \mu(x^{n+1} * ((x^{n+1} * y) * y))\\ &= \mu((x^{n+1} * y) * (x^{n+1} * y))\\ &= \mu(1) = \nu(1).\end{aligned}$$

It follows from (c1) and (c2) that

$$\begin{aligned}\nu(x^n * y) &\geq \min\{\nu((x^{n+1} * y) * (x^n * y)), \nu(x^{n+1} * y)\}\\ &\geq \min\{\nu(1), \nu(x^{n+1} * y)\}\\ &= \nu(x^{n+1} * y)\end{aligned}$$

so from Theorem 3.9 that μ is an $FnIO$-filter of S. □

References

[1] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Coll. Publ. Vol. XXV, Providence, 1967

[2] T. S. Blyth, *Pseudo-residuals in semigroups,* J. London Math. Soc. **40** (1965), 441–454.

[3] M. W. Chan and K. P. Shum, *Homomorphisms of implicative semigroups,* Semigroup Forum **46** (1993), 7–15.

[4] H. B. Curry, *Foundations of Mathematics Logic,* McGrow-Hill, New York, 1963.
[5] Y. B. Jun, *A note on ordered filters of implicative semigroups,* Bull. Korean Math. Soc. **34** (1997), no. 2, 185–191.
[6] Y. B. Jun, *Implicative ordered filters of implicative semigroups,* Comm. Korean Math. Soc. **14** (1999), no. 1, 47–55.
[7] Y. B. Jun, *Fuzzy implicative ordered filters in implicative semigroups,* Southeast Asian. Bull. Math. **26** (2003), 935–943.
[8] Y. B. Jun, *Some results on ordered filters of implicative semigroups,* Internat. J. Math. Math. Sci. **26** (2001), no. 12, 731–735.
[9] Y. B. Jun, *Folding theory applied to implicative ordered filters of implicative semigroups,* Southeast Asian. Bull. Math. (to appear).
[10] Y. B. Jun and K. H. Kim, *Positive implicative ordered filters of implicative semigroups,* Internat. J. Math. Math. Sci. **23** (2000), no. 12, 801–806.
[11] Y. B. Jun, Y. H. Kim and H. S. Kim, *Fuzzy positive implicative ordered filters of implicative semigroups,* Internat. J. Math. Math. Sci. **32** (2002), no. 5, 263–270.
[12] Y. B. Jun, J. Meng and X. L. Xin, *On ordered filters of implicative semigroups,* Semigroup Forum **54** (1997), 75–82.
[13] S. W. Kuresh, Y. B. Jun and W. P. Huang, *Fuzzy ordered filters in implicative semigroups,* Chinese Quartely J. Math. **13** (1998), no. 2, 53–57.
[14] W. C. Nemitz, *Implicative semilattices,* Trans. Amer. Math. Soc. **117** (1965), 128–142.

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