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INTUITIONISTIC FUZZY LIE IDEALS OF LIE ALGERBAS

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Abstract. The notion of intuitionistic fuzzy Lie ideal of a Lie algebra is introduced, and related properties are investigated. Characterizations of an intuitionistic fuzzy Lie ideal are provided. Using a collection of Lie ideals, an intuitionistic fuzzy Lie ideal is established.

1. Introduction

Fuzzy set theory formulated by Zadeh [8] in 1965 has evoked tremendous interest among researchers from all disciplines such as weather forecasting, linguistics, economics, computer sciences, operations research, graph theory, topological spaces, algebraic structures and so on. There is hardly any field in which FST cannot be applied. The fact that algebraic structures occupy a prominent place in mathematics with manifold applications in various disciplines, for example, theoretical physics, information sciences, coding theory, to name just a few, along with the fact that FST offers greater richness in application than the ordinary set theory motivates one to study various concepts/results of algebra in the broader framework of the fuzzy setting. Yehia [6, 7] introduced fuzzy sets in the realm of Lie algebra. He/She introduced the notion of fuzzy ideal and fuzzy Lie subalgebra in Lie algebras, and discussed several useful results. He/She also constructed fuzzy quotient Lie algebra by using fuzzy ideals. Jun et al. [5] discussed useful properties in fuzzy setting

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of a Lie algebra. Using a collection of ideals with additional properties, they established a fuzzy ideal. With relation to the ascending chain of ideals, they also stated a characterization for the set of values of any fuzzy ideal to be a well-ordered subset of the closed unit interval $[0, 1]$. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] is one among them. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. For more details on intuitionistic fuzzy sets, we refer the reader to [1, 2]. Since then, a great number of theoretical and practical results appeared in the area of IFSs. There are numerical applications of IFSs in various areas of computer science, for example, in artificial intelligence, as well as in medicine, chemistry, economics, astronomy, etc. In this paper, we apply the concept of an intuitionistic fuzzy set to Lie ideals in Lie algebras. We introduce the notion of an intuitionistic fuzzy Lie ideal of a Lie algebra, and investigate some related properties. We give characterizations of an intuitionistic fuzzy Lie ideal. Using a collection of Lie ideals, we make an intuitionistic fuzzy Lie ideal.

2. Preliminaries

A vector space $\mathscr{L}$ over a field F, with an operation $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$, denoted $(x, y) \mapsto [xy]$ and called the bracket or commutator of x and y, is called a *Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket operation is bilinear.

(L2) $[xx] = 0$ for all $x \in \mathscr{L}$.

(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in \mathscr{L}$.

A subspace K of a Lie algebra $\mathscr{L}$ is called a *Lie ideal* of $\mathscr{L}$ if $x \in \mathscr{L}$, $y \in K$ together imply $[xy] \in K$.

A mapping $\mu : \mathscr{L} \to [0,1]$, where $\mathscr{L}$ is an arbitrary nonempty set, is called a *fuzzy set* in $\mathscr{L}$. For any fuzzy set μ in $\mathscr{L}$ and any $t \in [0,1]$ we define two sets

$$U(\mu;t) = \{x \in \mathscr{L} \mid \mu(x) \geq t\} \text{ and } L(\mu;t) = \{x \in \mathscr{L} \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

A fuzzy set μ in a Lie algebra $\mathscr{L}$ is called a *fuzzy Lie ideal* of $\mathscr{L}$ if it satisfies

1. $(\forall x, y \in \mathscr{L})\ (\mu(x+y) \geq \min\{\mu(x), \mu(y)\})$.
2. $(\forall x \in \mathscr{L})\ (\forall r \in F)\ (\mu(rx) \geq \mu(x))$.
3. $(\forall x, y \in \mathscr{L})\ (\mu([xy]) \geq \mu(y))$.

As an important generalization of the notion of fuzzy sets in $\mathscr{L}$, Atanassov [1, 2] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a nonempty set $\mathscr{L}$ as objects having the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in \mathscr{L}\},$$

where the functions $\alpha_A : \mathscr{L} \to [0,1]$ and $\beta_A : \mathscr{L} \to [0,1]$ denote the *degree of membership* (namely $\alpha_A(x)$) and the *degree of nonmembership* (namely $\beta_A(x)$) of each element $x \in \mathscr{L}$ to the set A respectively, and

$$0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \tag{2.1}$$

for each $x \in \mathscr{L}$. For the sake of simplicity, we shall use the symbol $A = \langle \mathscr{L}, \alpha_A, \beta_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in \mathscr{L}\}$. Obviously, every fuzzy set A' corresponds to the following intuitionistic fuzzy set:

$$A' = \{\langle x, \alpha_{A'}(x), 1 - \alpha_{A'}(x)\rangle \mid x \in \mathscr{L}\}. \tag{2.2}$$

Obviously, for an IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A\rangle$ in $\mathscr{L}$, when

$$\beta_A(x) = 1 - \alpha_A(x), \text{ that is, } \alpha_A(x) + \beta_A(x) = 1 \tag{2.3}$$

for every $x \in \mathscr{L}$, the IFS A is a fuzzy set. Hence the notion of intuitionistic fuzzy set theory is a generalization of fuzzy set theory. Let

$A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ be an IFS in $\mathscr{L}$ and let $s, t \in [0,1]$ be such that $s + t \leq 1$. Then the set

$$\mathscr{L}_A^{(s,t)} := \{x \in \mathscr{L} \mid \alpha_A(x) \geq s,\ \beta_A(x) \leq t\}$$

is called an (s,t)*-level subset* of $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$. Note that

$$\begin{aligned} \mathscr{L}_A^{(s,t)} &= \{x \in \mathscr{L} \mid \alpha_A(x) \geq s, \beta_A(x) \leq t\} \\ &= \{x \in \mathscr{L} \mid \alpha_A(x) \geq s\} \cap \{x \in \mathscr{L} \mid \beta_A(x) \leq t\} \\ &= U(\alpha_A; s) \cap L(\beta_A; t). \end{aligned}$$

3. Intuitionistic fuzzy Lie ideals of Lie algebras

Throughout, $\mathscr{L}$ will be a Lie algebra over a field F, and denote by I the closed unit interval $[0,1]$. For any sets A and B, by $A \subset B$ we exclude the possibility that $A = B$. As the only exact intuitionistic fuzzification of the notion of Lie ideals in Lie algebras, we define the notion of intuitionistic fuzzy Lie ideal of a Lie algebra as follows.

Definition 3.1. An IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ in $\mathscr{L}$ is an *intuitionistic fuzzy Lie ideal* of $\mathscr{L}$ if it satisfies the following conditions for all $x, y \in \mathscr{L}$ and $r \in F$:

(I1) $\alpha_A(x+y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$, $\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\}$,
(I2) $\alpha_A(rx) \geq \alpha_A(x)$, $\beta_A(rx) \leq \beta_A(x)$,
(I3) $\alpha_A([xy]) \geq \alpha_A(y)$, $\beta_A([xy]) \leq \beta_A(y)$.

Example 3.2. Let $\mathscr{L}$ be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in \mathscr{L}$. Then $\mathscr{L}$ is a Lie algebra (see [4, p. 5]). Define an IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ in $\mathscr{L}$ by

$$\alpha_A\Big((x,y,z)\Big) = \begin{cases} t \in (0,1] & \text{if } x = y = z = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

$$\beta_A\Big((x,y,z)\Big) = \begin{cases} s \in [0,1) & \text{if } x = y = z = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2}$$

where $s + t \leq 1$. Then $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.

Example 3.3. Let H and K be two Lie ideals of $\mathscr{L}$. Define an IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ in $\mathscr{L}$ by

$$\alpha_A(x) = \begin{cases} t_1 & \text{if } x = h + k \text{ for some } h \in H \text{ and } k \in K, \\ t_2 & \text{otherwise,} \end{cases} \tag{3.3}$$

$$\beta_A(x) = \begin{cases} s_1 & \text{if } x = h + k \text{ for some } h \in H \text{ and } k \in K, \\ s_2 & \text{otherwise,} \end{cases} \tag{3.4}$$

where $t_1 > t_2$, $s_1 < s_2$ in $[0, 1]$ and $t_i + s_i \leq 1$ for $i = 1, 2$. Then $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.

Proposition 3.4. *If $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideals of $\mathscr{L}$. then $\alpha_A(0) = \sup_{x \in \mathscr{L}} \alpha_A(x)$ and $\beta_A(0) = \inf_{x \in \mathscr{L}} \beta_A(x)$.*

Proof. It is straightforward. □

Theorem 3.5. *Let $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. Then for each $m, n \in [0, 1]$ with $m \leq \alpha_A(0)$, $n \geq \beta_A(0)$ and $m + n \leq 1$, the (m, n)-level subset $\mathscr{L}_A^{(m,n)}$ is a Lie ideal of $\mathscr{L}$.*

Proof. Let $x, y \in \mathscr{L}_A^{(m,n)}$ and $r \in F$. Then

$$\alpha_A(x + y) \geq \min\{\alpha_A(x), \alpha_A(y)\} \geq m,$$
$$\beta_A(x + y) \leq \max\{\beta_A(x), \beta_A(y)\} \leq n,$$
$$\alpha_A(rx) \geq \alpha_A(x) \geq m,\ \beta_A(rx) \leq \beta_A(x) \leq n,$$

and so $x + y \in \mathscr{L}_A^{(m,n)}$ and $rx \in \mathscr{L}_A^{(m,n)}$. Hence $\mathscr{L}_A^{(m,n)}$ is a subspace of $\mathscr{L}$. Let $x \in \mathscr{L}$ and $y \in \mathscr{L}_A^{(m,n)}$. Then $\alpha_A([xy]) \geq \alpha_A(y) \geq m$ and $\beta_A([xy]) \leq \beta_A(y) \leq n$, which imply $[xy] \in \mathscr{L}_A^{(m,n)}$. Therefore $\mathscr{L}_A^{(m,n)}$ is a Lie ideal of $\mathscr{L}$. □

Theorem 3.6. *Let $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ be an IFS in $\mathscr{L}$ such that $\mathscr{L}_A^{(m,n)}$ is a Lie ideal of $\mathscr{L}$ for every $m, n \in [0, 1]$ with $m + n \leq 1$. Then $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.*

Proof. For any $x, y \in \mathscr{L}$, let $A(x) = (m_1, n_1)$ and $A(y) = (m_2, n_2)$, i.e., $\alpha_A(x) = m_1$, $\beta_A(x) = n_1$, $\alpha_A(y) = m_2$, $\beta_A(y) = n_2$, where $m_i + n_i \leq 1$ for $i = 1, 2$. Then $x \in \mathscr{L}_A^{(m_1,n_1)}$ and $y \in \mathscr{L}_A^{(m_2,n_2)}$. We may assume that $(m_1, n_1) \leq (m_2, n_2)$, i.e., $m_1 \leq m_2$ and $n_1 \geq n_2$, without loss of generality. Thus $y \in \mathscr{L}_A^{(m_2,n_2)} \subset \mathscr{L}_A^{(m_1,n_1)}$. It follows that $x+y \in \mathscr{L}_A^{(m_1,n_1)}$ so that

$$\alpha_A(x+y) \geq m_1 = \min\{m_1, m_2\} = \min\{\alpha_A(x), \alpha_A(y)\},$$

$$\beta_A(x+y) \leq n_1 = \max\{n_1, n_2\} = \max\{\beta_A(x), \beta_A(y)\}.$$

Now for any $x \in \mathscr{L}$, let $A(x) = (m, n)$, i.e., $\alpha_A(x) = m$ and $\beta_A(x) = n$, where $m + n \leq 1$. Then $x \in \mathscr{L}_A^{(m,n)}$, and so $rx \in \mathscr{L}_A^{(m,n)}$ for all $r \in F$. Hence $\alpha_A(rx) \geq m = \alpha_A(x)$ and $\beta_A(rx) \leq n = \beta_A(x)$. Finally for any $y \in \mathscr{L}$, let $A(y) = (s,t)$, i.e., $\alpha_A(y) = s$ and $\beta_A(y) = t$, where $s + t \leq 1$. Then $y \in \mathscr{L}_A^{(s,t)}$, and so $[xy] \in \mathscr{L}_A^{(s,t)}$ for all $x \in \mathscr{L}$. Hence $\alpha_A([xy]) \geq s = \alpha_A(y)$ and $\beta_A([xy]) \leq t = \beta_A(y)$. Consequently, $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. □

Corollary 3.7. *An IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ in $\mathscr{L}$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$ if and only if $U(\alpha_A; m)$ and $L(\beta_A; n)$ are Lie ideals of $\mathscr{L}$ for all $m \in [0, \alpha_A(0)]$ and $n \in [\beta_A(0), 1]$ with $m + n \leq 1$.*

Proof. Straightforward. □

Theorem 3.8. *Let w be a fixed element of $\mathscr{L}$. If $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$, then the set*

$$A^w := \{x \in \mathscr{L} \mid \alpha_A(x) \geq \alpha_A(w),\ \beta_A(x) \leq \beta_A(w)\}$$

is a Lie ideal of $\mathscr{L}$.

Proof. Let $x, y \in A^w$ and $r \in F$. Then

$$\alpha_A(x+y) \geq \min\{\alpha_A(x), \alpha_A(y)\} \geq \alpha_A(w),$$
$$\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\} \leq \beta_A(w),$$
$$\alpha_A(rx) \geq \alpha_A(x) \geq \alpha_A(w),\ \beta_A(rx) \leq \beta_A(x) \leq \beta_A(w).$$

Hence $x+y, rx \in A^w$. For every $x \in \mathscr{L}$ and $y \in A^w$, we have $\alpha_A([xy]) \geq \alpha_A(y) \geq \alpha_A(w)$ and $\beta_A([xy]) \leq \beta_A(y) \leq \beta_A(w)$. It follows that $[xy] \in A^w$. Therefore A^w is a Lie ideal of $\mathscr{L}$. □

Corollary 3.9. *If $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$, then the set*

$$A^0 := \{x \in \mathscr{L} \mid \alpha_A(x) = \alpha_A(0),\ \beta_A(x) = \beta_A(0)\}$$

is a Lie ideal of $\mathscr{L}$.

Proof. Straightforward. □

Theorem 3.10. *Let* K *be a nonempty subset of* $\mathscr{L}$ *and let* $B = \langle \mathscr{L}, \alpha_B, \beta_B \rangle$ *be an IFS in* $\mathscr{L}$ *defined by*

$$\alpha_B(x) := \begin{cases} t_1 & \text{if } x \in K, \\ t_2 & \text{otherwise,} \end{cases} \qquad \beta_B(x) := \begin{cases} s_1 & \text{if } x \in K, \\ s_2 & \text{otherwise,} \end{cases}$$

for all $x \in \mathscr{L}$ *where* $t_1 > t_2$, $s_1 < s_2$ *in* $[0, 1]$, *and* $t_i + s_i \leq 1$ *for* $i = 1, 2$. *Then* $B = \langle \mathscr{L}, \alpha_B, \beta_B \rangle$ *is an intuitionistic fuzzy Lie ideal of* $\mathscr{L}$ *if and only if* K *is a Lie ideal of* $\mathscr{L}$.

Proof. Assume that $B = \langle \mathscr{L}, \alpha_B, \beta_B \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. Let $x, y \in K$ and $r \in F$. Then

$$\begin{aligned} &\alpha_B(x + y) \geq \min\{\alpha_B(x), \alpha_B(y)\} = t_1, \\ &\beta_B(x + y) \leq \max\{\beta_B(x), \beta_B(y)\} = s_1, \\ &\alpha_B(rx) \geq \alpha_B(x) = t_1, \ \beta_B(rx) \leq \beta_B(x) = s_1, \end{aligned}$$

which imply that $\alpha_B(x+y) = t_1 = \alpha_B(rx)$ and $\beta_B(x+y) = s_1 = \beta_B(rx)$. Hence $x + y \in K$ and $rx \in K$. Now for any $x \in \mathscr{L}$ and $y \in K$, we have $\alpha_B([xy]) \geq \alpha_B(y) = t_1$ and $\beta_B([xy]) \leq \beta_B(y) = s_1$. It follows that $\alpha_B([xy]) = t_1$ and $\beta_B([xy]) = s_1$ so that $[xy] \in K$. Therefore K is a Lie ideal of $\mathscr{L}$. Conversely, suppose that K is a Lie ideal of $\mathscr{L}$. Note that

$$U(\alpha_B; t) = \begin{cases} \mathscr{L} & \text{if } 0 \leq t \leq t_2, \\ K & \text{if } t_2 < t \leq t_1, \\ \emptyset & \text{if } t_1 < t \leq 1, \end{cases} \quad L(\beta_B; s) = \begin{cases} \mathscr{L} & \text{if } s_2 \leq s \leq 1, \\ K & \text{if } s_1 \leq s < s_2, \\ \emptyset & \text{if } 0 \leq s < s_1. \end{cases}$$

Using Corollary 3.7, $B = \langle \mathscr{L}, \alpha_B, \beta_B \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. □

Theorem 3.11. *Let* $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ *be an IFS in* $\mathscr{L}$ *and*

$$\text{Im}(A) = \{(t_0, s_0), (t_1, s_1), \cdots, (t_n, s_n)\}$$

where $(t_i, s_i) < (t_j, s_j)$, *that is,* $t_i < t_j$ *and* $s_i > s_j$ *whenever* $i > j$. *Let* $\{K_i \mid i = 0, 1, \cdots, n\}$ *be a family of Lie ideals of* $\mathscr{L}$ *such that*

1. $K_0 \subset K_1 \subset \cdots \subset K_n = \mathscr{L}$,

2. *$A(K_i^*) = (t_i, s_i)$, i.e., $\alpha_A(K_i^*) = t_i$ and $\beta_A(K_i^*) = s_i$, where $K_i^* = K_i \setminus K_{i-1}$, $K_{-1} = \emptyset$ for $i = 0, 1, \cdots, n$.*

Then $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.

Proof. Let $x, y \in \mathscr{L}$. If $x \in K_i^*$ and $y \in K_i^*$ for every $i = 0, 1, \cdots, n$, then $x + y \in K_i$ and so $\alpha_A(x + y) \geq t_i = \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x + y) \leq s_i = \max\{\beta_A(x), \beta_A(y)\}$. If $x \notin K_i^*$ and $y \notin K_i^*$, then the following four cases arise:

1. $x \in \mathscr{L} \setminus K_i$ and $y \in \mathscr{L} \setminus K_i$.
2. $x \in K_{i-1}$ and $y \in K_{i-1}$.
3. $x \in \mathscr{L} \setminus K_i$ and $y \in K_{i-1}$.
4. $x \in K_{i-1}$ and $y \in \mathscr{L} \setminus K_i$.

But, in either case, we know that $\alpha_A(x + y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x + y) \leq \max\{\beta_A(x), \beta_A(y)\}$. If $x \in K_i^*$ and $y \notin K_i^*$, then either $y \in K_{i-1}$ or there exists $j \in \{0, 1, \cdots, n\}$ such that $i < j$ and $y \in K_j$. It follows that either $x + y \in K_i$ or $x + y \in K_j$. Hence $\alpha_A(x + y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\}$. If $x \notin K_i^*$ and $y \in K_i^*$, then by similar process we have $\alpha_A(x+y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\}$. If $x \in K_i^*$, then $rx \in K_i$ for all $r \in F$ and so $\alpha_A(rx) \geq \alpha_A(x)$ and $\beta_A(rx) \leq \beta_A(x)$. If $x \notin K_i^*$, then either $x \in K_{i-1}$ or $x \in K_j$ for some $j \in \{0, 1, \cdots, n\}$ with $i < j$. It follows that either $rx \in K_{i-1}$ or $rx \in K_j$ for all $r \in F$. Hence $\alpha_A(rx) \geq \alpha_A(x)$ and $\beta_A(rx) \leq \beta_A(x)$. For every $y \in \mathscr{L}$, if $y \in K_i^*$, then $[xy] \in K_i$ for all $x \in \mathscr{L}$. Hence $\alpha_A([xy]) \geq t_i = \alpha_A(y)$ and $\beta_A([xy]) \leq s_i = \beta_A(y)$. If $y \notin K_i^*$, then $y \in K_{i-1}$ or there exists $j \in \{i + 1, i + 2, \cdots, n\}$ such that $y \in K_j$. Hence, either $[xy] \in K_{i-1}$ or $[xy] \in K_j$ for all $x \in \mathscr{L}$. It follows that $\alpha_A([xy]) \geq \alpha_A(y)$ and $\beta_A([xy]) \leq \beta_A(y)$. Consequently, $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. □

Theorem 3.12. *Let $\{K_t \mid t \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a finite collection of Lie ideals of $\mathscr{L}$ such that*

1. $\mathscr{L} = \bigcup_{t \in \Lambda} K_t$,
2. $(\forall s, t \in \Lambda)$ $(t < s \Leftrightarrow K_s \subset K_t)$.

Then an IFS $A = \langle \mathscr{L}, \alpha_A, \beta_A \rangle$ *in* $\mathscr{L}$ *defined by*

$$\alpha_A(x) = \sup\{t \in \Lambda \mid x \in K_t\} \text{ and } \beta_A(x) = \inf\{t \in \Lambda \mid x \in K_t\}$$

for all $x \in \mathscr{L}$ *is an intuitionistic fuzzy Lie ideal of* $\mathscr{L}$.

Proof. According to Corollary 3.7, it is sufficient to show that the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are Lie ideals of $\mathscr{L}$ for every $t, s \in [0,1]$ with $t + s \leq 1$. In order to show that $U(\alpha_A; t)$ is a Lie ideal, we divide into the following two cases:

$$\text{(i) } t = \sup\{i \in \Lambda \mid i < t\} \text{ and (ii) } t \neq \sup\{i \in \Lambda \mid i < t\}.$$

Case (i) implies that

$$\begin{aligned} x \in U(\alpha_A; t) &\Leftrightarrow x \in K_i \text{ for all } i < t \\ &\Leftrightarrow x \in \bigcap_{i<t} K_i, \end{aligned}$$

so that $U(\alpha_A; t) = \bigcap_{i<t} K_i$, which is a Lie ideal of $\mathscr{L}$. For the case (ii), we claim that $U(\alpha_A; t) = \bigcup_{i \geq t} K_i$. If $x \in \bigcup_{i \geq t} K_i$, then $x \in K_i$ for some $i \geq t$. It follows that $\alpha_A(x) \geq i \geq t$ so that $x \in U(\alpha_A; t)$. This proves that $\bigcup_{i \geq t} K_i \subset U(\alpha_A; t)$. Now assume that $x \notin \bigcup_{i \geq t} K_i$. Then $x \notin K_i$ for all $i \geq t$. Since $t \neq \sup\{i \in \Lambda \mid i < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin K_i$ for all $i > t - \varepsilon$, which means that if $x \in K_i$ then $i \leq t - \varepsilon$. Thus $\alpha_A(x) \leq t - \varepsilon < t$, and so $x \notin U(\alpha_A; t)$. Therefore $U(\alpha_A; t) = \bigcup_{i \geq t} K_i$. Next we show that $L(\beta_A; s)$ is a Lie ideal of $\mathscr{L}$ for all $s \in [\beta_A(0), 1]$. We consider the following two cases:

$$\text{(iii) } s = \inf\{i \in \Lambda \mid s < i\} \text{ and (iv) } s \neq \inf\{i \in \Lambda \mid s < i\}.$$

For the case (iii) we have

$$\begin{aligned} x \in L(\beta_A; s) &\Leftrightarrow x \in K_i \text{ for all } s < i \\ &\Leftrightarrow x \in \bigcap_{s<i} K_i, \end{aligned}$$

and hence $L(\beta_A; s) = \bigcap_{s<i} K_i$, which is a Lie ideal of $\mathscr{L}$. For the case (iv), we will show that $L(\beta_A; s) = \bigcup_{s \geq i} K_i$. If $x \in \bigcup_{s \geq i} K_i$, then $x \in K_i$

for some $s \geq i$. It follows that $\beta_A(x) \leq i \leq s$ so that $x \in L(\beta_A; s)$. Hence $\bigcup_{s \geq i} K_i \subset L(\beta_A; s)$. Conversely, if $x \notin \bigcup_{s \geq i} K_i$ then $x \notin K_i$ for all $i \leq s$. Since $s \neq \inf\{i \in \Lambda \mid s < i\}$, there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \cap \Lambda = \emptyset$, which implies that $x \notin K_i$ for all $i < s + \varepsilon$, that is, if $x \in K_i$ then $i \geq s + \varepsilon$. Thus $\beta_A(x) \geq s + \varepsilon > s$, that is, $x \notin L(\beta_A; s)$. Therefore $L(\beta_A; s) \subset \bigcup_{s \geq i} K_i$ and consequently $L(\beta_A; s) = \bigcup_{s \geq i} K_i$. This completes the proof. □

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