

## ERROR BOUNDS OF TRAPEZOIDAL RULE ON SUBINTERVALS USING DISTRIBUTION

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ABSTRACT. We showed in [2] that if  $r \leq 2$ , then the average error between simple Trapezoidal rule and the composite Trapezoidal rule on two consecutive subintervals is proportional to  $h^{2r+3}$  using zero mean Gaussian distribution under the assumption that we have subintervals (for simplicity equal length) partitioning and that each subinterval has the length. In this paper, if  $r \geq 3$ , we show that zero mean Gaussian distribution of average error between simple Trapezoidal rule and the composite Trapezoidal rule on two consecutive subintervals is bounded by  $Ch^8$ .

### 1. Introduction

Many computational problems in science and engineering can only be solved approximately since the available information is partial. For instance, for problems defined on a space of functions, information about  $f$  is typically provided by few function values,  $N(f) = [f(x_0), f(x_1), \dots, f(x_n)]$ . Knowing  $N(f)$ , the solution is approximated by an algorithm. Therefore we have the error between the true and the approximate solutions. The error between the true and the approximate solutions can be reduced by acquiring more information.

The error between the true solution and the approximation depends on a problem setting. In the worst case setting, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions. In this paper, we concentrate on another setting, the average case setting. In this setting, we assume that the class  $F$  of input functions is equipped with a probability measure. The general references are [1], [5] and [7].

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## 2. Definitions

It is well known that the average case setting requires the space of functions to be equipped with a probability measure. In this paper, we choose a probability measure  $\mu_r$  which is a variant of an  $r$ -fold Wiener measure  $\omega_r$ . The probability measure  $\omega_r$  is a Gaussian measure with zero mean and correlation function given by  $M_{\omega_r}(f(x)f(y)) = \int_F f(x)f(y)\omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!} \frac{(y-t)_+^r}{r!} dt$ , where  $(z-t)_+^r = [\max\{0, (z-t)\}]^r$ . Equivalently,  $f$  distributed according to  $\omega_r$  can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since  $\omega_r$  is concentrated on functions with boundary conditions  $f(0) = f'(0) = \dots = f^{(r)}(0) = 0$ , we choose to study a slightly modified measure  $\mu_r$  that preserves basic properties of  $\omega_r$ , yet does not require any boundary conditions. More precisely, we assume that a function  $f$ , as a stochastic process, is given by

$$f(x) = f_1(x) + f_2(1-x), \quad x \in [0, 1],$$

where  $f_1$  and  $f_2$  are independent and distributed according to  $\omega_r$ . Then by [3-6], the corresponding probability measure  $\mu_r$  is a zero mean Gaussian with the correlation function given by

$$\begin{aligned} M_{\mu_r}(f(x)f(y)) &= \int_0^1 \frac{(x-t)_+^r (y-t)_+^r + (1-x-t)_+^r (1-y-t)_+^r}{r! r!} dt \\ &= \int_0^1 \frac{(x-t)_+^r (y-t)_+^r + (t-x)_+^r (t-y)_+^r}{r! r!} dt. \end{aligned}$$

We study the problem of approximating an integral  $I(f) = \int_0^1 f(x) dx$  for  $f \in F = C^r[0, 1]$ , assuming that the class of integrands is equipped with the probability measure  $\mu_r$ .

Assume that we have  $n$  subintervals (not necessarily equal length) partitioning  $[0, 1]$ . Let  $x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ . But for simplicity, we let  $x_i = ih$ , for  $i = 0, \dots, n$  where  $h = \frac{1}{n}$ . With this indexing, we get

$$I_i(f) \equiv \int_{x_{i-1}}^{x_{i+1}} f(x) dx \quad \text{and} \quad T_i(f) = h \{f(x_{i-1}) + f(x_{i+1})\}$$

while  $T_i$  is the basic Trapezoidal rule using  $f(x_{i-1})$  and  $f(x_{i+1})$ . Let  $\overline{T}_i$  be the composite Trapezoidal rule that uses  $f(x_{i-1})$ ,  $f(x_i)$  and  $f(x_{i+1})$ , i.e.,

$$\overline{T}_i(f) = \frac{h}{2} \{f(x_{i-1}) + 2f(x_i) + f(x_{i+1})\}.$$

Also let

$$Z_i(f) = \frac{1}{8}(\overline{T}_i(f) - T_i(f)).$$

### 3. Error bounds on two consecutive subintervals using distribution

In this section, we consider two consecutive subintervals. In order to find a new error bound for the subintervals, we need to compute the distributions of  $Z_i$ . In fact, they are Gaussian with zero-mean and are given in next theorems and lemmas.

**THEOREM 3.1.** For  $i \leq j$  and  $r \leq 2$ ,

$$M_{\mu_r}(Z_i Z_j) = \delta_{ij} C_r \cdot h^8,$$

where  $\delta_{ij}$  is the Kronecker delta, the constants  $C_r$  are independent of  $h$ 's and equal respectively:  $C_0 = \frac{1}{16}$ ,  $C_1 = \frac{1}{192}$  and  $C_2 = \frac{581}{61440}$ .

*Proof.* See [2]. □

**THEOREM 3.2.** For  $i \leq j$  and  $r \geq 3$ ,

$$M_{\mu_r}(Z_i Z_j) = C_{ij} \cdot h^8,$$

where  $C_{ij}$  is bounded from below by

$$\begin{aligned} & c_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p x_{i-1}^{2r-3-p} + (x_j - x_i)^p (1 - x_j)^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + c_2 [(x_{j-1} - x_i)^{r-2} h_i^{r-2} + (x_{j-1} - x_i)^{r-2} h_j^{r-2}] \\ & \geq c_r [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2} + (x_{j-1} - x_i)^{r-2} h_i^{r-2}] \\ & + c_r [(1 - x_j)^{r-1} (x_j - x_i)^{r-2} + (1 - x_j)^{r-1} (1 - x_i)^{r-2}] \\ & + c_r [(x_{j-1} - x_i)^{r-2} h_j^{r-2}] \end{aligned}$$

and  $C_{ij}$  is bounded from above by

$$\begin{aligned} & c_1 \sum_{p=0}^{r-2} \frac{(x_j - x_i)^p x_i^{2r-3-p} + (x_{j-1} - x_{i-1})^p (1 - x_{j-1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + c_2 [(x_j - x_{i-1})^{r-2} h_i^{r-2} + (x_j - x_{i-1})^{r-2} h_j^{r-2}] \\ & \leq c_{r'} [x_i^{r-1} x_j^{r-2} + (x_j - x_{i-1})^{r-2} h_i^{r-2}] \\ & + c_{r'} [(1 - x_{j-1})^{r-1} (1 - x_{i-1})^{r-2} + (x_j - x_{i-1})^{r-2} h_j^{r-2}] \end{aligned}$$

where  $c_1 = a_1 + b_1$ ,  $c_2 = a_2 + b_2$ ,  $c_r = \min\{\frac{c_1}{(r-1)! \cdot 2^{r-1}}, a_2 + \frac{b_1}{(r-1)! \cdot 2^{r-1}}, \frac{a_1}{(r-1)! \cdot 2^{r-1}} + b_2, c_2\}$  and  $c_{r'} = \max\{\frac{c_1}{(r-2)!}, a_2 + \frac{b_1}{(r-2)!}, \frac{a_1}{(r-2)!} + b_2, c_2\}$  if we let  $a_1 = \frac{1}{2^2 \cdot 3^2 \cdot (r-2)!}$ ,  $a_2 = \frac{2^r \cdot (r+1)}{2^4 \cdot 3 \cdot (r+2)! \cdot (r-2)!}$ ,  $b_1 = \frac{1}{(2^{10} \cdot 3^2 \cdot (r-2)!)}$  and  $b_2 = \frac{r}{3 \cdot 2^9 \cdot (r+2)! \cdot (r-2)!}$ .

Theorem 3.2 is the main result of this study. In order to prove that, we prepare two lemmas. Let  $X_i(f) = \frac{1}{8}(\overline{T}_i(f) - I_i(f))$  and  $Y_i(f) = \frac{1}{8}(I_i(f) - T_i(f))$ . Then  $Z_i(f) = X_i(f) + Y_i(f)$ .

LEMMA 3.3. For  $i \leq j$  and  $r \geq 3$ ,

$$M_{\mu_r}(X_i X_j) = C_{ij} \cdot h^8,$$

where  $C_{ij}$  is bounded from below by

$$\begin{aligned} & a_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p x_{i-1}^{2r-3-p} + (x_{j+1} - x_{i+1})^p (1 - x_{j+1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + a_2 [(x_{j-1} - x_{i+1})^{r-2} h^{r-2} + (x_{j-1} - x_{i-1})^{r-2} h^{r-2}] \\ & \geq a_r [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2} + (x_{j-1} - x_{i+1})^{r-2} h^{r-2}] \\ & + a_r [(1 - x_{j+1})^{r-1} (x_{j+1} - x_{i+1})^{r-2}] \\ & + a_r [(1 - x_{j+1})^{r-1} (1 - x_{i+1})^{r-2} + (x_{j-1} - x_{i+1})^{r-2} h^{r-2}] \end{aligned}$$

and  $C_{ij}$  is bounded from above by

$$\begin{aligned} & a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p x_{i+1}^{2r-3-p} + (x_{j+1} - x_{i+1})^p (1 - x_{j-1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + a_2 [(x_{j+1} - x_{i-1})^{r-2} h^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] \\ & \leq a_{r'} [x_{i+1}^{r-1} x_{j+1}^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] \\ & + a_{r'} [(1 - x_{j-1})^{r-1} (1 - x_{i-1})^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}]. \end{aligned}$$

where  $a_1 = \frac{1}{2^2 \cdot 3^2 \cdot (r-2)!}$ ,  $a_2 = \frac{2^r \cdot (r+1)}{2^4 \cdot 3 \cdot (r+2)! \cdot (r-2)!}$ ,  $a_r = \min\{\frac{a_1}{(r-1)! \cdot 2^{r-1}}, a_2\}$  and  $\lambda_{r'} = \max\{\frac{a_1}{(r-2)!}, a_2\}$ .

*Proof.* Let  $X_{i1} = X_i(f_1)$  and  $X_{i2} = X_i(f_2)$ . Then  $X_i(f) = X_{i1} + X_{i2}$ , and due to the independence of  $f_1$  and  $f_2$ , we have  $M_{\mu_r}(X_i X_j) = M_{\omega_r}(X_{i1} X_{j1}) + M_{\omega_r}(X_{i2} X_{j2})$ . For  $i \leq j$ ,

$$\begin{aligned} & M_{\omega_r}(X_{i1} X_{j1}) \\ &= \int_0^1 \left[ \frac{1}{8} \left( A_{i1}(t) - \int_{x_{i-1}}^{x_{i+1}} \frac{(x-t)_+^r}{r!} dx \right) \right] \left[ \frac{1}{8} \left( A_{j1}(t) - \int_{x_{j-1}}^{x_{j+1}} \frac{(y-t)_+^r}{r!} dy \right) \right] dt \\ &= \int_0^1 L_{i1}(t) \cdot L_{j1}(t) dt, \end{aligned}$$

where  $L_{i1}$  is the first term and  $L_{j1}$  is the second term in the above integral, and  $A_{i1}(t) = \overline{T}_i \left( \frac{(x-t)_+^r}{r!} \right)$ . Since  $L_{i1}(t) = 0$  when  $t \in [x_{i+1}, 1]$ , we have

$$M_{\omega_r}(X_{i1} X_{j1}) = \int_0^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt.$$

Similarly,

$$\begin{aligned} & M_{\omega_r}(X_{i2} X_{j2}) \\ &= \int_{x_{j-1}}^1 \left[ \frac{1}{8} \left( A_{i2}(t) - \int_{x_{i-1}}^{x_{i+1}} \frac{(t-x)_+^r}{r!} dx \right) \right] \left[ \frac{1}{8} \left( A_{j2}(t) - \int_{x_{j-1}}^{x_{j+1}} \frac{(t-y)_+^r}{r!} dy \right) \right] dt \\ &= \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt, \end{aligned}$$

where  $L_{i2}$  is the first term and  $L_{j2}$  is the second term in the above integral, and  $A_{i2}(t) = \overline{T}_i \left( \frac{(t-x)_+^r}{r!} \right)$ . Since  $L_{j2}(t) = 0$  when  $t \in [0, x_{j-1}]$ , we therefore have

$$M_{\mu_r}(X_i X_j) = \int_0^{x_{i-1}} L_{i1}(t) \cdot L_{j1}(t) dt + \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt.$$

First divide the integral in  $M_{\omega_r}(X_{i1} X_{j1})$  into two parts,

$$\begin{aligned} M_{\omega_r}(X_{i1} X_{j1}) &= \int_0^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt \\ &= \int_0^{x_{i-1}} L_{i1}(t) \cdot L_{j1}(t) dt + \int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt. \end{aligned}$$

We compute the first part  $\int_0^{x_{i-1}} L_{i1}(t) \cdot L_{j1}(t) dt$ . For  $t \in [0, x_{i-1}]$ ,

$$\begin{aligned} \int_0^{x_{i-1}} L_{i1}(t) \cdot L_{j1}(t) dt &= \frac{(2h_i)^4 (2h_j)^4}{2^6 \cdot 12^2} \int_0^{x_{i-1}} \frac{(\xi_t - t)^{r-2}}{(r-2)!} \frac{(\eta_t - t)^{r-2}}{(r-2)!} dt \\ &= A_{ij} h_i^4 h_j^4 \\ &= A_{ij} h^8, \end{aligned}$$

where  $\xi_t \in (x_{i-1}, x_{i+1})$  and  $\eta_t \in (x_{j-1}, x_{j+1})$ . By binomial expansion theorem,

$$\begin{aligned} A_{ij} &= \frac{1}{36} \cdot \frac{1}{(r-2)! \cdot (r-2)!} \int_0^{x_{i-1}} (\xi_t - t)^{r-2} (\eta_t - t)^{r-2} dt \\ &= \frac{a_1}{(r-2)!} \int_0^{x_{i-1}} \sum_{p=0}^{r-2} \binom{r-2}{p} (\xi_t \eta_t - (\xi_t + \eta_t)t + t^2)^{r-2-p} dt \\ &\geq \frac{a_1}{(r-2)!} \sum_{p=0}^{r-2} \frac{(r-2)!}{p! \cdot (r-2-p)!} \int_0^{x_{i-1}} (x_{j-1} - x_{i-1})^p t^{2r-4-2p} dt \\ &= a_1 \sum_{p=0}^{r-2} \frac{1}{p! \cdot (r-2-p)!} (x_{j-1} - x_{i-1})^p \int_0^{x_{i-1}} t^{2r-4-2p} dt \\ &= a_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p}{p! \cdot (r-2-p)!} \cdot \frac{x_{i-1}^{2r-3-p}}{2r-3-p} \end{aligned}$$

where  $a_1 = \frac{1}{36(r-2)!} = \frac{1}{2^2 \cdot 3^2 (r-2)!}$ . Let

$$g(t) = \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p}{p!} \cdot \frac{t^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p}.$$

Then  $A_{ij} \geq a_1 g(x_{i-1})$ . Since  $g(0) = 0$ ,

$$\begin{aligned} &g(x_{i-1}) \\ &= \int_0^{x_{i-1}} g'(t) dt \\ &= \int_0^{x_{i-1}} t^{r-2} \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p}{p!} \frac{t^{r-2-p}}{(r-2-p)!} dt \\ &= \int_0^{x_{i-1}} \frac{t^{r-2}}{(r-2)!} \sum_{p=0}^{r-2} \binom{r-2}{p} (x_{j-1} - x_{i-1})^p t^{r-2-p} dt \\ &= \int_0^{x_{i-1}} \frac{t^{r-2}}{(r-2)!} (x_{j-1} - x_{i-1} + t)^{r-2} dt \\ &= \left( \int_0^z + \int_z^{x_{i-1}} \right) t^{r-2} \frac{(x_{j-1} - x_{i-1} + t)^{r-2}}{(r-2)!} dt, \quad \text{where } z = x_{i-1}/2 \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^z t^{r-2} \frac{(x_{j-1} - x_{i-1})^{r-2}}{(r-2)!} dt + \int_z^{x_{i-1}} z^{r-2} \frac{(x_{j-1} - x_{i-1} + t)^{r-2}}{(r-2)!} dt \\
&= \frac{z^{r-1}}{r-1} \frac{(x_{j-1} - x_{i-1})^{r-2}}{(r-2)!} + z^{r-2} \frac{x_{j-1}^{r-1} - (x_{j-1} - x_{i-1} + X)^{r-1}}{(r-1)!} \\
&\geq \frac{z^{r-1}}{(r-1)!} (x_{j-1} - x_{i-1})^{r-2} + z^{r-2} \frac{(x_{i-1} - X)x_{j-1}^{r-2}}{(r-1)!} \\
&\geq \frac{x_{i-1}^{r-1}}{(r-1)!2^{r-1}} (x_{j-1} - x_{i-1})^{r-2} + \frac{x_{i-1}^{r-1}}{(r-1)!2^{r-1}} x_{j-1}^{r-2} \\
&= \frac{1}{(r-1)! \cdot 2^{r-1}} [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2}].
\end{aligned}$$

Thus,

$$\begin{aligned}
(A) \quad A_{ij} &\geq a_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p}{p!} \cdot \frac{x_{i-1}^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p} \\
&\geq \frac{a_1}{(r-1)! \cdot 2^{r-1}} [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2}].
\end{aligned}$$

In the same manner, we can easily show  $A_{ij}$  is also bounded from above by

$$\begin{aligned}
A_{ij} &\leq \frac{a_1}{(r-2)!} \sum_{p=0}^{r-2} \binom{r-2}{p} (x_{j+1} - x_{i+1})^p \frac{x_{i+1}^{2r-3-p}}{(2r-3-p)} \\
&= a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p! \cdot (r-2-p)!} \cdot \frac{x_{i+1}^{2r-3-p}}{(2r-3-p)}.
\end{aligned}$$

Therefore, since  $2r-3-p \geq 1$ ,

$$\begin{aligned}
(B) \quad A_{ij} &\leq a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p! \cdot (r-2-p)!} \cdot \frac{x_{i+1}^{2r-3-p}}{(2r-3-p)} \\
&\leq a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p! \cdot (r-2-p)!} x_{i+1}^{2r-3-p} \\
&= \frac{a_1}{(r-2)!} x_{i+1}^{r-1} \sum_{p=0}^{r-2} \binom{r-2}{p} (x_{j+1} - x_{i+1})^p x_{i+1}^{r-2-p} \\
&\leq \frac{a_1}{(r-2)!} x_{i+1}^{r-1} x_{j+1}^{r-2}.
\end{aligned}$$

We now compute the last part  $\int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt$ . For  $t \in [x_{i-1}, x_{i+1}]$ ,

$$\begin{aligned}
\int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt &= -\frac{(2h_j)^4}{2^3 \cdot 12} \int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot \frac{(\eta t - t)^{r-2}}{(r-2)!} dt \\
&= -\frac{h^4}{6} \int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot \frac{(\eta t - t)^{r-2}}{(r-2)!} dt
\end{aligned}$$

If we set  $z = \frac{x-x_{i-1}}{h_i}$  and  $u = \frac{t-x_{i-1}}{h_i}$ , then

$$\begin{aligned} & -\frac{h^4}{6} \int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot \frac{(\eta_t - t)^{r-2}}{(r-2)!} dt \\ &= -\frac{h^4}{6} \int_{x_{i-1}}^{x_i} \left[ \frac{1}{8} \left( \int_{x_{i-1}}^{x_{i+1}} A_{i1}(t) - \frac{(x-t)_+^r}{r!} dx \right) \right] \cdot \frac{(\eta_t - t)^{r-2}}{(r-2)!} dt \\ &= -\frac{h^4}{48} \int_0^1 \left[ \int_0^1 \frac{(2h_i)^r (z-u)_+^r}{r!} h_i dz - \frac{h_i}{2} \left\{ 2 \left( \frac{h_i^r (0-u)_+^r}{r!} + \frac{h_i^r (1-u)_+^r}{r!} \right) \right\} \right] \\ & \quad \times \frac{(\eta_t - t)^{r-2}}{(r-2)!} h_i du \\ &= -\frac{h^4}{48} \cdot \frac{2^r \cdot h_i^{r+2}}{r! \cdot (r-2)!} \\ & \quad \times \int_0^1 \left[ \int_0^1 (z-u)_+^r dz - \{(0-u)_+^r + (1-u)_+^r\} \right] (\eta_t - t)^{r-2} du \\ &= -\frac{2^r \cdot h^{r+6}}{48 \cdot r! \cdot (r-2)!} \\ & \quad \times \int_0^1 \left[ \int_0^1 (z-u)_+^r dz - \{(0-u)_+^r + (1-u)_+^r\} \right] (\eta_t - t)^{r-2} du. \end{aligned}$$

Let

$$\begin{aligned} B_{ij} &= -\frac{2^r}{48 \cdot r! \cdot (r-2)!} \\ & \quad \times \int_0^1 \left[ \int_0^1 (z-u)_+^r dz - \{(0-u)_+^r + (1-u)_+^r\} \right] (\eta_t - t)^{r-2} du. \end{aligned}$$

Then

$$\begin{aligned} \int_{x_{i-1}}^{x_{i+1}} L_{i1}(t) \cdot L_{j1}(t) dt &= B_{ij} \cdot h_i^{r+2} h_j^4 \\ &= B_{ij} \cdot h^{r+6}. \end{aligned}$$

Since  $t \in [x_{i-1}, x_{i+1}]$  and  $\eta_t \in [x_{j-1}, x_{j+1}]$ ,

$$(x_{j-1} - x_{i+1})^{r-2} \leq (\eta_t - t)^{r-2} \leq (x_{j+1} - x_{i-1})^{r-2}.$$

Thus

$$(C) \quad a_2(x_{j-1} - x_i)^{r-2} \leq B_{ij} \leq a_2(x_j - x_{i-1})^{r-2}$$



where

$$\begin{aligned}
 a_2 &= -\frac{2^r}{48 \cdot r! \cdot (r-2)!} \int_0^1 \left[ \int_0^1 (z-u)_+^r dz - \{(0-u)_+^r + (1-u)_+^r\} \right] du \\
 &= -\frac{2^r}{48 \cdot r! \cdot (r-2)!} \int_0^1 \left[ \int_u^1 (z-u)^r dz - \{(0-u)_+^r + (1-u)_+^r\} \right] du \\
 &= -\frac{2^r}{48 \cdot r! \cdot (r-2)!} \int_0^1 \left[ \frac{1}{r+1} (1-u)^{r+1} - (1-u)^r \right] du \\
 &= -\frac{2^r}{48 \cdot r! \cdot (r-2)!} \left[ \frac{1}{(r+1)(r+2)} - \frac{1}{(r+1)} \right] \\
 &= \frac{2^r \cdot (r+1)}{48 \cdot (r+2)! \cdot (r-2)!} \\
 &= \frac{2^r \cdot (r+1)}{2^4 \cdot 3 \cdot (r+2)! \cdot (r-2)!}.
 \end{aligned}$$

Therefore by (A) and (C),

$$\begin{aligned}
 &M_{\omega_r}(X_{i1}X_{j1}) \\
 &\geq \left[ a_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p}{p!} \cdot \frac{x_{i-1}^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p} \right] h^8 \\
 (1) \quad &+ [a_2(x_{j-1} - x_{i+1})^{r-2} h^{r-2}] h^8 \\
 &\geq a_r [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2}] h^8 \\
 &+ a_r [(x_{j-1} - x_{i+1})^{r-2} h^{r-2}] h^8,
 \end{aligned}$$

where  $a_r = \min\{\frac{a_1}{(r-1)! \cdot 2^{r-1}}, a_2\}$ . Moreover by (B) and (C),

$$\begin{aligned}
 &M_{\omega_r}(X_{i1}X_{j1}) \\
 (2) \quad &\leq \left[ a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p!} \cdot \frac{x_{i+1}^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p} \right] h^8 \\
 &+ [a_2(x_{j+1} - x_{i-1})^{r-2} h^{r-2}] h^8 \\
 &\leq a_{r'} [x_{i+1}^{r-1} x_{j+1}^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] h^8,
 \end{aligned}$$

where  $a_{r'} = \max\{\frac{a_1}{(r-2)!}, a_2\}$ .

Similar to the computation of  $M_{\omega_r}(X_{i1}X_{j1})$ , we can easily compute upper and lower bounds of  $M_{\omega_r}(X_{i2}X_{j2})$  by putting

$$\begin{aligned}
 M_{\omega_r}(X_{i2}X_{j2}) &= \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt \\
 &= \int_{x_{j-1}}^{x_{j+1}} L_{i2}(t) \cdot L_{j2}(t) dt + \int_{x_{j+1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt,
 \end{aligned}$$

with

$$\begin{aligned} \int_{x_{j+1}}^1 L_{i2}(t) \cdot L_{j2}(t) dt &= \frac{h_i^4 h_j^4}{36} \int_{x_{j+1}}^1 \frac{(\xi_t - t)^{r-2}}{(r-2)!} \frac{(\eta_t - t)^{r-2}}{(r-2)!} dt \\ &= A'_{ij} \cdot h_i^4 h_j^4 \\ &= A'_{ij} \cdot h^8, \end{aligned}$$

and

$$\begin{aligned} \int_{x_{j-1}}^{x_{j+1}} L_{i2}(t) \cdot L_{j2}(t) dt &= -\frac{h_i^4}{6} \int_{x_{j-1}}^{x_{j+1}} \frac{(\xi_t - t)^{r-2}}{(r-2)!} \cdot L_{j2}(t) dt \\ &= B'_{ij} \cdot h_i^4 h_j^{r+2} \\ &= B'_{ij} \cdot h^{r+6}, \end{aligned}$$

where  $\xi_t \in (x_{i-1}, x_{i+1})$ ,  $\eta_t \in (x_{j-1}, x_{j+1})$ . And we end up with

$$\begin{aligned} &M_{\omega_r}(X_{i2}X_{j2}) \\ &\geq \left[ a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p!} \cdot \frac{(1 - x_{j+1})^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p} \right] h^8 \\ (3) \quad &+ [a_2(x_{j-1} - x_{i-1})^{r-2} h^{r-2}] h^8 \\ &\geq a_r [(1 - x_{j+1})^{r-1} (x_{j+1} - x_{i+1})^{r-2}] h^8 \\ &+ a_r [(1 - x_{j+1})^{r-1} (1 - x_{i+1})^{r-2} + x_{j-1} - x_{i+1})^{r-2} h^{r-2}] h^8. \end{aligned}$$

And

$$\begin{aligned} &M_{\omega_r}(X_{i2}X_{j2}) \\ (4) \quad &\leq \left[ a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p}{p!} \cdot \frac{(1 - x_{j-1})^{2r-3-p}}{(r-2-p)!} \cdot \frac{1}{2r-3-p} \right] h^8 \\ &+ [a_2(x_{j+1} - x_{i-1})^{r-2} h^{r-2}] h^8 \\ &\leq a_{r'} [(1 - x_{j-1})^{r-1} (1 - x_{i-1})^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] h^8. \end{aligned}$$

From (1) and (3),  $C_{ij}$  is bounded from below by

$$\begin{aligned} &a_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p x_{i-1}^{2r-3-p} + (x_{j+1} - x_{i+1})^p (1 - x_{j+1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ &+ a_2 [(x_{j-1} - x_{i+1})^{r-2} h^{r-2} + (x_{j-1} - x_{i-1})^{r-2} h^{r-2}] \\ &\geq a_r [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2} + (x_{j-1} - x_{i+1})^{r-2} h^{r-2}] \\ &+ a_r [(1 - x_{j+1})^{r-1} (x_{j+1} - x_{i+1})^{r-2}] \\ &+ a_r [(1 - x_{j+1})^{r-1} (1 - x_{i+1})^{r-2} + (x_{j-1} - x_{i+1})^{r-2} h^{r-2}] \end{aligned}$$

and from (2) and (4),  $C_{ij}$  is bounded from above by

$$\begin{aligned} & a_1 \sum_{p=0}^{r-2} \frac{(x_{j+1} - x_{i+1})^p x_{i+1}^{2r-3-p} + (x_{j+1} - x_{i+1})^p (1 - x_{j-1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + a_2 [(x_{j+1} - x_{i-1})^{r-2} h^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] \\ \leq & a_{r'} [x_{i+1}^{r-1} x_{j+1}^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}] \\ & + a_{r'} [(1 - x_{j-1})^{r-1} (1 - x_{i-1})^{r-2} + (x_{j+1} - x_{i-1})^{r-2} h^{r-2}]. \end{aligned}$$

This completes the proof. □

LEMMA 3.4. For  $i \leq j$  and  $r \geq 3$ ,

$$M_{\mu_r}(Y_i Y_j) = C_{ij} \cdot h^8,$$

where  $C_{ij}$  is bounded from below by

$$\begin{aligned} & b_1 \sum_{p=0}^{r-2} \frac{(x_{j-1} - x_{i-1})^p x_{i-1}^{2r-3-p} + (x_j - x_i)^p (1 - x_j)^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + b_2 [(x_{j-1} - x_i)^{r-2} h_i^{r-2} + (x_{j-1} - x_i)^{r-2} h_j^{r-2}] \\ \geq & b_r [x_{i-1}^{r-1} (x_{j-1} - x_{i-1})^{r-2} + x_{i-1}^{r-1} x_{j-1}^{r-2} + (x_{j-1} - x_i)^{r-2} h_i^{r-2}] \\ & + b_r [(1 - x_j)^{r-1} (x_j - x_i)^{r-2} + (1 - x_j)^{r-1} (1 - x_i)^{r-2}] \\ & + b_r [(x_{j-1} - x_i)^{r-2} h_j^{r-2}] \end{aligned}$$

and  $C_{ij}$  is bounded from above by

$$\begin{aligned} & b_1 \sum_{p=0}^{r-2} \frac{(x_j - x_i)^p x_i^{2r-3-p} + (x_{j-1} - x_{i-1})^p (1 - x_{j-1})^{2r-3-p}}{p! (r-2-p)! (2r-3-p)} \\ & + b_2 [(x_j - x_{i-1})^{r-2} h_i^{r-2} + (x_j - x_{i-1})^{r-2} h_j^{r-2}] \\ \leq & b_{r'} [x_i^{r-1} x_j^{r-2} + (x_j - x_{i-1})^{r-2} h_i^{r-2}] \\ & + b_{r'} [(1 - x_{j-1})^{r-1} (1 - x_{i-1})^{r-2} + (x_j - x_{i-1})^{r-2} h_j^{r-2}] \end{aligned}$$

where  $b_1 = \frac{1}{(2^{10} \cdot 3^2 \cdot (r-2)!)}$ ,  $b_2 = \frac{r}{3 \cdot 2^9 \cdot (r+2)! \cdot (r-2)!}$ ,  $b_r = \min\{\frac{b_1}{(r-1)! \cdot 2^{r-1}}, b_2\}$  and  $b_{r'} = \max\{\frac{b_1}{(r-2)!}, b_2\}$ .

*Proof.* Since the proof is similar to Lemma 3.3, we omit it. □

We now prove Theorem 3.2 using Lemma 3.3 and Lemma 3.4.

*Proof of Theorem 3.2.* Let  $Z_{i1} = Z_i(f_1)$  and  $Z_{i2} = Z_i(f_2)$ . Then  $Z_i(f) = Z_{i1} + Z_{i2}$ , and due to the independence of  $f_1$  and  $f_2$ , we have  $M_{\mu_r}(Z_i Z_j) = M_{\omega_r}(Z_{i1} Z_{j1}) + M_{\omega_r}(Z_{i2} Z_{j2})$ . Since  $Z_i(f) = X_i(f) + Y_i(f)$ ,  $M_{\mu_r}(Z_i Z_j) = M_{\mu_r}(X_i X_j) + M_{\mu_r}(Y_i Y_j)$ . Therefore

$$\begin{aligned} & M_{\mu_r}(Z_i Z_j) \\ &= M_{\omega_r}(Z_{i1} Z_{j1}) + M_{\omega_r}(Z_{i2} Z_{j2}) \\ &= (M_{\omega_r}(X_{i1} X_{j1}) + M_{\omega_r}(X_{i2} X_{j2})) + (M_{\omega_r}(Y_{i1} Y_{j1}) + M_{\omega_r}(Y_{i2} Y_{j2})). \end{aligned}$$

Thus, Theorem 3.2 is an immediate consequence of Lemma 3.3 and Lemma 3.4. We complete the proof. □

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