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STABILITY OF A QUADRATIC TYPE FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate some results concerning the stability of the following quadratic type functional equation:

$$f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(z+x)+f(z-x)$$
$$=4f(x)+4f(y)+4f(z).$$

1. Introduction

In 1940, S. M. Ulam [29] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D. H. Hyers [11] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [22]. Since then, the stability problems of several functional equation have been extensively investigated by a number of authors (for instances, [1, 2, 3, 4, 5, 6, 8, 12, 14, 17, 18, 23, 24, 25, 26, 27]).

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In particular, one of the important functional equations studied is the following functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x)=ax^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 7, 15, 19].

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [27] for functions between a normed space and a Banach space. Afterwards, her result was extended by P. W. Cholewa [6] and S. Czerwik [7].

In this paper, we deal with the following quadratic type functional equation:

$$(1.1)\quad \begin{aligned} &f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(z+x)+f(z-x)\\ &\quad =4f(x)+4f(y)+4f(z). \end{aligned}$$

It is easy to see that the function $f(x)=ax^2$ is a solution of (1.1).

The main purpose of this paper is to offer the stability result for this equation.

2. The Required Result

For completeness, we will first present the following result.

Lemma 2.1. *Let X and Y be vector spaces. A function $Q: X \to Y$ satisfies the functional equation* (1.1) for all $x, y, z \in X-\{0\}$. Then Q is a quadratic.

Proof. In order to obtain our result, it suffices to show that

$$Q(0)=0,\ Q(-x)=Q(x),\ x\neq 0.$$

Putting $y=z=x\neq 0$ in (1.1), we arrive at

$$(2.1)\qquad Q(2x)+Q(0)=4Q(x).$$

Letting $z = x$ in (1.1) gives the equation

$$Q(x+y) + Q(x-y) + Q(y+x) + Q(y-x) + Q(2x) + Q(0) = 8Q(x) + 4Q(y), \tag{2.2}$$

and by interchanging x with y in (2.2), it follows that

$$Q(y+x) + Q(y-x) + Q(x+y) + Q(x-y) + Q(2y) + Q(0) = 4Q(x) + 8Q(y). \tag{2.3}$$

By subtracting (2.2) from (2.3), then we see that

$$Q(2y) - Q(2x) = 4Q(y) - 4Q(x). \tag{2.4}$$

Setting $x := x + y$ and $y := x - y$ in (2.4) yields

$$Q(2x+2y) - 4Q(x+y) = Q(2x-2y) - 4Q(x-y). \tag{2.5}$$

Further, we let $x = y$ in (2.5) and then use (2.1) to find that

$$Q(4x) = 16Q(x) - 7Q(0). \tag{2.6}$$

We substitute $x := 2x$ in (2.1) and then utilize (2.1) to get

$$Q(4x) = 16Q(x) - 5Q(0). \tag{2.7}$$

By considering (2.6) and (2.7), we note that

$$Q(0) = 0. \tag{2.8}$$

Now the relation (2.1) can be written as

$$Q(2x) = 4Q(x). \tag{2.9}$$

On the other hand, replacing z by x in (1.1), we obtain that

$$2Q(x+y) + Q(x-y) + Q(y-x) = 4Q(x) + 4Q(y). \tag{2.10}$$

If we take $2x$ instead of y in (2.10) and use (2.8), then we have

$$2Q(3x) = 19Q(x) - Q(-x). \tag{2.11}$$

In relation (2.10), we replace y by $-2x$:

$$Q(3x) + Q(-3x) = 4Q(x) + 14Q(-x). \tag{2.12}$$

From (2.11) and (2.12), its results:

$$Q(x) = Q(-x). \tag{2.13}$$

By (2.13), the relation (2.10) becomes

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X - \{0\}$. Therefore [9, Theorem 2.3] guarantees that Q is a quadratic. This completes the proof of the lemma. □

3. The stability of eq. (1.1)

In this section, let X be a real vector space and Y be a real Banach space. For given a function $f : X \to Y$, we set

$$\begin{aligned} Df(x,y,z) &:= f(x+y) + f(x-y) + f(y+z) + f(y-z) \\ &+ f(z+x) + f(z-x) - 4f(x) - 4f(y) - 4f(z) \end{aligned}$$

for all $x, y, z \in X$.

Let $\phi : X \times X \times X \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x, 2^i x)}{4^i}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{4^n} = 0$$

for all $x, y, z \in X - \{0\}$.

Theorem 3.1. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \phi(x,y,z) \tag{3.1}$$

for all $x, y, z \in X - \{0\}$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the inequality

$$\|f(x) - Q(x)\| \le \frac{1}{4}\sum_{i=0}^{\infty}\left[\frac{\phi(2^i x, 2^i x, 2^i x)}{3 \cdot 4^i} + \frac{\|f(0)\|}{4^i}\right] \tag{3.2}$$

holds for all $x \in X - \{0\}$.

Proof. Putting $y = z = x \neq 0$ in (3.1) and multiplying by $\frac{1}{3}$ on both sides

$$\|f(2x) - 4f(x)\| \leq \frac{\phi(x,x,x)}{3} + \|f(0)\|,$$

which implies that

$$\left\|\frac{f(2x)}{4} - f(x)\right\| \leq \frac{1}{4}\Big[\frac{\phi(x,x,x)}{3} + \|f(0)\|\Big]. \tag{3.3}$$

Replacing x by $2x$ in (3.3) and dividing by 4 and summing the resulting inequality with (3.3), we get

$$\left\|\frac{f(2^2x)}{4^2} - f(x)\right\| \leq \frac{1}{4}\Big[\frac{\phi(2x,2x,2x)}{3\cdot 4} + \frac{\phi(x,x,x)}{3} + \frac{\|f(0)\|}{4} + \|f(0)\|\Big]. \tag{3.4}$$

An induction argument now implies

$$\left\|\frac{f(2^nx)}{4^n} - f(x)\right\| \leq \frac{1}{4}\sum_{i=0}^{n-1}\Big[\frac{\phi(2^ix,2^ix,2^ix)}{3\cdot 4^i} + \frac{\|f(0)\|}{4^i}\Big]. \tag{3.5}$$

We divide (3.5) by 4^m and replace x by 2^mx to obtain that

$$\left\|\frac{f(2^n2^mx)}{4^{n+m}} - \frac{f(2^mx)}{4^m}\right\| \leq \frac{1}{4}\cdot\frac{1}{4^m}\sum_{i=0}^{n-1}\Big[\frac{\phi(2^{m+i}x,2^{m+i}x,2^{m+i}x)}{3\cdot 4^i} + \frac{\|f(0)\|}{4^i}\Big] \leq \frac{1}{4}\sum_{i=0}^{\infty}\Big[\frac{\phi(2^{m+i}x,2^{m+i}x,2^{m+i}x)}{3\cdot 4^{m+i}} + \frac{\|f(0)\|}{4^{m+i}}\Big]. \tag{3.6}$$

This show that $\{\frac{f(2^nx)}{4^n}\}$ is a Cauchy sequence in Y by taking the limit $m \to \infty$. Since Y is a Banach space, it follows that the sequence $\{\frac{f(2^nx)}{4^n}\}$ converges.

Therefore, we may define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^nx)}{4^n}$$

for all $x \in X - \{0\}$. By letting $n \to \infty$ in (3.5), we arrive at the formula (3.2).

From (3.1) it follows that

$$DQ(x, y, z) = \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y, 2^n z)\|$$
$$\le \lim_{n\to\infty} \frac{1}{4^n}\phi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in X - \{0\}$, that is, Q satisfies the functional equation (1.1). So Lemma 2.1 guarantees that Q is a quadratic.

We claim that Q is unique. Suppose that there exists another quadratic function $T : X \to Y$ satisfying the inequality (3.2). Since $T(2^n x) = 4^n T(x)$ and $Q(2^n x) = 4^n Q(x)$, we conclude that

$$\|T(x) - Q(x)\| = \frac{1}{4^n}\|T(2^n x) - Q(2^n x)\|$$
$$\le \frac{1}{4^n}[\|T(2^n x) - f(2^n x)\| + \|f(2^n x) - Q(2^n x)\|]$$
$$\le \frac{1}{2}\sum_{i=0}^{\infty}\Big[\frac{\phi(2^{n+i}x, 2^{n+i}x, 2^{n+i}x)}{3 \cdot 4^{n+i}} + \frac{\|f(0)\|}{4^{n+i}}\Big]$$

By letting $n \to \infty$ in this inequality, we have $T(x) = Q(x)$, which ends the proof of the theorem. □

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [22] of the functional equation (1.1).

Corollary 3.2. *Let X and Y be a normed space and a Banach space, respectively. Let θ, p be nonnegative real numbers with $p < 2$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X - \{0\}$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the inequality

$$\|f(x) - Q(x)\| \le \frac{1}{4 - 2^p}\theta\|x\|^p + \frac{1}{3}\|f(0)\|$$

for all $x \in X - \{0\}$.

Proof. In Theorem 3.1, defining $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, we obtain the desire result. □

The following corollary is the Hyers-Ulam stability [11] of the functional equation (1.1).

Corollary 3.3. *Let X and Y be a normed space and a Banach space, respectively, and $\epsilon \geq 0$ be a real numbers. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \leq \epsilon$$

for all $x, y, z \in X - \{0\}$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{9}\epsilon + \frac{1}{3}\|f(0)\|$$

for all $x \in X - \{0\}$.

Proof. Considering $\phi(x, y, z) := \epsilon$ in the theorem 3.1, we arrive at the conclusion of the corollary. □

Our problem is the following: If f is a function such that

$$\frac{f(x+y) + f(x-y) + f(y+z) + f(y-z) + f(z+x) + f(z-x)}{4f(x) + 4f(y) + 4f(z)}$$

is close to 1, then f can approximately by a quadratic function? We offer a positive answer to this problem([10], [16], [28]).

In the following, we consider G an abelian group and $\varphi : G \to (0, 1)$ a function such that

$$\sum_{n=0}^{\infty} \varphi(2^n x, 2^n y, 2^n z)$$

converges for all $x, y, z \in G - \{0\}$.

We denote

$$\varphi_0(x, y, z) = \prod_{n=0}^{\infty} [1 - \varphi(2^n x, 2^n y, 2^n z)],$$

$$\varphi_1(x, y, z) = \prod_{n=0}^{\infty} [1 + \varphi(2^n x, 2^n y, 2^n z)]$$

for all $x, y, z \in G - \{0\}$.

Theorem 3.4. *Suppose that $f : G \to [0, \infty)$ is a function such that $f(x) = 0$ if and only if $x = 0$, and*

$$\left|\frac{f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(z+x)+f(z-x)}{4f(x)+4f(y)+4f(z)} - 1\right| \leq \varphi(x,y,z) \tag{3.7}$$

for all $x, y, z \in G - \{0\}$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the relation

$$\varphi_0(x,x,x) \leq \frac{Q(x)}{f(x)} \leq \varphi_1(x,x,x) \tag{3.8}$$

holds for all $x \in G - \{0\}$.

Proof. By considering $y = z = x \neq 0$ in (3.7), then we see that

$$\left|\frac{f(2x)}{4f(x)} - 1\right| \leq \varphi(x,x,x). \tag{3.9}$$

Let us define a function F_n by

$$F_n(x) = \frac{f(2^n x)}{4^n}, \quad n \in \mathbb{N}.$$

Then we have

$$\frac{F_{n+1}(x)}{F_n(x)} = \frac{f(2^{n+1}x)}{4f(2^n x)}.$$

From this and (3.9), we can show that

$$\left|\frac{F_{n+1}(x)}{F_n(x)} - 1\right| \leq \varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in G - \{0\}$, which means that

$$1 - \varphi(2^n x, 2^n x, 2^n x) \leq \frac{F_{n+1}(x)}{F_n(x)} \leq 1 + \varphi(2^n x, 2^n x, 2^n x).$$

Hence we can deduce that

$$\prod_{k=m}^{n-1}[1 - \varphi(2^k x, 2^k x, 2^k x)] \leq \frac{F_n(x)}{F_m(x)} \leq \prod_{k=m}^{n-1}[1 + \varphi(2^k x, 2^k x, 2^k x)]. \tag{3.10}$$

Now taking the logarithm in (3.10), we obtain

$$\sum_{k=m}^{n-1} \log[1 - \varphi(2^k x, 2^k x, 2^k x)] \le \log F_n(x) - \log F_m(x) \le \sum_{k=m}^{n-1} \log[1 + \varphi(2^k x, 2^k x, 2^k x)] \tag{3.11}$$

for $n > m$.

From hypothesis, the series

$$\sum_{n=0}^{\infty} \log[1 - \varphi(2^n x, 2^n y, 2^n z)] \text{ and } \sum_{n=0}^{\infty} \log[1 + \varphi(2^n x, 2^n y, 2^n z)]$$

converge for all $x, y, z \in G - \{0\}$. Then it follows from (3.11) that $\{\log F_n(x)\}$ is a Cauchy sequence for all $x \in G - \{0\}$.

Here we can define a function Q by

$$Q(x) = \exp[\lim_{n\to\infty} \log F_n(x)]$$

for all $x \in G - \{0\}$. Then we get

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \quad x \neq 0,$$

and we note that $Q(0) = 0$.

To show that Q is a quadratic, take $x := 2^n x$, $y := 2^n y$, $z := 2^n z$ in (3.7) and let $n \to \infty$. Then we find that

$$\left| \frac{Q(x+y) + Q(x-y) + Q(y+z) + Q(y-z) + Q(z+x) + Q(z-x)}{4Q(x) + 4Q(y) + 4Q(z)} - 1 \right| = 0$$

for all $x, y, z \in G - \{0\}$, that is, Q satisfies the functional equation (1.1). Thus the lemma 2.1 implies that Q is a quadratic.

Now we prove that the inequality (3.8) holds. Taking the limit as $n \to \infty$ in (3.10), we conclude that

$$\prod_{k=m}^{\infty} [1 - \varphi(2^k x, 2^k x, 2^k x)] \le \frac{Q(x)}{F_m(x)} \le \prod_{k=m}^{\infty} [1 + \varphi(2^k x, 2^k x, 2^k x)].$$

By setting $m = 0$ in the above relation, then we lead to

$$\varphi_0(x, x, x) \leq \frac{Q(x)}{f(x)} \leq \varphi_1(x, x, x), \ x \neq 0.$$

It remains to show that Q is unique. Suppose that Q_1 is a quadratic function such that

$$\varphi_0(x, x, x) \leq \frac{Q_1(x)}{f(x)} \leq \varphi_1(x, x, x), \ x \neq 0$$

for all $x, y \in G - \{0\}$. Then we have

$$\varphi_0(2^n x, 2^n x, 2^n x) \leq \frac{4^n Q_1(x)}{f(2^n x)} \leq \varphi_1(2^n x, 2^n x, 2^n x).$$

Letting $n \to \infty$ yields

$$Q_1(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \ x \neq 0.$$

Hence it follows that $Q_1(x) = Q(x), \ x \neq 0$. According to the lemma 2.1, we obtain that $Q_1(0) = Q(0)$, which completes the proof of the theorem. □

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References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.

[2] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. 80 (1980), 411-416.

[3] I.-S. Chang and Y.-S. Jung, *Stability of a functional equation deriving from cubic and quadratic functions*, J. Math. Anal. Appl. 283(2) (2003), 491-500.

[4] I.-S. Chang, K.-W. Jun and Y.-S. Jung, *The modified Hyers-Ulam-Rassias stability of a cubic type functional equation*, Math. Ineq. Appl. 8(4) (2005), 675-683.

[5] I.-S. Chang, E. H. Lee and H.-M. Kim, *On Hyers-Ulam-Rassias stability of a quadratic functional equation*, Math. Ineq. Appl. 6(1) (2003), 87-95.

[6] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., 27 (1984), 76-86.

[7] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, 62 (1992), 59-64.

[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431-436.

[9] ______, *On the Hyers-Ulam-Rassias stability of the quadractic mappings*, Nonlinear Funct. Anal. Appl., 9(3) (2004), 415-428.

[10] R. Ger, *Superstability is not natural*, Rocznik Naukowo-dydaktyczny WSP w Krakowie, Prace Math., 159 (1993), 109-123.

[11] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci., 27 (1941), 222-224.

[12] D. H. Hyers, G. Isac and Th. M. Rassias, *"Stability of Functional Equations in Several Variables"*, Birkhäuser, Basel, 1998.

[13] K.-W. Jun and H.-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl., 274(2) (2002), 867-878.

[14] K.-W. Jun, H.-M. Kim and I.-S. Chang, *On the Hyers-Ulam stability of an Euler-Lagrange type functional equation*, J. Comp. Anal. Appl., 7(1) (2005), 21-33.

[15] S.-M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., 222 (1998), 126-137.

[16] ______, *On the stability of gamma functional equation*, Results. Math., 33 (1998), 306-309.

[17] Y.-S. Jung and I.-S. Chang, *The stability of a cubic type functional equation with the fixed point alternative*, J. Math. Anal. Appl., 306(2) (2005), 752-760.

[18] Y.-S. Jung and K.-H. Park, *On the stability of the functional equation* $f(x+y+xy) = f(x)+f(y)+xf(y)+yf(x)$, J. Math. Anal. Appl., 274(2) (2002), 659-666.

[19] Pl. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., 27 (1995), 368-372.

[20] B. Margolis and J. B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., 126, 74 (1968), 305-309.

[21] V. Radu, *The fixed point alternative and the stability of functional equations*, Seminar on Fixed Point Theory Cluj-Napoca, (to appear in vol. IV on 2003).

[22] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978), 297-300.

[23] ______, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., 251 (2000), 264-284.

[24] ______, *On the stability of functional equations and a problem of Ulam*, Acta Math. Appl., 62 (2000), 23-130.

[25] Th. M. Rassias (Ed.), *"Functional Equations and inequalities"*, Kluwer Academic, Dordrecht/ Boston/ London, 2000.

[26] Th. M. Rassias and J. Tabor, *What is left of Hyers-Ulam stability?*, Journal of Natural Geometry, 1 (1992), 65-69.

[27] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano, 53 (1983), 113-129.

[28] T. Trif, *On the stability of a general gamma-type functional equation*, Pulb. Math. Debrecen, 60/1-2 (2002), 47-61.

[29] S. M. Ulam, *Problems in Modern Mathematics*, (1960) Chap. VI, Science ed., Wiley, New York.

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