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APPLICATIONS OF CRITICAL POINT THEOREMS TO NONLINEAR BEAM PROBLEMS

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Abstract. Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. We consider nonlinear beam equations, $Lu + bu^+ = f$, in H, where H is the Hilbert space spanned by eigenfunctions of L. We reveal the existence of multiple solutions of the nonlinear beam problems by critical point theorems.

1. 1. Introduction

In this paper we investigate the existence of multiple solutions of the nonlinear beam equation in an interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$u_{tt} + u_{xxxx} + bu^+ = f(x,t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{1.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{1.2}$$

$$u \; is \; \pi - periodic \; in \; t \; and \; even \; in \; x \; and \; t, \tag{1.3}$$

where the nonlinearity $-(bu^+)$ crosses an eigenvalue λ_{10}. This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

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Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

with (1.2) and (1.3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding eigenfunctions $\phi_{mn} (m, n \geq 0)$ given by

$$\phi_{mn} = \cos 2mt \cos(2n+1)x$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17$$

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H the Hilbert space defined by

$$H = \left\{u \in L^2(Q) : \quad u \text{ is even in } x \text{ and } t \quad \right\}.$$

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H. Hence equation (1.1) with (1.2) and (1.3) is equivalent to

$$Lu + bu^+ = f \quad \text{in} \quad H. \tag{1.4}$$

In [6], the authors showed by degree theory that equation (1.4) with constant load $1+\epsilon h$ (h is bounded) has at least two solutions. In [1], the authors showed by a variational reduction method that equation (1.4) with constant load $1+\epsilon h$ (h is bounded) has at least three solutions when condition (1.3) is replaced by

$$u \textit{ is } \pi - \textit{periodic in } t \textit{ and even in } x. \tag{1.3'}$$

In section 2, we show by critical points theory that equation (1.4) with $f = s\phi_{00} (s > 0)$ has a positive solution and at least two change sign solutions. In section 3, we investigate the existence of nontrivial solutions of the nonlinear beam equation.

2. An Application of Critical Point Theory

In this section we investigate the multiplicity of solutions of a nonlinear beam equation

$$Lu + bu^+ = s\phi_{00} \quad \text{in} \quad H, \tag{2.1}$$

where we $3 < b < 15$ and s is real. To show the existence of solutions of equation (2.1) we use critical point theory.

In this section we suppose that the eigenfuction $\phi_{mn}(m, n \geq 0)$ are normalized with respect to L^2-norm. Now we define a subspace H_0 of the Hilbert space H as follows

$$H_0 = \left\{ u \in H : u = \sum h_{mn}\phi_{mn}, \ \sum |\lambda_{mn}| h_{mn}^2 < \infty \right\}$$

with norm

$$|||u||| = \left[\sum |\lambda_{mn}| h_{mn}^2 \right]^{\frac{1}{2}}.$$

Then this normed space is complete and we have the following simple properties.

Proposition 2.1. *(i) $Lu \in H_0$ implies $u \in H_0$.*
(ii) $|||u||| \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u.
(iii) $\|u\| = 0$ iff $|||u||| = 0$.

Proof. (i) Let

$$Lu = \sum \lambda_{mn} h_{mn} \phi_{mn} + \sum \lambda_{mn} \tilde{h}_{mn} \psi_{mn}.$$

Then

$$\infty > |||Lu|||^2 = \sum |\lambda_{mn}|(\lambda_{mn}^2 h_{mn}^2 + \lambda_{mn}^2 \tilde{h}_{mn}^2) \geq \sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2) = |||u|||^2,$$

because $|\lambda_{mn}| \geq 1$ for all m, n. (ii) and (iii) are trivial.

LEMMA 2.1. *Let δ be not an eigenvalue of L. Let $u \in H$. Then $(L+\delta)^{-1}u \in H_0$.*

Proof. Suppose that δ be not an eigenvalue of L and finite. We recall that

$$\lambda_{mn} = (2n+1)^4 - 4m^2 = (4n^2+4n+1)^2 - (2m)^2.$$

For a fixed integer n, we define

$$\begin{aligned} \lambda_n^+ &= \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 8n^2 + 8n + 1, \\ \lambda_n^- &= \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3. \end{aligned}$$

When $n \longrightarrow \infty$, $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$. Hence we know that the number of $\{\lambda_{mn} : |\lambda_{mn}| < |\delta|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum h_{mn}\phi_{mn}.$$

Then

$$(L+\delta)^{-1}u = \sum \frac{1}{\lambda_{mn}+\delta} h_{mn}\phi_{mn}.$$

Hence we have

$$\begin{aligned} |||(L+\delta)^{-1}u|||^2 &= \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}+\delta)^2} h_{mn}^2 \\ &\leq C \sum h_{mn}^2 \end{aligned}$$

for some C, which means that

$$|||(L+\delta)^{-1}u||| \leq C_1 \|u\|, \quad C_1 = \sqrt{C}.$$

∎

With the above Lemma 1.1, we can obtain the following proposition.

PROPOSITION 2.2. *Let $w(x,t) \in H$ and δ be not an eigenvalue of L. Then all solutions in H of*

$$Lu + \delta u^+ = w(x,t) \quad in \quad H$$

belong to H_0.

With aid of Proposition 2.2 it is enough that we investigate the existence of solutions of (2.1) in the subspace H_0 of H

$$Lu + bu^+ = s\phi_{00} \ in \ H_0. \tag{2.2}$$

Now we define the functional on H_0

$$I_b(u) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) + \frac{b}{2}|u^+|^2 - s\phi_{00}u]dtdx. \tag{2.3}$$

Then the solutions of (2.2) coincide with the critical points of I_b. First we prove the continuity and Fréchet differentiability of I_b.

PROPOSITION 2.3. *The functional I_b is continuous in H_0 and Fréchet differentiable in H_0.*

For the proof we refer [1].

Next we shall use a variational reduction method to apply the mountain pass theorem.

Let $P : H_0 \longrightarrow V$ denote the orthogonal projection of H_0 onto V and $I - P : H_0 \longrightarrow W$ denote that of H onto W, where V is spanned by eigenfunctions ϕ_{00}, ϕ_{10} and W is the orthogonal compliment of V.

LEMMA 2.2. *Let $3 < b < 15$, $f \in V$. Let $v \in V$ be given. Then there exists a unique solution $z \in W$ of the equation*

$$Lz + (I - P)[b(v + z)^+ - f] = 0 \quad \text{in} \quad W. \tag{2.4}$$

If $z = \theta(v)$, then θ is continuous on V and we have $DI_b(v+\theta(v))(w) = 0$ for all $w \in W$. In particular $\theta(v)$ satisfies a uniform Lipschitz in v with respect to the L^2 norm (also the norm $|||\cdot|||$). If $\tilde{I}_b : V \longrightarrow R$ is defined by $\tilde{I}_b(v) = I_b(v + \theta(v))$, then $\tilde{I}_b$ has a continuous Fréchet derivative $D\tilde{I}_b$ with respect to v and

$$D\tilde{I}_b(v)(h) = DI_b(v + \theta(v))(h) \quad \text{for all} \quad h \in V.$$

If v_0 is a critical point of $\tilde{I}_b$, then $v_0 + \theta(v_0)$ is a solution of (2.2) and conversely every solution of (2.2) is of this form.

Proof. Let $3 < b < 15, \delta = \frac{1}{2}b$, and $g(\xi) = b\xi^+$. If $g_1(\xi) = g(\xi) - \delta\xi$, then equation (2.4) is equivalent to

$$z = (L + \delta)^{-1}(I - P)[-g_1(v + z)]. \tag{2.5}$$

By Lemma 1.1 the right hand side of (2.5) defines, for fixed $v \in V$, a Lipschitz mapping of $(I - P)H$ into itself with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I - P)H$ (also $z \in (I - P)H_0$) which satisfies (2.5). If $\theta(v)$ denotes the unique $z \in (I - P)H$ which solves (2.5) then θ is continuous (with respect to the norm $\|\cdot\|$) in V. In fact, if $z_1 = \theta(v_1)$

and $z_2 = \theta(v_2)$, then we have

$$\begin{aligned}
&\|z_1 - z_2\| \\
= \ & \|(L+\delta)^{-1}(I-P)(-g_1(v_1+z_1) + g_1(v_2+z_2))\| \\
= \ & \|(L+\delta)^{-1}(I-P)([(\delta-b)(v_1+z_1)^+ - \delta(v_1+z_1)^-] \\
& \quad -[(\delta-b)(v_2+z_2)^+ - \delta(v_2+z_2)^-])\| \\
\leq \ & \gamma\|(v_1+z_1) - (v_2+z_2)\| \\
\leq \ & \gamma(\|v_1 - v_2\| - \|z_1 - z_2\|).
\end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq c\|v_1 - v_2\|, \qquad c = \frac{\gamma}{1-\gamma},$$

which shows that $\theta(v)$ satisfies a uniform Lipschitz condition in v with respect to the L^2 norm. With the above inequality we have

$$\begin{aligned}
&|||z_1 - z_2||| \\
= \ & |||(L+\delta)^{-1}(I-P)[(-b(z_1+v_1)^+ + \delta(z_1+v_1)) \\
& \quad -(-b_1(z_2+v_2)^+ + \delta(z_2+v_2))]||| \\
\leq \ & \frac{\sqrt{17}}{14}\|(I-P)[(-b(z_1+v_1)^+ + \delta(z_1+v_1)) - (-b(z_2+v_2)^+ + \delta(z_2+v_2) \\
\leq \ & \frac{4\sqrt{17}}{4}(\|z_1 - z_2\| + \|v_1 - v_2\|) \\
\leq \ & \frac{4\sqrt{17}}{4}(c+1)|||v_1 - v_2|||.
\end{aligned}$$

This shows that $\theta(v)$ satisfies a uniform Lipschitz condition in v with respect to the norm $|||\cdot|||$.

Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (2.4) we see that

$$\int_Q (-z_t w_t + z_{xx} w_{xx} + b(v+z)^+ - fw)\,dt\,dx = 0.$$

Since

$$\int_Q v_t w_t = 0 \quad \text{and} \quad \int_Q v_{xx} w_{xx} = 0,$$

we have

$$DI_b(v + \theta(v))(w) = 0 \quad \text{for} \quad w \in W. \tag{2.6}$$

Let W_1 be the subspace of H_0 which is the closure of the span of functions ϕ_{mn} and whose eigenvalues are $\lambda_{mn} \leq -15$ and let W_2 be the subspace of H_0 which is the closure of the span of functions ϕ_{mn} and whose eigenvalues are $\lambda_{mn} \geq 17$. Let $v \in V$ and consider the function h : $W_1 \times W_2 \longrightarrow R$ defined by

$$h(w_1, w_2) = I_b(v + w_1 + w_2).$$

The function h has continuous partial Fréchet derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_i h(w_1, w_2)(y_i) = DI_b(v + w_1 + w_2)(y_i)$$

for $y_i \in W_i$, $\quad i = 1, 2$. Therefore, if we set $\theta(v) = \theta_1(v) + \theta_2(v)$ with $\theta_i(v) \in W_i$ for $i = 1, 2$, it follows from (2.6) that

$$D_i h(\theta_1(v), \theta_2(v)) = 0, \quad i = 1, 2. \tag{2.7}$$

If w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$\begin{aligned} & [D_2 h(w_1, w_2) - D_2 h(w_1, y_2)](w_2 - y_2) \\ = \; & (DI_b(v + w_1 + w_2) - DI_b(v + w_1 + y_2))(w_2 - y_2) \\ = \; & \int_Q [-|(w_2 - y_2)_t|^2 + |(w_2 - y_2)^2_{xx} + b((v + w_1 + w_2)^+ \\ & -(v + w_1 + y_2)^+)(w_2 - y_2)] dtdx. \end{aligned}$$

Since $(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) \geq 0$ for arbitrary ξ_1 and ξ_2, and

$$\int_Q [-|(w_2 - y_2)_t|^2 + (w_2 - y_2)^2_{xx}] dtdx = |||w_2 - y_2|||^2,$$

it follows that

$$(D_2h(w_1, w_2) - D_2h(w_1, y_2))(w_2 - y_2) \geq |||w_2 - y_2|||^2.$$

Therefore, h is strictly convex with respect to the second variable. Similarly, using the fact that $(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) \leq \frac{b}{2}(\xi_2 - \xi_1)^2$, we see that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned}&(D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1)\\&\leq -|||w_1 - y_1|||^2 + b\|w_1 - y_1\|^2 \leq (-1 + \frac{b}{30})|||w_1 - y_1|||^2,\end{aligned}$$

where $-30 + b < 0$. Therefore, h is strictly concave with respect to the first variable. From (2.7) it follows that

$$I_b(v + \theta_1(v) + \theta_2(v)) \leq I_b(v + \theta_1(v) + y_2) \tag{2.8}$$

for $y_2 \in W_2$ with equality if and only if $y_2 = \theta_2(v)$ and

$$I_b(v + \theta_1(v) + \theta_2(v)) \geq I_b(v + y_1 + \theta_2(v)) \tag{2.9}$$

for $y_1 \in W_1$ with equality iff $y_1 = \theta_1(v)$.

Since h is strictly concave (convex) with respect to its first (second) variable, Theorem 2.3 of [1] implies that $\tilde{I}_b$ is C^1 with respect to v and

$$D\tilde{I}_b(v)(h) = DI_b(v + \theta(v))(h), \quad h \in V. \tag{2.10}$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{I}_b(v_0) = 0$. From (2.10) it follows that $DI_b(v_0 + \theta(v_0))(v) = 0$ for all $v \in V$. Since (2.6) holds for all $w \in W$ and H_0 is the direct sum of V and W, it follows that $DI_b(v_0 + \theta(v_0)) = 0$ in H_0. Therefore, $u = v_0 + \theta(v_0)$ is a solution of (2.2).

Conversely our reasoning shows that if u is a solution of (1.4) and $v = Pu$, then $D\tilde{I}_b(v) = 0$ in V. ∎

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

LEMMA 2.3. *Let $3 < b < 15$. Then we have: (i) Let $s = 1 + b$. Then equation (2.2) has a positive v_p and there exists a small open neighborhood B_p of v_p in C_1 such that in B_p, v_p is a strict local point of minimum of $\tilde{I}_b$. (ii) Let $s = -1$. Then equation (2.2) has a negative solution v_n and there exists a small open neighborhood B_n of v_n in C_3 such that in B_n, v_n is a saddle point of $\tilde{I}_b$.*

Proof. (i) Let $s = b + 1$ $(f = (b+1)\phi_{00})$. Then equation (2.2) has a positive solution $u_p = \phi_{00}$ which is of the form $u_p = v_p + \theta(v_p)$ (in this case $\theta(v_p) = 0$) and $I + \theta$, where I is an identity map on V, is continuous. Since v_p is in the interior, $IntC_1$, of C_1, there exists a small open neighborhood B_p of v_p in C_1. We note that $\theta(v) = 0$ in B_p.

Therefore, if $v = v_p + v' \in B_p$, then we have

$$\begin{aligned}
\tilde{I}_b(v) &= I_b(v_p + v') \\
&= \int_Q [\frac{1}{2}(-|(v_p+v')_t|^2 + |(v_p+v')_{xx}|^2) + \frac{b}{2}|(v_p+v')^+|^2 \\
&\quad -f(v_p+v')]dtdx \\
&= \int_Q [\frac{1}{2}(-|v'_t|^2 + |v'_{xx}|^2) + \frac{b}{2}v'^2 - fv']dtdx \\
&\quad + \int_Q [-(v_p)_t v'_t + (v_p)_{xx} v'_{xx} + b v_p v']dtdx \\
&\quad + \int_Q [\frac{1}{2}(-|(v_p)_t|^2 + |(v_p)_{xx}|^2) + \frac{b}{2}v_p^2 - fv_p]dtdx \\
&= \int_Q [\frac{1}{2}(-|v'_t|^2 + |v'_{xx}|^2) + \frac{b}{2}v'^2 - fv_p]dtdx \\
&\quad + \int_Q [-(v_p)_t v'_t + (v_p)_{xx} v'_{xx} + b v_p v']dtdx + C,
\end{aligned}$$

where

$$\begin{aligned}
C &= \int_Q [\frac{1}{2}(-|(v_p)_t|^2 + |(v_p)_{xx}|^2) + \frac{b}{2}v_p^2 - fv_p]dtdx \\
&= I_b(u_p) = \tilde{I}_b(v_p).
\end{aligned}$$

Let $v' = c_1\phi_{00} + c_2\phi_{10}$. Then for $v = v_p + v' \in B_p$, we have

$$\begin{aligned}
\tilde{I}_b(v) - \tilde{I}_b(v_p) &= \int_Q [\frac{1}{2}(-|v'_t|^2 + |v'_{xx}|^2) + \frac{b}{2}v'^2 - fv']dtdx \\
&\quad + \int_Q [-(v_p)_t v'_t + (v_p)_{xx} v'_{xx} + b v_p v']dtdx \\
&= \frac{1}{2}[(b+1)c_1^2 + (b-3)c_2^2]
\end{aligned}$$

Since $3 < b < 15$, it follows that v_p is a strict local point of minimum of $\tilde{I}_b$.

(ii) Let $s = 1$ ($f = -\phi_{00}$). Then equation (2.2) has a negative solution $u_n = -\phi_{00}$ which is of the form $u_n = v_n + \theta(v_n)$, where $\theta(v_n) = 0$ and $I + \theta$ is continuous in$V$. Since v_n is in the interior, $IntC_3$, of C_3, there exists a small open neighborhood B_n of v_n in C_3. We note that $\theta(v) = 0$ in B_n. Therefore, if $v = v_n + v' \in B_n$, then we have

$$\begin{aligned}
\tilde{I}_b(v) &= I_b(v+v') \\
&= \int_Q [\frac{1}{2}(-|(v_n+v')_t|^2 + |(v_n+v')_{xx}|^2) - f(v_n+v')]dtdx \\
&= \int_Q [\frac{1}{2}(-|v_t'|^2 + |v_{xx}'|^2) - fv']dtdx \\
&\quad + \int_Q [-(v_n)_t v_t' + (v_n)_{xx} v_{xx}']dtdx \\
&\quad + \int_Q [\frac{1}{2}(-|(v_n)_t|^2 + |(v_n)_{xx}|^2) - fv_n]dtdx \\
&= \int_Q [\frac{1}{2}(-|v_t'|^2 + |v_{xx}'|^2) - fv']dtdx \\
&\quad + \int_Q [-(v_n)_t v_t' + (v_n)_{xx} v_{xx}']dtdx + C,
\end{aligned}$$

where

$$\begin{aligned}
C &= \int_Q [\frac{1}{2}(-|(v_t)_t|^2 + |(v_n)_{xx}|^2) - fv_n]dtdx \\
&= I_b(u_n) = \tilde{I}_b(v_n).
\end{aligned}$$

Let $v' = c_1\phi_{00} + c_2\phi_{10}$. Then for $v = v_n + v' \in B_n$ we have

$$\begin{aligned}
\tilde{I}_b(v) - \tilde{I}_b(v_n) &= \int_Q [\frac{1}{2}(-|v_t'|^2 + |v_{xx}'|^2)]dtdx \\
&\quad + \int_Q [-(v_n)_t v_t' + (v_n)_{xx} v_{xx}' + bv_n v' + fv']dtdx \\
&= \frac{1}{2}[c_1^2 - 3c_2^2]
\end{aligned}$$

The above equation implies that v_n is a saddle point of $\tilde{I}_b$. ∎

LEMMA 2.4. *Let $f \in V$. For $-1 < b < 15$ the functional $\tilde{I}_b$, defined on V, satisfies the Palais-Smale condition : Any sequence $\{v_n\} \subset V$*

for which $\tilde{I}_b(v_n)$ is bounded and $D\tilde{I}_b(v_n) \longrightarrow 0$ possesses a convergent subsequence.

Proof. If $\tilde{I}_b(v_n)$ is bounded and $D\tilde{I}_b(v_n) \longrightarrow 0$ in V for any sequence $\{v_n\} \subset V$, then since V is 2 dimensional and spanned by smooth functions we have with $u_n = v_n + \theta(v_n)$

$$Lu_n + bu_n^+ = DI_b(u_n) + f \quad \text{in} \quad H. \tag{2.11}$$

Assuming [P.S.] condition does not hold, that is $|||v_n||| \longrightarrow +\infty$, we see that $|||u_n||| \longrightarrow +\infty$. Dividing by $|||u_n|||$ and taking $w_n = |||u_n|||^{-1}u_n$ we have

$$Lw_n + bw_n^+ = |||u_n|||^{-1}(DI_b(u_n) + f). \tag{2.12}$$

Since $DI_b(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $|||u_n||| \longrightarrow +\infty$, the right hand side of (2.12) converges to 0 in H_0 as $n \longrightarrow \infty$. Moreover (2.12) shows that $|||Lw_n|||$ is bounded. Since L^{-1} is a compact operator, passing to a subsequence we get : $w_n \longrightarrow w_0$ in H_0. Since $|||w_n||| = 1$, it follows that $|||w_0||| = 1$. Taking the limit of both sides of (2.12), we find

$$Lw_0 + bw_0^+ = 0$$

with $|||w_0||| \neq 0$. This contradicts to the fact that for $-1 < b < 15$ the following equation

$$Lu + bu^+ = 0 \quad \text{in} \quad H_0$$

has only the trivial solution (cf. [1]). ∎

Let $3 < b < 15$ and $s = b+1$. We consider the functional $\tilde{I}_b(v) - \tilde{I}_b(v_p)$. Let $v' = c_1\phi_{00} + c_2\phi_{10}$. If $v = v_p + v' \in C_1$, then we have

$$\begin{aligned}\tilde{I}_b(v) - \tilde{I}_b(v_p) &= \int_Q [\frac{1}{2}(-|v'_t|^2 + |v'_{xx}|^2) + \frac{b}{2}v'^2 - (b+1)\phi_{00}v']dtdx \\ &\quad + \int_Q [-(v_p)_t v'_t + (v_p)_{xx}v'_{xx} + bv_p v']dtdx \\ 3 &= \frac{1}{2}[(b+1)c_1^2 + (b-3)c_2^2]\end{aligned}$$

If $v = v_p + v' \in C_3$, then we have

$$\begin{aligned}\tilde{I}_b(v) - \tilde{I}_b(v_p) &= \int_Q [\frac{1}{2}(v_p + v')L(v_p + v') - (b+1)\phi_{00}(v_p + v')]dtdx \\ &\quad - \int_Q [\frac{1}{2}v_p L v_p + \frac{b}{2}|v_p^+|^2 - (b+1)\phi_{00}v_p]dtdx \\ &= \frac{1}{2}(c_1^2 - 3c_2^2) - b(c_1 + \frac{1}{2})\end{aligned}$$

LEMMA 2.5. *Let $3 < b < 15$ and $s = b+1$. Then we have $\tilde{I}_b(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$ along a boundary ray of C_3.*

Proof. Let $3 < b < 15$. Let $v' = c_1\phi_{00} + c_2\phi_{10}$. If $v = v_p + v' \in C_3$, then we have

$$\tilde{I}_b(v) = \frac{1}{2}(c_1^2 - 3c_2^2) - b(c_1 - \frac{1}{2}).$$

We note that $v_p + v' \in \partial C_3$ if and only if $c_2 = \pm(c_1 + 1), c_1 \leq -1$. Hence if $v \in \partial C_3$, then we have

$$\tilde{I}_b(v) = \frac{1}{2}[c_1^2 - 3(c_1 + 1)^2] - b(c_1 - \frac{1}{2}),$$

where $\tilde{I}_b(v) \longrightarrow -\infty$ as $c_1 \longrightarrow -\infty$. This completes the lemma. ∎

The following theorem is the main theorem of this section.

THEOREM 2.1. *Let $3 < b < 15$ and $s = b + 1$. Then $\tilde{I}_b(v)$ has a critical point in* Int C_1 *and at least one critical point in* $\text{Int}V \backslash (C_1 \cup C_3)$. *Therefore equation (2.1) has a positive solution and at least one sign changing solution.*

Proof. By Lemma 2.3, there exists a small open neighborhood B_p of v_p in C_1 such that in B_p, v_p is a strict local point of minimum of $\tilde{I}_b$, where v_p is a critical point of $\tilde{I}_b$. Also $\tilde{I}_b(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$ along a boundary ray of C_3 (Lemma 2.5) and $\tilde{I}_b \in C^1(V, R)$ satisfies Palais-Smale condition.

Since $\tilde{I}_b(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$ along a boundary ray of C_3, we can choose $v_0 \in \partial C_3$ such that $\tilde{I}_b(v_0) < \tilde{I}_b(v_p)$. Let Γ be the set of all paths in V joining v_p and v_0. We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{I}_b(v).$$

The fact that in B, v_p is a strict local point of minimum of $\tilde{I}_b$, the fact that $\tilde{I}_b(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$ along a boundary ray of C_3, the fact that $\tilde{I}_b$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem (cf. [2]) imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{I}_b(v)$$

is a critical value of $\tilde{I}_b$. When $3 < b < 15$ and $s = b + 1$, equation (2.1) has a unique positive solution v_p and no negative solution. Hence there exists a critical point v_3, sign changing solution, of $\tilde{I}_b$ such that

$$\tilde{I}_b(v_3) = c.$$

■

If u_0 is a solution of $Lu + bu^+ = s\phi_{00}$, in H_0, then $ku_0 (k \geq 0)$ is a solution of $Lu + bu^+ = ks\phi_{00}$, in H_0. Therefore we have the following.

THEOREM 2.2. *Let $3 < b < 15$ and $s > 0$. Then $\tilde{I}_b(v)$ has a critical point in* Int C_1 *and at least one critical point in* Int$V \backslash (C_1 \cup C_3)$. *Therefore equation (2.1) has a positive solution and at least one sign changing solution.*

3. An Application of Linking Theory

In this section we consider the following beam equation

$$u_{tt} + u_{xxxx} + bu^+ = f(x, t, u) \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{3.1}$$

with the conditions (1.2), (1.3), where $u^+ = \max\{u, 0\}$ and f is defined by

$$(1) \qquad f(x, t, s) = \begin{cases} |s|^{p-2}s, & s \geq 0 \\ |s|^{q-2}s, & s < 0 \end{cases}$$

where $p, q > 2$ and $p \neq q$.

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) | u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn} | m, n = 0, 1, 2, \cdots\}$ is an orthogonal base of H and H consists of the functions

$$u(x, t) = \sum_{m,n=0}^{\infty} a_{mn}\phi_{mn}(x, t)$$

with the norm given by

$$\|u\|^2 = \sum_{m,n=0}^{\infty} a_{mn}^2.$$

We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues, by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\cdots < \Lambda_1^- = -3 < \Lambda_1^+ = 1 < \Lambda_2^+ = 17 < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

Let $I_b : H \to R$ be defined by

$$I_b(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \int_\Omega F(Au)dx$$

where $A = R + S$ and $F(s) = \int_0^s f(x,t,\tau)d\tau$. Then it is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - Af(Au).$$

Following the idea of Hofer (see [2]) one can show that

PROPOSITION 3.1. *$I_b \in C^{1,1}(H,R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R+S)(u)$ is a weak solution of (P), that is,*

$$\int_\Omega (w(v_{tt} + v_{xxxx}) + b[w]^+ v) dxdt = \int_\Omega f(w) v dxdt \text{ for all smooth } v \in H.$$

The following theorem is the uniqueness result for problem (3.1).

PROPOSITION 3.2. *$b < -\Lambda_1^-$ and*

$$f(x,t,s) = \begin{cases} |s|^{p-2}s, & s \geq 0 \\ 0, & s \leq 0, \end{cases}$$

then problem (3.1) has only trivial solution.

Proof. Let $Lu = u_{tt} + u_{xxxx}$ and we rewrite (3.1) as

$$\begin{aligned} Lu - \Lambda_1^+ u &= f(x,t,u) - \Lambda_1^+ u - bu^+ \\ &= (u^+)^{p-1} - \Lambda_1^+ u - bu^+ \\ &= (u^+)^{p-1} - (\Lambda_1^+ + b)u^+ + \Lambda_1^+ u^-. \end{aligned}$$

Multiplying across by e_1^+ and integrating over Ω,

$$\begin{aligned} 0 &= < [L - \Lambda_1^+]u, e_1^+ > \\ &= \int_\Omega [(u^+)^{p-1} - (\Lambda_1^+ + b)u^+ + \Lambda_1^+ u^-] e_1^+ dx \geq 0, \end{aligned}$$

since the condition $b < -\Lambda_1^-$ imply that $-(\Lambda_1^+ + b)u^+ \geq 0$, $(u^+)^{p-1} \geq 0$, and $\Lambda_1^+ u^- \geq 0$ for all real valued function u. and $e_1^+(x) > 0$ for all $x \in \Omega$. Therefore the only possibility to hold (1) is that $u \equiv 0$. ∎

Remark. $b < -\Lambda_1^-$ and f is defined by equation (1), then problem (3.1) has no positive solutions.

In this section, we suppose $b > 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (3.1). Here we suppose that f is defined by equation (1).

In the following, we consider the following sequence of subspaces of $L^2(R^N)$:

$$H_n = (\oplus_{i=1}^{n} H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^{n} H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ.

LEMMA 3.1. *The functional I_b satisfies $(P.S.)^*_\gamma$ condition, with respect to (H_n), for all γ.*

Proof. Let (k_n) be any sequence in N with $k_n \to \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n, $I_b(u_n) \to \gamma$ and $\nabla(I_b)\mid_{H_{k_n}}(u_n) \to 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \to \infty$ and set $\hat{u_n} = u_n/t_n$. Up to a subsequence $\hat{u_n} \rightharpoonup \hat{u}$ in H for some $\hat{u}$ in H. Moreover

$$\begin{aligned} 0 &\leftarrow\ < \nabla(I_b)_{H_{k_n}}(u_n), \hat{u_n} > -\frac{2}{t_n} I_b(u_n) \\ &= \frac{2}{t_n}\int_\Omega F(Au_n)dx - \frac{1}{t_n}\int_\Omega f(Au_n)Au_n dx \\ &= \int_\Omega -\frac{p-2}{p}(t_n)^{p-1}[(A\hat{u_n})^+]^p + \frac{q+2}{q}(t_n)^{q-1}[(A\hat{u_n})^-]^q dx. \end{aligned}$$

Since $t_n \to \infty$, $(A\hat{u_n})^+ \to 0$ and $(A\hat{u_n})^- \to 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightharpoonup u$ for some $u \in H$. We know that

$$\nabla(I_b)_{H_{k_n}}(u_n) = P^+u_n - P^-u_n + bA(Au_n)^+ - Af(Au_n).$$

Since A is the compact operator, $P^+u_n - P^-u_n$ converges strongly, hence $u_n \to u$ strongly and $\nabla I_b(u) = 0$. ∎

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the Theorem via a linking argument.

First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^{\infty} H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+$$

LEMMA 3.2. *There exists R such that $\sup_{v \in Z_1 \oplus Z_2, \|v\|=R} I_b(v) \le 0$.*

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx.$$

Since

$$\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx = \int_\Omega \frac{b}{2}([Sv]^+)^2 - \frac{1}{p}([Sv]^+)^p - \frac{1}{q}([Sv]^-)^q dx,$$

there exists R such that $\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx \le 0$ for all $\|v\| = R$. Hence

$$I_b(v) \le -\frac{1}{2}\|v\|^2 \le 0$$

∎

LEMMA 3.3. *There exists ρ such that $\inf_{u \in Z_2 \oplus Z_3, \|u\|=\rho} I_b(u) > 0$.*

It is easy to prove the lemma (cf.[6])

DEFINITION 3.1. Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:

$$\begin{aligned}
B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\
S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\
\triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\
\Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}.
\end{aligned}$$

THEOREM 3.1. *If $\Lambda_i^- \leq -b$ then problem (3.1) has at least one nontrivial solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1 and Lemma 3.2, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)).$$

Moreover $(P.S.)^*_\gamma$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \leq I_b(u) \leq \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$.

■

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