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ANALYTIC FUNCTIONS OF k SINGLE INTEGRALS

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Abstract. In this paper we establish the existence theorem of the operator-valued function space integral $K_\lambda(F)$ for $\lambda \in \mathbb{C}^{\sim}_{+,\lambda_0}$ where F is of the form (1.1). Our result is a generalization for Johnson and Skoug result in [8]

1. Introduction and Preliminaries

In 1992, Kun Soo Chang and Kun Sik Ryu [3] proved the existence of an analytic operator valued function space integral as an $\mathcal{L}(L_p, L_{p'})$ $(1 < p < 2)$ for certain functionals involving some Borel measures. G.W.Johnson and D.L.Skoug [8] established the operator valued function space integral as an $\mathcal{L}(L_p, L_{p'})$ $(1 < p \leq 2)$ theory, for certain functionals involving Lebesgue measure.

In this paper we study the existence of the operator valued function space integral for the functionals F of the form

$$(1.1)F(x) = f\Big(\int_0^t \theta_1(s, x(s))\, d\eta_1(s), \cdots, \int_0^t \theta_k(s, x(s))\, d\eta_k(s)\Big)$$

where η_j is a measure on $(0, t)$, satisfied the conditions (1) and (2) of 5 in this section, $\theta_j \in L_{\alpha r;\eta}$, $j = 1, \cdots, k$ and

$$f(z_1, \cdots, z_k) = \sum_{n_1,\cdots,n_k=0}^{\infty} a_{n_1,\cdots,n_k} z_1^{n_1} \cdots z_k^{n_k}$$

is an entire function of k complex variables of growth (ρ, σ).

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Now we present some necessary notations and lemma from [3].

Notations and Definitions.

1. Let $\mathbb{C}_+^\sim$ denote the nonzero complex numbers with nonnegative real part. Let C_0 be the space of $\mathbb{R}^l$-valued continuous functions x on $[0,t]$ such that $x(0) = 0$ and let $C = C([0,t],\mathbb{R}^l)$ be the space of $\mathbb{R}^l$-valued continuous functions x on $[0,t]$. m_w will denote Wiener measure on C_0.

2. If $1 < p < 2$ is given, let α be in $(1,\infty)$ such that $\alpha = p/(2-p)$. In our theorems, l will be a positive integer restricted so that $l < 2\alpha$. For$1 < p < 2$, let r be a real number such that $2\alpha/(2\alpha - l) < r < \infty$. The number $l/2\alpha$ will occur often and so it is worthwhile to give a symbol for it ; $\delta = l/2\alpha$. Note that $0 < r'\delta < 1$ where r and r' are conjugate indices.

3. For $1 \le p < \infty$, $L_p(\mathbb{R}^l)$ be the space of Borel measurable $\mathbb{C}$-valued functions on $\mathbb{R}^l$ such that $|\psi|^p$ is integrable with respect to Lebesgue measure m_L on $\mathbb{R}^l$. Let $\mathcal{L}(L_p, L_{p'})$ be the space of bounded linear operators from $L_p(\mathbb{R}^l)$ into $L_{p'}(\mathbb{R}^l)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $L_\infty(\mathbb{R}^l)$ be the space of Borel measurable $\mathbb{C}$-valued functions which are essentially bounded.

4. Let $1 \le p \le 2$ be given. For λ in $\mathbb{C}_+^\sim$, ψ in $L_p(\mathbb{R}^l)$, ξ in $\mathbb{R}^l$ and a positive real number s, let

$$(C_{\lambda/s}\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{l/2} \int_{\mathbb{R}^l} \psi(u) \exp\left(\frac{-\lambda\|u-\xi\|^2}{2s}\right) dm_L(u).$$

If $p = 1$, $C_{\lambda/s}$ is in $\mathcal{L}(L_1, L_\infty)$ and $\|C_{\lambda/s}\| \le (\frac{|\lambda|}{2\pi s})^{l/2}$. And as a function of λ, $C_{\lambda/s}$ is analytic in $\mathbb{C}_+$ and weakly continuous in $\mathbb{C}_+^\sim$. If $1 < p \le 2$ from [2,8], $C_{\lambda/s}$ is in $\mathcal{L}(L_p, L_{p'})$ and $\|C_{\lambda/s}\| \le (\frac{|\lambda|}{2\pi s})^{\delta}$. And as a function of λ, $C_{\lambda/s}$ is analytic in $\mathbb{C}_+$ and strongly continuous in $\mathbb{C}_+^\sim$.

5. Let $t > 0$ be given. $M(0,t)$ denote the space of complex Borel measures η on the open interval $(0,t)$. Every measure η in $M(0,t)$ has a unique decomposition, $\eta = \mu + \nu$ into a continuous part μ and a discrete part $\nu = \sum_{p=1}^\infty \omega_p \delta_{\tau_p}$ where (ω_p) is a summable sequence in $\mathbb{C}$ and δ_{τ_p} is

the Dirac measure [9]. $M(0,t)^*$ will denote the subset of $M(0,t)$ which satisfies the following conditions;

1. If μ is the continuous part of η in $M(0,t)^*$, then the Radon-Nikodym derivative $d|\mu|/dm_L$ exists and is essentially bounded where m_L is the Lebesgue measure on $(0,t)$.
2. If $\nu=\sum_{p=1}^{\infty}\omega_p\delta_{\tau_p}$ is the discrete part of η in $M(0,t)^*$, then $\sum_{p=1}^{\infty}|\omega_p|\tau_p^{-r'\delta}$ converges.

6. For $1<p\leq 2$ and $\eta\in M(0,t)$, let $L_{\alpha r;\eta}([0,t]\times\mathbb{R}^l)=L_{\alpha r;\eta}$ be the space of all complex valued Borel measurable functional θ on $[0,t]\times\mathbb{R}^l$ such that

$$\|\theta\|_{\alpha r;\eta}=\{\int_{(0,t)}\|\theta(s,\cdot)\|_{\alpha}^{r}d|\eta|(s)\}^{1/r}<\infty.$$

7. Let $1<p\leq 2$ be given and θ be in $L_\alpha(\mathbb{R}^l)$. From Lemma 1.3 in [8], a function $M_\theta : L_{p'}(\mathbb{R}^l)\longrightarrow L_p(\mathbb{R}^l)$ defined by $M_\theta(f)=f\theta$, is in $\mathcal{L}(L_{p'},L_p)$ and $\|M_\theta\|\leq\|\theta\|_\alpha$. It will be convenient to let $\theta(s)$ denote $M_{\theta(s,\cdot)}$ for θ in $L_{\alpha r;\eta}$.

8. Let $0<k<1$ be given and m be in $\mathbb{N}$. For $a<s_1<s_2<\cdots<s_m<b$,

$$\int_a^b\int_a^{s_m}\cdots\int_a^{s_2}\{(s_1-a)(s_2-s_1)\cdots(b-s_m)\}^{-k}ds_1ds_2\cdots ds_m = \frac{(b-a)^{m-(m+1)k}\{\Gamma(1-k)\}^{m+1}}{\Gamma((m+1)(1-k))}$$

where Γ is the gamma function. Throughout this paper, this value is denoted by $E(a,b;m;k)$.

9. Let X,Y be two Banach spaces, $\mathcal{L}(X,Y)$ a space of bounded linear operators from X into Y and (Ω,m) be a measure space. Let $G:\Omega\to\mathcal{L}(X,Y)$ be a function such that for each $x\in X$, $\{G(s)\}(x)$ is Bochner integrable with respect to m. Then there exists a linear operator J from X into Y such that

$$J(x)=(B)\int_\Omega\{G(s)\}(x)dm(s)$$

for x in X where $(B)\int_\Omega \{G(s)\}(x)dm(s)$ refers to the Bochner integral. Here this linear operator J is denoted by $(BS)\int_\Omega G(s)dm(s)$ and it is called the Bochner integral in the strong operator sense. When $X = Y$, J is called the strong integral of G.

Definition 1.1. Let $1 \le p \le 2$ be given. Let F be a function from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0$, $\psi \in L_p(\mathbb{R}^l)$ and $\xi \in \mathbb{R}^l$, we consider the expression

$$(1.2)\quad (K_\lambda(F)\psi)(\xi) = \int_{C_0} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(t) + \xi)\, dm_w(x).$$

If for m_L-a.e. $\xi \in \mathbb{R}^l$, $(K_\lambda(F)\psi)(\xi)$ exists in $L_{p'}(\mathbb{R}^l)$ and if the correspondence $\psi \to (K_\lambda(F)\psi)$ gives an element of $\mathcal{L}(L_p, L_{p'})$, we say that the operator valued function space integral $K_\lambda(F)$ exists for λ. Suppose there exists $\lambda_0 (0 < \lambda_0 \le \infty)$ such that $K_\lambda(F)$ exists for all $0 < \lambda < \lambda_0$ and there exists an $\mathcal{L}(L_p, L_{p'})$-valued function which is analytic in $\mathbb{C}_{+,\lambda_0} = \mathbb{C}_+ \cap \{z \in \mathbb{C} | |z| < \lambda_0\}$ and agree with $K_\lambda(F)$ on $(0, \lambda_0)$, then this $\mathcal{L}(L_p, L_{p'})$-valued function is called the operator-valued function space integral of F associated with λ and in this case, we say that $K_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$. If $K_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$ and is a strongly continuous function in $\mathbb{C}^\sim_{+,\lambda_0} = \mathbb{C}^\sim_+ \cap \{z \in \mathbb{C} | |z| < \lambda_0\}$, we say that $K_\lambda(F)$ exists for $\lambda \in \mathbb{C}^\sim_{+,\lambda_0}$. When λ is pure imaginary, $K_\lambda(F)$ is called the analytic operator valued Feynman integral of F.

Lemma 1.2. *(Lemma 1.1 in [3]) Let $\eta \in M(0,t)^*$ and suppose that $\theta \in L_{\alpha r;\eta}$. Let*

$$F(y) = \int_{(0,t)} \theta(s, y(s))\, d\eta(s)$$

for any $y \in C$ for which the integral exists. Then for every $\lambda > 0$, $F(\lambda^{-1/2}x + \xi)$ is defined for $m_w \times m_L$-a.e. $(x, \xi) \in C_0 \times \mathbb{R}^l$.

2. Analytic operator valued function space integral

In this section, we consider the functions F of the form

$$(2.1) \qquad F(x) := \prod_{u=1}^{m} \int_{(0,t)} \theta_u(s, x(s))\, d\eta_u(s)$$

where $\eta_u \in M(0,t)^*$ and $\theta_u \in L_{\alpha r;\eta_u}$ for $u = 1, 2, \cdots, m$.

For every $\lambda > 0$, $F(\lambda^{-1/2}x + \xi)$ is defined for $m_w \times m_L$- a.e. $(x, \xi) \in C_0 \times \mathbb{R}^l$ by Lemma 1.2.

Theorem 2.1. *Let $1 < p < 2$. Let F be given by (2.1). Suppose that η_u are purely continuous($\eta_u = \mu_u$). Then $K_\lambda(F)$ exists for $\lambda \in \mathbb{C}_+^{\sim}$. Moreover, for all $\lambda \in \mathbb{C}_+^{\sim}$,*

$$(2.2)\, K_\lambda(F) = m!(BS) \int_{\Delta_m(\rho)} L(\lambda; s_1, \cdots, s_m) d \prod_{u=1}^{m} \mu_{\rho(u)}(s_{\rho(u)})$$

where ρ ranges through the group S_m of permutations of $\{1, 2, \cdots, m\}$ and $\Delta_m(\rho) = \{(s_1, \cdots, s_m) \in (0,t)^m | 0 < s_{\rho(1)} < s_{\rho(2)} < \cdots < s_{\rho(m)} < t\}$ and for $(s_1, \cdots, s_m) \in \Delta_m(\rho)$, $L(\lambda; s_1, \cdots, s_m) = C_{\lambda/s_{\rho(1)}} \circ \theta_{\rho(1)}(s_{\rho(1)}) \circ C_{\lambda/(s_{\rho(2)} - s_{\rho(1)})} \circ \theta_{\rho(2)}(s_{\rho(2)}) \circ \cdots \circ C_{\lambda/(t - s_{\rho(m)})}$. Moreover, for all $\lambda \in \mathbb{C}_+^{\sim}$,

(2.3)

$$\begin{aligned} \|K_\lambda(F)\| \ \leq \ & (m!)^{1/r'} \left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} \|g\|_r^m \Big(\prod_{u=1}^{m} ess \sup d|\mu_{\rho(u)}|/dm_L\Big)^{1/r'} \\ & \times E(0, t; m; r'\delta)^{1/r'} \end{aligned}$$

where $g : [0, t] \to \mathbb{R}$ is given by

$$(2.4) \qquad g(s) = \max\{\|\theta_1(s, \cdot)\|_\alpha, \cdots, \|\theta_m(s, \cdot)\|_\alpha\}$$

Proof. It can be proved by the similar method as in the proof of Theorem 2.1 in [8]. □

Theorem 2.2. *Let $1 < p < 2$ and let $\lambda_0 > 0$ be given. Let $F(x) = \sum_{k=0}^{\infty} a_k F_k(x)$ and $F_k(x) = \prod_{j=1}^{m_k} \int_0^t \theta_{k,j}(s, x(s)) d\eta_{k,j}(s)$ where $\eta_{k,j}$ are* *purely continuous. Suppose that* (a_n) *is a sequence of complex numbers such that for every* λ *in* $\mathbb{C}^{\sim}_{+,\lambda_0}$, $\sum_{k=0}^{\infty}|a_k|b_k(|\lambda|)<\infty$ *where*

(2.5)

$$b_k(|\lambda|) = (m_k!)^{1/r'}(\frac{|\lambda|}{2\pi})^{(m_k+1)\delta}\|g_k\|_r^{m_k}(\prod_{j=1}^{m_k} ess\sup d|\eta_{k,j}|/dm_L)^{1/r'} \times E(0,t;m_k;r'\delta)^{1/r'}$$

and

$$g_k(s)=\max\{\|\theta_{k,1}(s,\cdot)\|_\alpha,\cdots,\|\theta_{k,m_k}(s,\cdot)\|_\alpha\}.$$

Then for every λ *in* $(0,\lambda_0)$, *the series* $\sum_{k=0}^{\infty}a_kF_k(\lambda^{-1/2}x+\xi)$ *converges absolutely for a.e.* $(x,\xi)\in C_0\times\mathbb{R}^l$. *Also Then* $K_\lambda(F)$ *exists for* $\lambda\in\mathbb{C}^{\sim}_{+,\lambda_0}$ *and* $K_\lambda(F)=\sum_{k=0}^{\infty}a_kK_\lambda(F_k)$. *The series* $\sum_{k=0}^{\infty}a_kK_\lambda(F_k)$ *converges in operator norm and* $\|K_\lambda(F)\|\le\sum_{k=0}^{\infty}|a_k|b_k(|\lambda|)$.

Proof. It can be proved by the same method as in the proof of Theorem 2.5 in [3] and Theorem 3.1 in [8].

□

3. Analytic functions of k single integrals

In this section, we study the existence of the operator valued function space integral for the functionals F of the form

$$(3.1) F(x)=f\Big(\int_0^t\theta_1(s,x(s))\,d\eta_1(s),\cdots,\int_0^t\theta_k(s,x(s))\,d\eta_k(s)\Big)$$

where η_j are continuous measures, $\theta_j\in L_{\alpha r;\eta}$, $j=1,\cdots,k$ and

$$(3.2)\qquad f(z_1,\cdots,z_k)=\sum_{n_1,\cdots,n_k=0}^{\infty}a_{n_1,\cdots,n_k}z_1^{n_1}\cdots z_k^{n_k}$$

is an entire function of k complex variables of growth (ρ,σ).

Let f be given by (3.2). Let the domain D be the polycylinder $D=D(R_1,\cdots,R_k)=\{(z_1,\cdots,z_k):|z_j|<R_j<\infty, j=1,\cdots,k\}$ and let $D_R=D_R(R_1,\cdots,R_k)=\{(z_1,\cdots,z_k):(\frac{z_1}{R},\cdots,\frac{z_k}{R})\in D\}$.

Let $M_f(R) = \sup_{D_R} |f(z_1, \cdots, z_k)|$.

Definition 3.1. The entire function f is of order ρ if

$$\rho = \overline{\lim_{R\to\infty}} \frac{\ln\ln M_f(R)}{\ln R} \quad (0 \le \rho \le \infty).$$

The entire function f of positive order ρ is of type σ if

$$\sigma = \overline{\lim_{R\to\infty}} \frac{\ln M_f(R)}{R^\rho} \quad (0 \le \sigma \le \infty).$$

Theorem 3.2. *The equations*

$$\rho = \overline{\lim_{n_1+\cdots+n_k\to\infty}} \left\{ \frac{(n_1 + \cdots + n_k)\ln(n_1 + \cdots + n_k)}{-\ln|a_{n_1,\cdots,n_k}|} \right\}$$

and

$$(e\rho\sigma)^{1/\rho} = \overline{\lim_{n_1+\cdots+n_k\to\infty}} \{(n_1 + \cdots + n_k)^{1/\rho}[|a_{n_1,\cdots,n_k}|R_1^{n_1}\cdots R_k^{n_k}]^{1/(n_1+\cdots+n_k)}\}$$

are valid.[4]

Theorem 3.3. *Let $1 < p < 2$. Let $F(x)$ be of the form (3.1) where η_j are continuous measures, $\theta_j \in L_{\alpha r;\eta}$, $j = 1, \cdots, k$. Let f be given by (3.2) an entire function of growth $(\frac{2\alpha}{l}, \sigma)$ where $\sigma = \sigma_D < \infty$*

1. *growth $(\frac{2\alpha}{l}, 0)$. In this case, $K_\lambda(F)$ exists for $\lambda \in \mathbb{C}_+^\sim$.*
2. *order $f = \frac{2\alpha}{l}$, type $f = \sigma = \sigma_{D(R_1,\cdots,R_k)} \in (0, \infty)$, $R_1, \cdots, R_k$ any k positive numbers. In this case, $K_\lambda(F)$ exists for $\lambda \in \mathbb{C}_{+,\lambda_0}^\sim$ where*

$$(3.3)\quad \lambda_0^\delta = \frac{(2\pi\delta)^\delta \min\{R_1, \cdots, R_k\}}{\sigma^\delta \|g\|_r (\Gamma(1 - r'\delta))^{\frac{1}{r'}} (ess\sup \frac{d\eta}{dm_L})^{\frac{1}{r'}}} \left(\frac{1 - r'\delta}{t}\right)^{\frac{1-r'\delta}{r'}}$$

and where $g(s) = \max\{\|\theta_1(s, \cdot)\|_\alpha, \cdots, \|\theta_k(s, \cdot)\|_\alpha\}$, $ess\sup \frac{d\eta}{dm_L} = \max\{ess\sup \frac{d\eta_1}{dm_L}, \cdots, ess\sup \frac{d\eta_k}{dm_L}\}$.

Proof. Let

$$F_{n_1,\cdots,n_k}(x) = a_{n_1,\cdots,n_k}\left[\int_{(0,t)} \theta_1(s, x(s))\, d\eta_1(s)\right]^{n_1} \cdots \left[\int_{(0,t)} \theta_k(s, x(s))\, d\eta_k(s)\right]^{n_k}$$

Let $N = n_1 + \cdots + n_k$. From the essentially same method as in [3,6,8] we can easily check that $K_\lambda(F_{n_1,\cdots,n_k})$ exists for $\lambda \in \mathbb{C}_+^\sim$. Now , using the dominated convergence theorem, we will prove that $K_\lambda(F) =$

$K_\lambda(\sum_{n_1,\cdots,n_k=0}^{\infty} F_{n_1,\cdots,n_k})$ exists. By Theorem 2.2, we see that it will suffice to establish the convergence of the series

(3.4)

$$\sum_{n_1,\cdots,n_k=0}^{\infty} \frac{|a_{n_1,\cdots,n_k}|(N!)^{\frac{1}{r'}}|\lambda|^{(N+1)\delta}\|g\|_r^N(ess\sup\frac{d\eta}{dm_L})^{\frac{N}{r'}}t^{\frac{N(1-r'\delta)}{r'}}}{(2\pi)^{(N+1)\delta}t^\delta} \times \frac{[\Gamma(1-r'\delta)]^{\frac{N+1}{r'}}}{\Gamma[(N+1)(1-r'\delta)]^{\frac{1}{r'}}}$$

for the appropriate λ. Now $\Gamma(z) = z^{z-1/2}e^{-z}\sqrt{2\pi}(1+o(1))$ and hence for positive z sufficiently large

$$\frac{1}{\Gamma(z)} < \frac{2e^z\sqrt{z}}{\sqrt{2\pi}z^z}. \tag{3.5}$$

By Stirling's formula

$$N! = N\Gamma(N) \le (\frac{N}{e})^N(2\pi N)^{1/2}\exp(\frac{1}{12N}). \tag{3.6}$$

Since $r'\delta < 1$, $2N+1-2r'\delta(N+1) = 2(N+1)(1-r'\delta)-1 > 0$. So

$$\frac{N^{N/r'+1/2r'}(N+1)^{1/2r'}N^{-(N+1)\delta}}{(N+1)^{(N+1)(1-r'\delta)/r'}} = (\frac{N}{N+1})^{(2N+1-2r'N\delta-2r'\delta)/2r'} \le 1 \tag{3.7}$$

Combining (3.5) (3.6) and (3.7) we see that for sufficiently large N

$$\{\frac{N!}{\Gamma[(N+1)(1-r'\delta)]}\}^{1/r'} \le \frac{2^{1/r'}\exp(\frac{12N+1}{12Nr'})N^{(N+1)\delta}}{e^{(N+1)\delta}(1-r'\delta)^{\frac{2(N+1)(1-r'\delta)-1}{2r'}}}. \tag{3.8}$$

(1) $0 < \text{order} f < \frac{2\alpha}{l}$: By Theorem 3.2 there exists a d in $(0, \frac{2\alpha}{l})$ such that $|a_{n_1,\cdots,n_k}| < N^{-N/d}$ for sufficiently large N. Using this fact and (3.8) for sufficiently large N the series (3.4) is dominated by the series

(3.9)

$$\sum_{n_1,\cdots,n_k=0}^{\infty} \frac{2^{1/r'}|\lambda|^{(N+1)\delta}\|g\|_r^N t^{N(\frac{1-r'\delta}{r'})}(ess\sup\frac{d\eta}{dm_L})^{\frac{N}{r'}}\exp(\frac{12N+1}{12Nr'})}{(2\pi e)^{(N+1)\delta}t^\delta N^{N/d}N^{(-N\delta)}} \times \frac{N^\delta[\Gamma(1-r'\delta)]^{\frac{N+1}{r'}}}{(1-r'\delta)^{(2(N+1)(1-r'\delta)-1)/2r'}}.$$

Since $\frac{1}{d} - \delta > 0$,

$$\left(\frac{N^\delta}{N^{(\frac{1}{d}-\delta)N}}\right)^{1/N} = \frac{(N^{1/N})^\delta}{N^{(\frac{1}{d}-\delta)}} \to 0$$

as $N \to \infty$. We denote A is the right hand expression of the series (3.9). Then $\lim_{N\to\infty} A^{1/N} = 0$ throughout $Re\lambda \geq 0$, uniformly on compact subsets. Hence the series (3.4) is convergent throughout $Re\lambda \geq 0$, uniformly on compact subsets.
(2) f has order $\frac{2\alpha}{l}$, type $\sigma = \sigma_{D(R_1,\cdots,R_k)} \in [0,\infty)$, $R_1,\cdots,R_k$ and k fixed positive numbers. By the second part of Theorem 3.2 and order $f = \frac{2\alpha}{l} = \frac{1}{\delta}$ we have $\overline{\lim}_{N\to\infty}\{N^{N\delta}[|a_{n_1,\cdots,n_k}|R_1^{n_1}\cdots R_k^{n_k}]\}^{1/N} = (\frac{e\sigma}{\delta})^{\delta}$. In this case (3.4) is dominated by the series

(3.10)

$$\sum_{n_1,\cdots,n_k=0}^{\infty}\{\frac{2^{1/r'}|\lambda|^{(N+1)\delta}\|g\|_r^N t^{N(\frac{1-r'\delta}{r'})}(ess\sup\frac{d\eta}{dm_L})^{\frac{N}{r'}}\exp(\frac{12N+1}{12Nr'})}{(2\pi e)^{N\delta}(2\pi et)^{\delta}N^{N/d}[\min\{R_1,\cdots,R_k\}]^N}$$
$$\times\frac{N^{\delta}[\Gamma(1-r'\delta)]^{\frac{N+1}{r'}}}{(1-r'\delta)^{(2(N+1)(1-r'\delta)-1)/2r'}}\}\times\{N^{N\delta}|a_{n_1,\cdots,n_k}|R_1^{n_1}\cdots R_k^{n_k}\}.$$

By root test, for all $Re\lambda \geq 0$ such that

$$\frac{|\lambda|^{\delta}\|g\|_r[\Gamma(1-r'\delta)]^{\frac{1}{r'}}(ess\sup\frac{d\eta}{dm_L})^{\frac{1}{r'}}(\frac{e\sigma}{\delta})^{\delta}}{(2\pi e)^{\delta}\min\{R_1,\cdots,R_k\}}\times(\frac{t}{1-r'\delta})^{(1-r'\delta/r')}<1,$$

the above series will converges. If $\sigma = 0$, we obtain the convergence of (3.4) throughout $Re\lambda \geq 0$, uniformly on compact subsets, while if $\sigma > 0$ we obtain (3.3). □

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