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NATURAL ORTHONORMAL BASES ASSOCIATED WITH FINITE FRAMES

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Abstract. In this paper we show that for each finite frame for a Hilbert space there are two orthonormal elements related to the optimal lower and upper bounds of the frame. Based on this we show that an orthonormal basis is naturally associated with every finite frame. We then analyze the relationship between such an orthonormal basis and the given finite frame.

1. Introduction

A basis for a Hilbert space is a very useful tool. Every element can be expanded into a series through the elements belonging to a basis. In order to be a basis a subset of a Hilbert space should have the properties of completeness and minimality. Minimality, however, renders a basis too restrictive. Since a frame needs not be minimal it is more flexible than a basis.

A sequence $\{f_i\}_{i\in I}$ of elements in a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ such that for every $f \in H$

$$A||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2.$$

The constants A and B are called a *lower* and an *upper bound* of the frame, respectively. If $A{=}B$, then the frame is called *tight*; and if $A{=}B{=}1$, then it is called a *normalized tight frame*. As frame bounds are not unique the optimal frame bounds are often considered. The *optimal lower frame bound* is the supremum of all the lower frame bounds, and the *optimal upper frame bound* is the infimum of all the upper frame bounds. One can easily see that optimal frame bounds are actually frame bounds. One of the fundamental results from the frame theory is

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that if $\{f_i\}_{i\in I}$ is a frame for a Hilbert space H, then there exists a dual sequence $\{g_i\}_{i\in I}$ such that for every $f \in H$, $f = \sum_{i\in I}\langle f, g_i\rangle f_i$; that is, one can obtain a series expansion for every element in terms of the frame vectors ([7], [8]). A frame is thus complete; that is, the linear span of a frame is dense.

It has been shown that frames are very flexible under perturbations, hence more adaptable for applications than bases ([4], [5]). To analyze frames one can consider subsequences of frames. A subsequence of a frame is called a *frame sequence* if it is a frame for its closed linear span. A frame is said to have *subframe property* if every subsequence is a frame sequence, and it is called a *Riesz frame* if it has a subframe property and a common lower bound for all the frame sequences. Many results have been obtained about the structures of frames ([1], [2], [3]). In this paper we study the function defined by $\varphi(f) = \sum_{i\in I}|\langle f, f_i\rangle|^2$ with the given frame $\{f_i\}_{i\in I}$, and analyze the structure of the frame. Because of certain technical difficulties we consider only finite frames. It is well known that every finite subset of a Hilbert space is a frame sequence (see [7] for proof).

2. Main results

Let H be a Hilbert space and $\mathcal{F} = \{f_i\}_{i=1}^n$ be a sequence in H. We define an associated continuous function $\varphi_{\mathcal{F}} : H \to \mathbb{R}$ by $\varphi_{\mathcal{F}}(f) = \sum_{i=1}^n |\langle f, f_i\rangle|^2$. Note that if the sequence $\mathcal{F} = \{f_i\}_{i=1}^n$ is a frame sequence with bounds A and B, then for every $f \in$ span $\mathcal{F}$, $A||f||^2 \le \varphi_{\mathcal{F}}(f) \le B||f||^2$. So, if it is a tight frame, $\varphi_{\mathcal{F}}(f) = A||f||^2 = B||f||^2$; and if it is a normalized tight frame, $\varphi_{\mathcal{F}}(f) = ||f||^2$.

We use the following elementary lemma to prove our first main theorem.

Lemma 2.1. *Let H be a Hilbert space and $f \in H$ be given. If L is a subspace of H containing f, then for every $g \in H$,*

$$\langle f, g\rangle = \langle f, P_L g\rangle,$$

where P_L is the orthogonal projection onto L.

Proof. For every $g \in H$, we can write $g = P_L g + (I - P_L)g$. Hence for given $f \in H$, we have

$$\begin{aligned}\langle f, g\rangle &= \langle f, P_L g + (I - P_L)g\rangle \\ &= \langle f, P_L g\rangle + \langle f, (I - P_L)g\rangle \\ &= \langle f, P_L g\rangle.\end{aligned}$$

□

Theorem 2.2. *Let H be a finite dimensional Hilbert space spanned by $\mathcal{F} = \{f_i\}_{i=1}^n$ and let $A = \inf\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$ and $B = \sup\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$. Then, there exist vectors $u, v \in H$ such that*
(1) u and v are orthonormal,
(2) $\varphi_{\mathcal{F}}(u) = A$, $\varphi_{\mathcal{F}}(v) = B$; that is, $\sum_{i=1}^n |\langle u, f_i\rangle|^2 = A, \sum_{i=1}^n |\langle v, f_i\rangle|^2 = B$.

Proof. Since H is a finite dimensional Hilbert space, the unit sphere of H is compact. Hence we can choose unit vectors $u_0, v_0 \in H$ such that $\varphi_{\mathcal{F}}(u_0) = A$, $\varphi_{\mathcal{F}}(v_0) = B$.

Let $L = \text{span}\{u_0, v_0\}$ and $h_i = P_L f_i$, for $i = 1, 2, \cdots, n$, where P_L is the orthogonal projection onto L. Then by Lemma 2.1,

$$\langle u_0, f_i\rangle = \langle u_0, P_L f_i\rangle = \langle u_0, h_i\rangle$$

and

$$\langle v_0, f_i\rangle = \langle v_0, P_L f_i\rangle = \langle v_0, h_i\rangle,$$

for $i = 1, 2, \cdots, n$.
Hence,

$$\begin{aligned}\sum_{i=1}^n |\langle u_0, h_i\rangle|^2 + \sum_{i=1}^n |\langle v_0, h_i\rangle|^2 &= \sum_{i=1}^n |\langle u_0, f_i\rangle|^2 + \sum_{i=1}^n |\langle v_0, f_i\rangle|^2 \\ &= A + B.\end{aligned}$$

Now, we can choose unit vectors $u_1, v_1 \in L$ such that $\langle u_0, v_1\rangle = \langle u_1, v_0\rangle = 0$. Then $\{u_0, v_1\}$ and $\{u_1, v_0\}$ are orthonormal bases for L. By the Parseval's identity,

$$\sum_{i=1}^n |\langle u_0, h_i\rangle|^2 + \sum_{i=1}^n |\langle v_1, h_i\rangle|^2 = \sum_{i=1}^n \left(|\langle u_0, h_i\rangle|^2 + |\langle v_1, h_i\rangle|^2\right) = \sum_{i=1}^n ||h_i||^2,$$

and

$$\sum_{i=1}^n |\langle u_1, h_i\rangle|^2 + \sum_{i=1}^n |\langle v_0, h_i\rangle|^2 = \sum_{i=1}^n \left(|\langle u_1, h_i\rangle|^2 + |\langle v_0, h_i\rangle|^2\right) = \sum_{i=1}^n ||h_i||^2.$$

Hence,

$$\sum_{i=1}^n |\langle u_0, h_i\rangle|^2 + \sum_{i=1}^n |\langle v_1, h_i\rangle|^2 = \sum_{i=1}^n |\langle u_1, h_i\rangle|^2 + \sum_{i=1}^n |\langle v_0, h_i\rangle|^2.$$

On the other hand,

$$\sum_{i=1}^{n} |\langle u_0, h_i\rangle|^2 + \sum_{i=1}^{n} |\langle v_1, h_i\rangle|^2 \leq A + B,$$

and

$$\sum_{i=1}^{n} |\langle u_1, h_i\rangle|^2 + \sum_{i=1}^{n} |\langle v_0, h_i\rangle|^2 \geq A + B.$$

Therefore, $\sum_{i=1}^{n} |\langle u_1, h_i\rangle|^2 = A$ and $\sum_{i=1}^{n} |\langle v_1, h_i\rangle|^2 = B$.

Setting $u = u_0$ and $v = v_1$ (or $u = u_1$ and $v = v_0$) , we thus have
(1) $\langle u, v\rangle = 0$,
(2) $\varphi_{\mathcal{F}}(u) = \varphi_{\mathcal{F}}(u_0) = A$, $\varphi_{\mathcal{F}}(v) = \varphi_{\mathcal{F}}(v_1) = B$. □

Theorem 2.2 implies that if $\mathcal{F} = \{f_i\}_{i=1}^{n}$ is a frame, we can find two orthonormal vectors which determine the optimal lower and upper bounds of the frame.

Corollary 2.3. *Let H be a two dimensional Hilbert space spanned by $\mathcal{F} = \{f_i\}_{i=1}^{n}$ with an orthonormal basis $\{u, v\}$, and let $A = \inf\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$ and $B = \sup\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$. Then $\varphi_{\mathcal{F}}(u) = A$ if and only if $\varphi_{\mathcal{F}}(v) = B$.*

Proof. Suppose $\varphi_{\mathcal{F}}(u) = A$ and choose a unit vector v_1 such that $\varphi_{\mathcal{F}}(v_1) = B$. Then by the same argument as in the proof for Theorem 2.2, we can show that $\varphi_{\mathcal{F}}(v) = \varphi_{\mathcal{F}}(v_1) = B$ since $\langle u, v\rangle = 0$. The converse is proved similarly. □

Theorem 2.4. *Let H be a finite dimensional Hilbert space spanned by $\mathcal{F} = \{f_i\}_{i=1}^{n}$ and let $A = \inf\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$ and $B = \sup\{\varphi_{\mathcal{F}}(f) : f \in H, ||f|| = 1\}$. If unit vectors $u, v \in H$ are orthogonal and satisfy $\varphi_{\mathcal{F}}(u) = A$, $\varphi_{\mathcal{F}}(v) = B$, then*
(1) $\sum_{i=1}^{n} \langle u, f_i\rangle \overline{\langle v, f_i\rangle} = 0$,
(2) $\varphi_{\mathcal{F}}(su + tv) = |s|^2 A + |t|^2 B = \varphi_{\mathcal{F}}(su) + \varphi_{\mathcal{F}}(tv)$, for all $s, t \in \mathbb{C}$ with $|s|^2 + |t|^2 = 1$.

Proof. Let $C = \sum_{i=1}^{n}\langle u, f_i\rangle\overline{\langle v, f_i\rangle}$. Then for every $s,t \in \mathbb{C}$ with $|s|^2+|t|^2=1$,

$$\begin{aligned}
\varphi_{\mathcal{F}}(su+tv) &= \sum_{i=1}^{n}|\langle su+tv, f_i\rangle|^2 = \sum_{i=1}^{n}\langle su+tv, f_i\rangle\overline{\langle su+tv, f_i\rangle}\\
&= \sum_{i=1}^{n}\Big(s\langle u, f_i\rangle + t\langle v, f_i\rangle\Big)\Big(\bar{s}\overline{\langle u, f_i\rangle}+\bar{t}\overline{\langle v, f_i\rangle}\Big)\\
&= |s|^2\sum_{i=1}^{n}|\langle u, f_i\rangle|^2 + |t|^2\sum_{i=1}^{n}|\langle v, f_i\rangle|^2 + 2\mathrm{Re}\Big(s\bar{t}\sum_{i=1}^{n}\langle u, f_i\rangle\overline{\langle v, f_i\rangle}\Big)\\
&= |s|^2A+|t|^2B+2\mathrm{Re}\Big(s\bar{t}C\Big).
\end{aligned}$$

So, if we show $C=0$, the proof is completed. To get a contradiction, assume that $C\neq 0$. Let $D=\sqrt{(B-A)^2+4|C|^2}$, $a=\sqrt{(D+B-A)/(2D)}$, and $b=-(C/|C|)\sqrt{(D-B+A)/(2D)}$. Then $D>B-A\geq 0$ and $|a|^2+|b|^2=1$. So, $||au+bv||^2=|a|^2+|b|^2=1$. But,

$$\begin{aligned}
\varphi_{\mathcal{F}}(au+bv) &= |a|^2A+|b|^2B+2\mathrm{Re}\Big(a\bar{b}C\Big)\\
&= \frac{A(D+B-A)}{2D}+\frac{B(D-B+A)}{2D}-\frac{2|C|\sqrt{D^2-(B-A)^2}}{2D}\\
&= \frac{(A+B)D+2AB-(A^2+B^2)-4|C|^2}{2D}\\
&= \frac{(A+B)D-D^2}{2D}\\
&= \frac{A+B-D}{2}\\
&< \frac{A+B-(B-A)}{2}=A
\end{aligned}$$

This is a contradiction. Hence, $C=\sum_{i=1}^{n}\langle u, f_i\rangle\overline{\langle v, f_i\rangle}=0$. □

Note that if $\mathcal{F}=\{f_i\}_{i=1}^{n}$ is a frame sequence, then A and B in Theorem 2.4 are the optimal lower and upper bounds of $\mathcal{F}=\{f_i\}_{i=1}^{n}$, respectively. So, Theorem 2.4 shows certain relationship between the optimal frame bounds of the frame $\mathcal{F}$ and the associated continuous function $\varphi_{\mathcal{F}}$. The following theorem shows a more detailed analysis.

Theorem 2.5. *Let H be a finite dimensional Hilbert space and $\mathcal{F}=\{f_i\}_{i=1}^{n}$ be a frame for H with optimal frame bounds $A, B(A<B)$. Then there exist an orthonormal basis $\{u_1,u_2,...,u_m\}$ for H and positive*

constants $A_1, A_2, ..., A_m$ *such that*
(1) $A = A_1 \le A_2 \le ... \le A_m = B$,
(2) $\varphi_{\mathcal{F}}(a_1u_1 + a_2u_2 + ... + a_mu_m) = |a_1|^2A_1 + |a_2|^2A_2 + \cdots + |a_m|^2A_m$, *for any* $a_1, a_2, \cdots, a_m \in \mathbb{C}$ *with* $|a_1|^2 + |a_2|^2 + \cdots + |a_m|^2 = 1$.

Proof. Let $A_1 = A, B_1 = B, L_1 = H$, and $\mathcal{F}_1 = \mathcal{F}$. Choose a unit vector $u_1 \in L_1$ such that $\varphi_{\mathcal{F}_1}(u_1) = A_1$, and put $L_2 = \text{span}\{u_1\}^\perp \cap L_1$. Then $\mathcal{F}_2 = \{P_{L_2}f_i\}_{i=1}^n$ is a frame for L_2. Let A_2 and B_2 be the optimal lower and upper bounds of $\mathcal{F}_2$, respectively. Then, $A_1 \le A_2$. Since $\varphi_{\mathcal{F}_1}(u_1) = A_1$, as in Corollary 2.3, there is a unit vector $v_1 \in L_1$ such that $\varphi_{\mathcal{F}_1}(v_1) = B_1$ and $u_1 \perp v_1$. Hence $v_1 \in L_2$, and so $B_2 = B_1$.

Choose a unit vector $u_2 \in L_2$ such that $\varphi_{\mathcal{F}_2}(u_2) = A_2$, and put $L_3 = \text{span}\{u_2\}^\perp \cap L_2$. We now obtain a frame $\mathcal{F}_3 = \{P_{L_3}f_i\}_{i=1}^n$ for L_3. Let A_3 and B_3 be the optimal bounds for L_3. Then $A_2 \le A_3$, and by a similar argument as above $B_3 = B_2$.

Continuing this process, we obtain a sequence of vectors $u_1, u_2, \cdots, u_m$ and nonnegative constants $A_1, A_2, \cdots, A_m$ such that
(2.1) $u_k \in L_k$ with $\varphi_{\mathcal{F}_k}(u_k) = A_k$, $k = 1, 2, \cdots, m$,
where for every $k = 2, 3, \cdots, m$,
(2.2) $L_k = \text{span}\{u_{k-1}\}^\perp \cap L_{k-1} = \text{span}\{u_1, u_2, \cdots, u_{k-1}\}^\perp \cap H$,
(2.3) $\mathcal{F}_k = \{P_{L_k}f_i\}_{i=1}^n$,
and A_k is the optimal lower bound of $\mathcal{F}_k$. So, $A_1 \le A_2 \le \cdots \le A_m$. We know also that B is the optimal upper bound of $\mathcal{F}_k$ for every $k = 1, 2, \cdots, m$. Since H is finite dimensional, we may assume that the dimension of L_m is 1. Then $A_m = B$, whence we get (1). To show (2) let, for each pair $j < k$, $L_{j,k} = \text{span}\{u_j, u_k\}$ and $\mathcal{F}_{j,k} = \{P_{L_{j,k}}f_i\}_{i=1}^n$ with optimal bounds $A_{j,k} \le B_{j,k}$. Then $A_{j,k} = A_j = \varphi_{\mathcal{F}_j}(u_j) = \varphi_{\mathcal{F}_{j,k}}(u_j)$. Since $u_j \perp u_k$ and $L_{j,k}$ is of dimension 2, by Corollary 2.3 $B_{j,k} = \varphi_{\mathcal{F}_{j,k}}(u_k) = \varphi_{\mathcal{F}}(u_k) = \varphi_{\mathcal{F}_k}(u_k) = A_k$. It now follows from Theorem 2.4 applied to $L_{j,k}$ and $\mathcal{F}_{j,k}$ that

$$\sum_{i=1}^n \langle u_j, f_i\rangle\overline{\langle u_k, f_i\rangle} = \sum_{i=1}^n \langle u_j,\ P_{L_{j,k}}f_i\rangle\overline{\langle u_k,\ P_{L_{j,k}}f_i\rangle} = 0.$$

Hence,

$$\begin{aligned}
\varphi_{\mathcal{F}}(a_1u_1 + a_2u_2 + \cdots + a_mu_m) &= \sum_{i=1}^{n} \mid \langle a_1u_1 + a_2u_2 + \cdots + a_mu_m, f_i\rangle \mid^2 \\
&= \sum_{i=1}^{n} \mid \sum_{j=1}^{m} a_j\langle u_j, f_i\rangle \mid^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} a_i\overline{a_k}\langle u_j, f_i\rangle\overline{\langle u_k, f_i\rangle} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} \mid a_k \mid^2 \mid \langle u_k, f_i\rangle \mid^2 \\
&= \sum_{k=1}^{m} \mid a_k \mid^2 \sum_{i=1}^{n} \mid \langle u_k, f_i\rangle \mid^2 \\
&= \sum_{k=1}^{m} \mid a_k \mid^2 \varphi_{\mathcal{F}}(u_k) = \sum_{k=1}^{m} \mid a_k \mid^2 A_k.
\end{aligned}$$

□

Corollary 2.6. *Let H be a finite dimensional Hilbert space and $\mathcal{F} = \{f_i\}_{i=1}^n$ be a frame for H. Then there exist an orthonormal basis $\{u_1, u_2, \cdots, u_m\}$ and positive constants $A_1, A_2, \cdots, A_m$ such that for every $f \in H$, $\varphi_{\mathcal{F}}(f) = \sum_{k=1}^m \mid \langle f, u_k\rangle \mid^2 A_k$.*

Proof. Since $\varphi_{\mathcal{F}}(f) = \| f \|^2 \varphi_{\mathcal{F}}(f/\|f\|)$ and $f/\|f\| = \sum_{k=1}^m \langle f/\|f\|, u_k\rangle u_k$ with $\sum_{k=1}^m |\langle f/\|f\|, u_k\rangle|^2 = 1$, by Theorem 2.5 $\varphi_{\mathcal{F}}(f/\|f\|) = \sum_{k=1}^m |\langle f/\|f\|, u_k\rangle|^2 A_k$. Therefore, $\varphi_{\mathcal{F}}(f) = \sum_{k=1}^m |\langle f, u_k\rangle|^2 A_k$. □

3. Epilogue

Let $\{f_i\}_{i=1}^n$, $\{u_k\}_{k=1}^m$, and $\{A_k\}_{k=1}^m$ be as in Theorem 2.5. Then, it can be shown that A_k's are the eigenvalues of the frame operator S defined by $Sf = \sum_{i=1}^n \langle f, f_i\rangle f_i$ for $f \in H$. Furthermore, u_k's are eigenvectors of S corresponding to the eigenvalues A_k. On the other hand, since S is a self-adjoint operator, there exists an orthonormal basis for H consisting of eigenvectors of S. The same conclusion as in Theorem 2.5 can be derived from this fact. The process we adopt in this paper, however, reveals intrinsically the relationship between $\varphi_{\mathcal{F}}$ and the associated orthonormal basis.

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