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ASYMPTOTIC OPTION PRICING UNDER A PURE JUMP PROCESS

SEONGJOO SONG[1]

ABSTRACT

This paper studies the problem of option pricing in an incomplete market. The market incompleteness comes from the discontinuity of the underlying asset price process which is, in particular, assumed to be a compound Poisson process. To find a reasonable price for a European contingent claim, we first find the unique minimal martingale measure and get a price by taking an expectation of the payoff under this measure. To get a closed-form price, we use an asymptotic expansion. In case where the minimal martingale measure is a signed measure, we use a sequence of martingale measures (probability measures) that converges to the equivalent martingale measure in the limit to compute the price. Again, we get a closed form of asymptotic option price. It is the Black-Scholes price and a correction term, when the distribution of the return process has nonzero skewness up to the first order.

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1. INTRODUCTION

Although the Black-Scholes model is simple and easy to understand, it is evident that the real stock price process does not follow the Black-Scholes geometric Brownian motion. The most evident feature of the nonnormality of the stock returns is the larger kurtosis, and sometimes nonzero skewness is also observed. Volatility smile is another evidence of failure of the Black-Scholes assumptions. There has been extensive investigation in finance literature on alternatives to the Black-Scholes model, and many practical and empirically relevant models have been proposed, which overcome several drawbacks of the Black-Scholes model. However, with many of those alternatives including jump-diffusion models, the

Received September 2006; accepted January 2007.
[1]Department of Statistics, The University of Seoul, 90 Jeonnong-dong, Dongdaemun-gu, Seoul 130-743, Korea (e-mail: ssong@uos.ac.kr)

market becomes incomplete. In an incomplete market, there exist many equivalent martingale measures, and thus, no unique price of a contingent claim exists. Therefore, the problem of option pricing is not a simple mathematical question under the market incompleteness.

Since no unique price exists, some authors found a range of prices of contingent claims. El Karoui and Quenez (1995) determined a price range for the actual market price of a contingent claim. Eberlein and Jacod (1997) found a price range for a general payoff function when the asset price process is discontinuous and showed that the upper limit of this price range is extremely high. Pricing method based on utility consideration has been studied by many authors in the past few years, as Rouge and El Karoui (2000) or Frittelli (2000).

Several criteria on choosing a specific equivalent martingale measure have also been proposed, since we obtain one price from each equivalent martingale measure. For example, Delbaen and Schachermayer (1996) suggested the variance-optimal martingale measure as the 'most natural' martingale measure for pricing and hedging. Elliott and Madan (1998) developed the extended Girsanov principle for discrete time continuous state processes to select an equivalent martingale measure. Föllmer and Schweizer (1991) introduced a minimal martingale measure which preserves the structure of the real statistical measure as far as possible. A martingale measure is called minimal if any square integrable martingale under the original measure that are independent of the asset price process remain as martingales under the new martingale measure. They also constructed the risk minimization strategy that minimizes the intrinsic risk using this measure. The minimal martingale measure has been studied in various literature since then, geared towards problems of local risk minimization or mean-variance hedging (For review, see, for instance, Prigent, 2003).

In this paper, we will utilize the minimal martingale measure to get a closed form asymptotic option price when the underlying price process is a jump process. More specifically, the log of the stock price process is modeled by a compound Poisson process. While a pure jump part is used to model only abnormal components of the stock return in a typical jump-diffusion model, jumps drive the whole process under our model. In that sense, it is similar to pure jump processes in Eberlein and Keller (1995), Madan and Seneta (1990) and Carr *et al.* (2002). Although Poisson processes have been used more often with diffusion processes, they are also used to model the underlying price process on their own, (for example, Frey, 2000; Kirch and Runggaldier, 2004; Song and Mykland, 2006) as they could have such real market features that prices change at discrete random points in time. A compound Poisson process has a very simple structure and yet, it can capture fat tails or the asymmetry of the return distribution. It is also intuitive in the sense that it jumps at random times and there are finitely many jumps in any finite interval. Here, we consider a sequence of compound Poisson processes whose limit is the Black-Scholes geometric Brownian motion. We use this convergence to obtain a first-order option price.

Since the famous paper by Cox *et al.* (1979), convergence of the sequence of discrete time price processes to a continuous time process has been widely studied. Unlike most of the previous literature dealing with a set of fixed time points, we deal with a set of random time points with the time interval going to 0 as the jump intensity goes to infinity. The compound Poisson process we have can be considered as a generalization of a binomial tree model, as an extension of the model by Rachev and Ruschendorf (1994), since it randomizes the jump time and the jump distribution from a binomial tree model in a particular way that the limit is a Brownian motion.

The purpose of this paper is to find an appropriate, closed-form option price in an incomplete market. Assuming that the log of the underlying price process is compound Poisson, we first find the unique minimal martingale measure and take the expected value of the payoff of the option if the minimal measure is a proper probability measure. To get a closed-form solution for the price, we use an asymptotic method. The price we propose here is an asymptotic price, which has a term of the order of square root of the jump intensity. See Section 3 for details. When the minimal martingale measure is not a proper probability measure, we consider a different sequence of martingale measures. As stated in Elliott and Madan (1998), requirements for minimality are inconsistent with those of being a probability law. Thus, with a jump process, the minimal martingale measure may not be a proper probability measure. In such cases, we find a sequence of equivalent martingale measures that converges weakly to the equivalent martingale measure in the limit. These measures are proper probability measures. Then we get a price of an option by taking the expected value of the payoff under this measure and by an asymptotic expansion.

The remainder of the paper is organized as follows. Section 2 describes the detailed model and the convergence of the underlying asset price process. Section 3 finds the minimal martingale measure and finds a price under that measure. Then, we find a sequence of equivalent martingale measures converging to the equivalent martingale measure in the limit in Section 4 and obtain a price under this measure. Proofs are in Appendix. Section 5 contains concluding comments.

2. THE MODEL

Consider a sequence of pure jump processes that converges to a geometric Brownian motion. Each element of the sequence is a discontinuous process, indexed by n. A larger n means that the degree of discontinuity is smaller, *i.e.*, the process is closer to a geometric Brownian motion.

We suppose that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^{(n)})_{0\le t\le T}, \mathbf{P}^{(n)})$ satisfying the usual conditions, for each n. The log stock price process is defined on this probability space and follows a compound Poisson process such as

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{2.1}$$

where $N^{(n)}$ is a Poisson process with rate λ_n, and $Z_i^{(n)}$'s are *i.i.d.* random variables, distributed as $Z^{(n)}$, that are independent of $N^{(n)}$. The filtration, $\{\mathcal{F}_t^{(n)}\}$, is generated by the stock price process $S^{(n)}$ defined above. We also assume the initial stock price $S_0^{(n)}$ is the same as S_0 for all n. As n goes to ∞, we assume that λ_n goes to ∞ and $Z^{(n)}$ converges to 0 in distribution. $N_t^{(n)}$ is the number of jumps in the log stock price process up to time t, and $Z_i^{(n)}$ represents the size of the i^{th} jump of $\log S^{(n)}$.

The jump intensity, λ_n, is related to the level of the trading activity of an individual stock. A heavily traded stock is modeled with a large λ_n, and a less heavily traded stock is modeled with a smaller λ_n. According to the level of trading activity of a stock, we determine the value of λ_n so that the model fits with the data. Each jump occurs when there is a trading that changes the underlying stock price.

We define the jump size distribution $Z^{(n)}$ more precisely as follows.

$$Z^{(n)} \overset{\mathcal{D}}{=} \frac{1}{\sqrt{\lambda_n}} Q + \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right), \tag{2.2}$$

where Q is a random variable with $EQ = 0$, $EQ^2 = \sigma^2$, $EQ^3 = k_3$ and $EQ^4 = k_4$, under $\mathbf{P}^{(n)}$, for all n. Q has a distribution that does not depend on n and it has finite moments of all orders. μ is a nonzero constant. $\overset{\mathcal{D}}{=}$ means that both sides of the equality have the same distribution. It is clear that $Z^{(n)}$ converges to 0 in probability as well as in distribution as n goes to ∞, $E(Z^{(n)}) = (1/\lambda_n)(\mu - \sigma^2/2)$, $E|Z^{(n)}|^p = O(\lambda_n^{-p/2})$ for $p = 2, 3$ and 4, and $E|Z^{(n)}|^p = o(\lambda_n^{-2})$ for $p > 4$.

The model (2.1) can be written in a form of the stochastic differential equation as follows.

$$dS_t^{(n)} = S_{t-}^{(n)} dR_t^{(n)} = S_{t-}^{(n)} d\tilde{R}_t^{(n)} + S_{t-}^{(n)} dA_t^{(n)}, \tag{2.3}$$

where $R_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} (e^{Z_i^{(n)}} - 1)$, $A_t^{(n)} = \lambda_n t \cdot E(e^{Z^{(n)}} - 1)$ and $\tilde{R}_t^{(n)} = R_t^{(n)} - A_t^{(n)}$. Note that $\tilde{R}^{(n)}$ is a martingale.

Consider the asymptotics as n goes to ∞. The conditions above assure that as n goes to ∞, $\log S^{(n)}$ converges in distribution to $\log S$ that is

$$\log S_t = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t, \tag{2.4}$$

where B is a Brownian motion under the limiting measure $\mathbf{P}$ (see, for example, Song and Mykland, 2006). In the limit, the assumptions of Black-Scholes model are all satisfied including the geometric Brownian motion. Thus, the price of an option on S will be uniquely determined by Black-Scholes PDE in the limit.

We consider a market with two securities, a stock as a risky asset and a cash bond as a riskless asset. We also assume that the interest rate r is 0 without loss of generality. Then consider a European style option whose payoff is $H(S_T^{(n)})$ with the expiration time T, and we assume that $H(S_T^{(n)})$ is in $L^2(\mathbf{P}^{(n)})$. Throughout the paper, we denote $C(x,t)$ the solution of the Black-Scholes PDE at time $t < T$, with the terminal condition, $H(x) = C(x,T)$. C_S, C_{SS} and C_{SSS} denote the first, second and third derivatives of $C(x,t)$ with respect to x, respectively, and $C_S^{(p)}$ is used for the p^{th} derivative of $C(x,t)$ with respect to x, for $p > 3$. For each n, we compute $C(S_t^{(n)}, t)$ by plugging in the corresponding stock price process $S^{(n)}$. In other words, $C(S_t^{(n)}, t)$ is calculated by the Black-Scholes PDE, but it may be different from what we observe from the market.

3. Minimal Martingale Measure

Since there may be many different equivalent martingale measures, there have been attempts to find an appropriate equivalent martingale measure for pricing purpose under the market incompleteness. One of the widely accepted measures is the minimal martingale measure, introduced by Föllmer and Schweizer (1991). In their paper, the minimal martingale measure was defined for asset price processes with continuous sample paths. Later, the minimal martingale measure was found in more general cases, for example, Chan (1999) found it for general Lévy processes and Lee (2002) found it for processes including non-Lévy type jump processes. Following the method given in Lee (2002), we can find the unique minimal martingale measure of the underlying process, $S^{(n)}$, as follows. If we define

$$Y_t^{(n)} = \exp\left\{c_n^* \lambda_n t E(\exp(Z^{(n)}) - 1)\right\} \prod_{i=1}^{N_t^{(n)}} \left\{1 - c_n^*(\exp(Z_i^{(n)}) - 1)\right\}, \qquad (3.1)$$

then $\mathbf{P}^{*(n)}$ defined by $d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)} = Y_T^{(n)}$ is the unique minimal martingale measure of $S^{(n)}$, when $c_n^* = E(\exp(Z^{(n)}) - 1)/E(\exp(Z^{(n)}) - 1)^2$ and[2]

$$c_n^*(\exp(Z_i^{(n)}) - 1) < 1, \quad \text{almost surely.} \qquad (3.2)$$

When the underlying price process has jumps, the minimal martingale measure may be a signed measure. To avoid the case where we end up with a signed measure, we need some assumptions on the jump size distribution. The condition (3.2) is the assumption needed for $\mathbf{P}^{*(n)}$ to be a proper probability measure.

EXAMPLE 3.1. Suppose that Q in (2.2) is a binary random variable such as

$$Q = \begin{cases} \sigma\sqrt{\frac{1-p}{p}}, & \text{w.p. } p, \\ -\sigma\sqrt{\frac{p}{1-p}}, & \text{w.p. } 1-p \end{cases}$$

with $0 < p < 1$. Then $EQ = 0$, $EQ^2 = \sigma^2$ and $EQ^3 = \{\sigma^3(1-2p)\}/\{\sqrt{(1-p)p}\}$. If $p \neq 1/2$, Q has a nonzero skewness. Define p^* as

$$p^* = \frac{1 - e^{-\frac{1}{\sqrt{\lambda_n}}b + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)}}{e^{\frac{1}{\sqrt{\lambda_n}}a + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)} - e^{-\frac{1}{\sqrt{\lambda_n}}b + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)}}.$$

Then $c_n^*(\exp(Z^{(n)}) - 1)$ is always less than 1, if $0 < p^* < 1$. $0 < p^* < 1$ becomes equivalent to the following condition.

$$0 < p^* < 1 \Leftrightarrow \begin{cases} p < 1 - \dfrac{\left(\mu - \frac{1}{2}\sigma^2\right)^2}{\lambda_n \sigma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)^2}, & \text{if } \mu < \frac{1}{2}\sigma^2, \\[2ex] p > 1 - \dfrac{\sigma^2}{\sigma^2 + \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right)^2}, & \text{if } \mu > \frac{1}{2}\sigma^2 \end{cases}$$

[2]We remark that condition (3.2) is equivalent to $Z_i^{(n)} < \log(1 + 1/c_n^*)$ if $c_n^* > 0$ and $Z_i^{(n)} > \log(1 + 1/c_n^*)$ if $c_n^* < -1$. If $-1 \leq c_n^* \leq 0$, (3.2) is always satisfied. Moreover, the probability that the assumption (3.2) is satisfied converges to 1 as n goes to ∞. In fact, we can easily show that $c_n^*(\exp(Z^{(n)}) - 1)$ is $O_p(\lambda_n^{-1/2})$.

and if $\mu = \sigma^2/2$, $0 < p^* < 1$ always. Thus, when λ_n is large enough, the condition (3.2) holds for almost all values of p regardless of μ and σ.

When the condition (3.2) holds, the minimal martingale measure defined in (3.1) converges to the equivalent martingale measure in the limit jointly with the underlying asset price process (see, for example, Prigent, 2003). This is also a special case of Theorem 4.1. Proof of the next theorem is omitted.

THEOREM 3.1. *Subject to the assumption (3.2),* $(\log Y^{(n)}, \log S^{(n)})$ *converges jointly in distribution to* $(\xi, \log S)$ *where* $\xi_t = -(\mu/\sigma)B_t - (\mu^2/2\sigma^2)t$. *B is the Brownian motion that drives the price process S in the limit.*

Since the fair price of a contingent claim in a complete market is the expected value of the payoff under the unique equivalent martingale measure, we can try to price an option as the expected value of the payoff under the minimal martingale measure. An option price computed as the expectation of the payoff under the minimal martingale measure is known to converge the true price when the limiting model is complete (see, Prigent, 2003). In the next theorem, we obtain the first order correction term to the Black-Scholes option price as well as showing that the expectation under the minimal martingale measure converges to the Black-Scholes price.

THEOREM 3.2. *Suppose we want to price a European style payoff* $H(S_T^{(n)}) = C(S_T^{(n)}, T)$ *that satisfies Assumption A.1. Then using the minimal martingale measure, we get a price of* $C(S_T^{(n)}, T)$ *as*

$$E^*C(S_T^{(n)}, T) = C(S_0, 0) + \frac{k_3 T}{\sqrt{\lambda_n}} \left\{ \frac{1}{2}\left(1 - \frac{\mu}{\sigma^2}\right) C_{SS}(S_0, 0) S_0^2 + \frac{1}{6} C_{SSS}(S_0, 0) S_0^3 \right\} + o\left(\lambda_n^{-\frac{1}{2}}\right).$$

PROOF. See Appendix. □

4. MEASURES CONVERGING TO THE MINIMAL MARTINGALE MEASURE

Note that both papers by Chan (1999) and Lee (2002) impose conditions to avoid the case where the minimal martingale measure becomes a signed measure. In Section 3, we also assumed the condition (3.2) to avoid this case. But we may have more general jump distributions so that the measure defined in (3.1) is a signed measure. For example, if Q in (2.2) is a normal random variable with mean 0 and variance σ^2, then the condition (3.2) does not hold when $\mu > 0$. And it is not in general a good idea to use a signed measure to price a contingent claim, because, for example, it may end up with a negative price. Therefore, in such cases, we may want to use proper probability measures that have some desirable properties. Here, we consider a sequence of martingale measures that converges weakly to the equivalent martingale measure in the limit.

As we saw in (2.4), the limiting log stock price process is a Brownian motion with drift under $\mathbf{P}$. It is well known that

$$\log S_t = \log S_0 - \frac{1}{2}\sigma^2 t + \sigma B_t^*,$$

where B^* is a standard Brownian motion under $\mathbf{P}^*$, the unique equivalent martingale measure. The Radon-Nikodym derivative, $d\mathbf{P}^*/d\mathbf{P}$, is

$$\frac{d\mathbf{P}^*}{d\mathbf{P}}|_{\mathcal{F}_t} = \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right).$$

Note that $r = 0$ and $\{\mathcal{F}_t\}$ is a filtration generated by the Brownian motion B. We define $\tilde{Y}^{(n)}$ as

$$\tilde{Y}_t^{(n)} = \left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}} e^{(\lambda_n - \lambda_n^*)t} \prod_{i=1}^{N_t^{(n)}} g_n(Q_i), \tag{4.1}$$

where $g_n(Q)$ satisfies the following.

ASSUMPTION 4.1. (Assumption on $g_n(Q)$).

(i) $g_n > 0$, *$\mathcal{F}$-measurable,*

(ii) $E(g_n(Q)) = 1$,

(iii) $E(g_n(Q)(e^{Z^{(n)}} - 1)) = 0$,

(iv) $\lambda_n E(\log g_n(Q)) \longrightarrow -\frac{1}{2}\frac{\mu^2}{\sigma^2}$,

(v) $\lambda_n \mathrm{Var}(\log g_n(Q)) \longrightarrow \frac{\mu^2}{\sigma^2}$.

The first and second assumptions in Assumption 4.1 are needed for the changed measure to be a proper probability measure. The second assumption $E(g_n(Q)) = 1$ is also needed for the changed measure to be equivalent to $\mathbf{P}^{(n)}$. By assuming $E(g_n(Q)(e^{Z^{(n)}} - 1)) = 0$, $S^{(n)}$ is a martingale under the changed measure.

EXAMPLE 4.1. We see examples of $g_n(Q)$ satisfying Assumption 4.1 in the following.

1) If the condition (3.2) holds, $Y^{(n)}$ defined in (3.1) is a special case of $\tilde{Y}^{(n)}$ in (4.1) with
$$g_n(Q) = \frac{1 - c_n^*(e^{Z^{(n)}} - 1)}{1 - c_n^* E(e^{(Z^{(n)}} - 1)}.$$

2) Suppose $Q \sim N(0, \sigma^2)$ and $\mu > 0$. Then we can use
$$g_n(Q) = \exp\left(-\frac{1}{2\sigma^2}\left(\frac{\mu^2}{\lambda_n} + \frac{2\mu}{\sqrt{\lambda_n}}Q\right)\right).$$

3) Suppose $Q = X - \sigma$, where $X \sim \mathrm{Exp}(1/\sigma)$. Then we can use $g_n(Q) = \exp(cQ + b)$ where $c\sigma < 1$, $b = c\sigma + \log(1 - c\sigma)$, and
$$1 - c\sigma = \frac{\sigma}{\sqrt{\lambda_n}} \frac{\exp\left(\frac{\sigma}{\sqrt{\lambda_n}} - \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right)\right)}{\exp\left(\frac{\sigma}{\sqrt{\lambda_n}} - \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right)\right) - 1}.$$

The next theorem shows the joint convergence of $\tilde{Y}^{(n)}$ and $S^{(n)}$.

THEOREM 4.1. *Suppose that $\tilde{Y}^{(n)}$ is defined as (4.1). Then subject to Assumption 4.1, $\mathbf{P}^{*(n)}$ defined by $(d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)})|_{\mathcal{F}_t^{(n)}} = \tilde{Y}_t^{(n)}$ is an equivalent martingale measure of $S^{(n)}$. Moreover, assuming that $\log\lambda_n^* - \log\lambda_n = (\eta/\sqrt{\lambda_n}) + o(\lambda_n^{-1/2})$, $(\log\tilde{Y}^{(n)}, \log S^{(n)})$ converges jointly in distribution to $(\xi, \log S)$ where $\xi_t = -(\mu/\sigma)B_t + \eta\tilde{W}_t - (1/2)\{(\mu^2/\sigma^2) + \eta^2)\}t$, and $\tilde{W}$ is a Brownian motion under $\mathbf{P}$ which is independent of B. In particular, if $\eta = 0$, $\mathbf{P}^{*(n)}$ converges weakly to the unique equivalent martingale measure in the limit.*

PROOF. See Appendix. □

REMARK 4.1. We can also show that the sequence $\{\mathbf{P}^{*(n)}\}$ is contiguous to the sequence $\{\mathbf{P}^{(n)}\}$. By Theorem 4.1, $d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)}$ converges weakly to $\exp[(-\mu/\sigma)B_t + \eta\tilde{W}_t - (1/2)\{(\mu^2/\sigma^2) + \eta^2\}t]$ where $E(\exp[(-\mu/\sigma)B_t + \eta\tilde{W}_t - (1/2)\{(\mu^2/\sigma^2) + \eta^2\}t]) = 1$. By Corollary V.1.12 in Jacod and Shiryaev (1987), $\{\mathbf{P}^{*(n)}\}$ is contiguous to $\{\mathbf{P}^{(n)}\}$. For the relationship between contiguity and the convergence of martingale measures, see Hubalek and Schachermayer (1998).

THEOREM 4.2. *Suppose we choose an equivalent martingale measure* $\mathbf{P}^{*(n)}$ *defined in Theorem 4.1 for the pricing purpose. For a European style payoff* $H(S_T^{(n)}) = C(S_T^{(n)}, T)$ *that satisfies Assumption A.1, its price is*

$$E^*C(S_T^{(n)}, T) = C(S_0, 0) + \frac{k_3 T}{\sqrt{\lambda_n}} \left\{ \frac{1}{2} C_{SS}(S_0, 0) S_0^2 + \frac{1}{6} C_{SSS}(S_0, 0) S_0^3 \right\} + \frac{T}{2\sqrt{\lambda_n}} \left\{ \sqrt{\lambda_n} E(Q^2 \log g_n(Q)) \right\} C_{SS}(S_0, 0) S_0^2 + o\left(\lambda_n^{-\frac{1}{2}}\right). \quad (4.2)$$

PROOF. See Appendix. □

To get an explicit first-order price in the above theorem, we need the leading term of $\sqrt{\lambda_n} E(Q^2 \log g_n(Q))$. For example, if $Q = X - \sigma$, where $X \sim \text{Exp}(1/\sigma)$ or $Q \sim N(0, \sigma^2)$ and $\mu > 0$, then using $g_n(Q)$ defined in Example 4.1,

$$\sqrt{\lambda_n} E(Q^2 \log g_n(Q)) = -\frac{k_3 \mu}{\sigma^2} + o(1).$$

For more general distributions, suppose we take

$$g_n(Q) = \frac{1 + a_n(\exp(c_n Q + b_n) - 1)}{E(1 + a_n(\exp(c_n Q + b_n) - 1))}$$

for some constants $a_n = O(1)$, $b_n = O(\lambda_n^{-1})$ and $c_n = O(\lambda_n^{-1/2})$. Then, if $g_n(Q) > 0$, we can easily show that

$$\sqrt{\lambda_n} E(Q^2 \log g_n(Q)) = -\frac{k_3 \mu}{\sigma^2} + o(1),$$

when $g_n(Q)$ satisfies Assumption 4.1. Also note that $g_n(Q) = \exp(c_n Q + b_n)$ for $b_n = O(\lambda_n^{-1})$ and $c_n = O(\lambda_n^{-1/2})$ will give the same result.

Therefore, if we use such $g_n(Q)$ as above, the price of a European contingent claim $C(S_T^{(n)}, T)$ becomes

$$E^*C(S_T^{(n)}, T) = C(S_0, 0) + \frac{k_3 T}{\sqrt{\lambda_n}} \left\{ \frac{1}{2}\left(1 - \frac{\mu}{\sigma^2}\right) C_{SS}(S_0, 0) S_0^2 + \frac{1}{6} C_{SSS}(S_0, 0) S_0^3 \right\} + o\left(\lambda_n^{-\frac{1}{2}}\right),$$

which is the same as what we have obtained in Theorem 3.2 under the minimal martingale measure.

5. CONCLUDING REMARK

In this paper, we have studied the problem of option pricing under a pure jump model. Since the unique price in a complete market is the expected value of the payoff under the unique equivalent martingale measure, we tried to price an option by choosing an appropriate martingale measure among many possible measures. For a sequence of compound Poisson processes that converges to a Brownian motion with drift, we first found the unique minimal martingale measure and found a closed-form first-order price using an asymptotic expansion. When the minimal martingale measure is a signed measure, we found a sequence of probability measures converging to the unique equivalent martingale measure in the limit. These measures are also martingale measures. Then we found a price based on this. Again, we used an asymptotic expansion to get the closed form solution for the price. It is not included in this paper, but we also considered the case where we have a diffusion term that is independent of the jump term in the underlying price process and obtained a similar result.

APPENDIX

Proof of Theorem 3.2

This is a special case of Theorem 4.2. By taking $g_n(Q) = \{1 - c_n^*(\exp(Z^{(n)}) - 1)\}/\{1 - c_n^* E(\exp(Z^{(n)}) - 1)\}$ in Theorem 4.2, we get the result. Here, $\sqrt{\lambda_n} \times E(Q^2 \log g_n(Q))$ is $-(k_3\mu/\sigma^2) + o(1)$.

Proof of Theorem 4.1

For all $t \in [0, T]$, $\tilde{Y}_t^{(n)} > 0$ and

$$E(\tilde{Y}_t^{(n)}) = E\left\{E\left(\left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}} e^{(\lambda_n - \lambda_n^*)t} \prod_{i=1}^{N_t^{(n)}} g_n(Q_i) | N_t^{(n)}\right)\right\}$$
$$= e^{(\lambda_n - \lambda_n^*)t} E\left(\left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}}\right) = 1,$$

since we assume $E(g_n(Q)) = 1$. Similarly, we can show that $\tilde{Y}^{(n)}$ is a martingale under $\mathbf{P}^{(n)}$. Thus, $\mathbf{P}^{*(n)}$ defined by $d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)} = \tilde{Y}^{(n)}$ is an equivalent probability measure of $\mathbf{P}^{(n)}$. Using the independence of $N^{(n)}$ and Q under $\mathbf{P}^{(n)}$

and $E(g_n(Q)) = 1$, we can show that the Poisson process $N^{(n)}$ and Q are independent under $\mathbf{P}^{*(n)}$. It is also easy to check that $\{S_t^{(n)}\}$ is a martingale under $\mathbf{P}^{*(n)}$, because for any $0 \le u \le t \le T$,

$$\begin{aligned}
&E^*\left(S_t^{(n)}|\mathcal{F}_u^{(n)}\right) \\
&= S_u^{(n)} E^*\left(\prod_{i=N_u^{(n)}+1}^{N_t^{(n)}} e^{Z_i^{(n)}}|\mathcal{F}_u^{(n)}\right) \\
&= S_u^{(n)} \frac{E\left(\prod_{i=N_u^{(n)}+1}^{N_t^{(n)}} e^{Z_i^{(n)}} \tilde{Y}_t^{(n)}|\mathcal{F}_u^{(n)}\right)}{\tilde{Y}_u^{(n)}} \\
&= S_u^{(n)} e^{(\lambda_n-\lambda_n^*)(t-u)} \\
&\quad \times E\left\{E\left(\left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}-N_u^{(n)}} \prod_{i=N_u^{(n)}+1}^{N_t^{(n)}} e^{Z_i^{(n)}} g_n(Q_i)|N_t^{(n)}, \mathcal{F}_u^{(n)}\right)|\mathcal{F}_u^{(n)}\right\} \\
&= S_u^{(n)} e^{(\lambda_n-\lambda_n^*)(t-u)} E\left(\left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}-N_u^{(n)}}\right) \\
&= S_u^{(n)}.
\end{aligned}$$

We used the assumption $E(g_n(Q)(e^{Z^{(n)}} - 1)) = 1$. Note that we assume that the interest rate is 0. Moreover, we want $\log(d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)}) = O_p(1)$ because we want $\mathbf{P}^{*(n)}$ to converge to the minimal martingale measure. Define a random variable $\tilde{X}^{(n)}$ as

$$\tilde{X}_i^{(n)} := \frac{\log g_n(Q_i) - E(\log g_n(Q))}{\sqrt{\operatorname{Var}(\log g_n(Q))}}.$$

We show the following lemma before we see the limit of $\log(d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)})$.

LEMMA A.1. $((N_t^{(n)} - \lambda_n t)/\sqrt{\lambda_n}, (1/\sqrt{\lambda_n})\sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)})$ *converges in distribution to a two-dimensional Brownian motion under* $\mathbf{P}$, $(\tilde{W}, W^1)$.

PROOF. Define $M_t^{(n)}$ and $\tilde{M}_t^{(n)}$ to be $(N_t^{(n)} - \lambda_n t)/\sqrt{\lambda_n}$ and $(1/\sqrt{\lambda_n}) \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)}$, respectively. Then, they are square-integrable martingales satisfying $\langle M^{(n)}, M^{(n)}\rangle_t = t$, $\langle \tilde{M}^{(n)}, \tilde{M}^{(n)}\rangle_t = t$ and $\langle M^{(n)}, \tilde{M}^{(n)}\rangle_t = 0$. Consider local square integrable martingales $M_\epsilon^{(n)}$ and $\tilde{M}_\epsilon^{(n)}$ which include all the jumps of the original martingales greater than ϵ in absolute value. Since the jump size of

$(N_t^{(n)} - \lambda_n t)/\sqrt{\lambda_n}$ at any time point is either 0 or $1/\sqrt{\lambda_n}$, it becomes arbitrarily small for large n. Thus, we can define $M_\epsilon^{(n)} = 0$. In the case of $\tilde{M}^{(n)}$, define

$$\tilde{M}_{\epsilon,t}^{(n)} = \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)} I_{\{|\tilde{X}_i^{(n)}|>\epsilon\sqrt{\lambda_n}\}} - \sqrt{\lambda_n} t E\left(\tilde{X}^{(n)} I_{\{|\tilde{X}^{(n)}|>\epsilon\sqrt{\lambda_n}\}}\right).$$

Then,

$$|\Delta\tilde{M}_t^{(n)} - \Delta\tilde{M}_{\epsilon,t}^{(n)}| = \frac{1}{\sqrt{\lambda_n}} |\tilde{X}_{N_t^{(n)}}^{(n)}| I_{\{|\tilde{X}_{N_t^{(n)}}^{(n)}|\le\epsilon\sqrt{\lambda_n}\}} \le \epsilon$$

and

$$\langle \tilde{M}_\epsilon^{(n)}, \tilde{M}_\epsilon^{(n)} \rangle_t = tE\left(\left(\tilde{X}^{(n)}\right)^2 I_{\{|\tilde{X}^{(n)}|>\epsilon\sqrt{\lambda_n}\}}\right).$$

$\langle \tilde{M}_\epsilon^{(n)}, \tilde{M}_\epsilon^{(n)} \rangle_t$ converges to 0 as n goes to ∞ because $(\tilde{X}^{(n)})^2$ is integrable and $P(|\tilde{X}^{(n)}| > \epsilon\sqrt{\lambda_n})$ goes to 0. Now, Rebolledo's theorem(Andersen *et al.*, 1993, p. 83) completes the proof. □

On the other hand,

$$\begin{aligned}
\log \frac{d\mathbf{P}^{*(n)}}{d\mathbf{P}^{(n)}}|_{\mathcal{F}_t^{(n)}} &= N_t^{(n)}(\log\lambda_n^* - \log\lambda_n) + (\lambda_n - \lambda_n^*)t + \sum_{i=1}^{N_t^{(n)}} \log g_n(Q_i) \\
&= \sqrt{\lambda_n \mathrm{Var}(\log g_n(Q))} \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)} \\
&\quad + \frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}} \sqrt{\lambda_n}(\log\lambda_n^* - \log\lambda_n + E\log g_n(Q)) \\
&\quad + \lambda_n t \left\{ E\log g_n(Q) - \frac{1}{2}(\log\lambda_n^* - \log\lambda_n)^2 \right. \\
&\quad \left. - \frac{1}{6}(\log\lambda_n^* - \log\lambda_n)^3 + \cdots \right\}.
\end{aligned} \tag{A.1}$$

Since we are assuming $\log\lambda_n^* - \log\lambda_n$ is $(\eta/\sqrt{\lambda_n}) + o(\lambda_n^{-1/2})$ and Assumption 4.1,

$$\log \tilde{Y}_t^{(n)} = \log \frac{d\mathbf{P}^{*(n)}}{d\mathbf{P}^{(n)}}|_{\mathcal{F}_t^{(n)}} \xrightarrow{\mathcal{D}} \sqrt{\frac{\mu^2}{\sigma^2}} W_t^1 + \eta\tilde{W}_t - \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \eta^2\right)t,$$

where W^1 and $\tilde{W}$ are the Brownian motions given in Lemma A.1. Note that $E(\exp[\sqrt{(\mu^2/\sigma^2)}W_t^1 + \eta\tilde{W}_t - (1/2)\{(\mu^2/\sigma^2) + \eta^2\}t]) = 1$.

Now, we need the joint convergence of $\log \tilde{Y}_t^{(n)}$ with the underlying price process $S^{(n)}$. Let us consider the joint convergence of $(N_t^{(n)} - \lambda_n t)/\sqrt{\lambda_n}$, $(1/\sqrt{\lambda_n}) \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)}$ and $\log S_t^{(n)} - (\mu - \sigma^2/2)t$. Define $M_t^{(n)}$, $\tilde{M}_t^{(n)}$ and $\hat{M}_t^{(n)}$ to be $(N_t^{(n)} - \lambda_n t)/\sqrt{\lambda_n}$, $(1/\sqrt{\lambda_n}) \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)}$ and $\log S_t^{(n)} - (\mu - \sigma^2/2)t$, respectively. They are square-integrable martingales and we know from Lemma A.1, $M^{(n)}$ and $\tilde{M}^{(n)}$ jointly converge to $\tilde{W}$ and W^1, independent Brownian motions under $\mathbf{P}$.

Note that Assumption 4.1 implies that $\log g_n(Q) = O_p(\lambda_n^{-1/2})$, because if we take $K_\epsilon > (|\alpha| + 1)/\epsilon$, then

$$P\left(|\sqrt{\lambda_n} \log g_n(Q)| \geq K_\epsilon\right) \leq \frac{\sqrt{\lambda_n E(\log g_n^2(Q))}}{K_\epsilon} \leq \frac{|\alpha| + 1}{K_\epsilon} < \epsilon$$

for a large enough n.

In Assumption 4.1, we have $E(e^{Z^{(n)}} g_n(Q)) = 1$, which is equivalent to $E(e^{(1/\sqrt{\lambda_n})Q}\, g_n(Q)) = e^{(-1/\lambda_n)(\mu - \sigma^2/2)}$. Since

$$e^{-\frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)} = 1 - \frac{\mu}{\lambda_n} + \frac{\sigma^2}{2\lambda_n} + o(\lambda_n^{-1}),$$

$$E\left(e^{\frac{1}{\sqrt{\lambda_n}}Q + \log g_n(Q)}\right) = 1 + E(\log g_n(Q)) + \frac{\sigma^2}{2\lambda_n} + \frac{1}{\sqrt{\lambda_n}} E(Q \log g_n(Q)) + \frac{1}{2}\mathrm{Var}(\log g_n(Q)) + o(\lambda_n^{-1}),$$

we have

$$E(\log g_n(Q)) + \frac{1}{\sqrt{\lambda_n}} E(Q \log g_n(Q)) + \frac{1}{2}\mathrm{Var}(\log g_n(Q)) = -\frac{\mu}{\lambda_n} + o(\lambda_n^{-1}).$$

Thus, because $\lambda_n E(\log g_n(Q)) \longrightarrow -\mu^2/2\sigma^2$ and $\lambda_n \mathrm{Var}(\log g_n(Q)) \longrightarrow \mu^2/\sigma^2$,

$$\sqrt{\lambda_n} E(Q \log g_n(Q)) \longrightarrow -\mu. \tag{A.2}$$

Using Assumption 4.1 and (A.2), we can show that quadratic variations converge as follows.

$$\langle \hat{M}^{(n)}, \hat{M}^{(n)} \rangle_t = \sigma^2 t + \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right)^2 t \underset{n\to\infty}{\longrightarrow} \sigma^2 t,$$

$$\langle M^{(n)}, \hat{M}^{(n)} \rangle_t = \frac{1}{\lambda_n}\left(\mu - \frac{1}{2}\sigma^2\right) t \underset{n\to\infty}{\longrightarrow} 0,$$

$$\langle \tilde{M}^{(n)}, \hat{M}^{(n)} \rangle_t = \frac{E(Q \log g_n(Q))t}{\sqrt{\mathrm{Var}(\log g_n(Q))}} \underset{n\to\infty}{\longrightarrow} -\frac{\mu\sigma}{|\mu|} t = -\mathrm{sgn}(\mu)\sigma t.$$

$\text{sgn}(\mu)$ is 1 if $\mu > 0$, and -1 if $\mu < 0$. Note that we assume $\mu \neq 0$. If we define

$$\hat{M}_{\epsilon,t}^{(n)} = \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)} I_{\{|Z_i^{(n)}|>\epsilon\}} - \lambda_n t E\left(Z^{(n)} I_{\{|Z^{(n)}|>\epsilon\}}\right),$$

then $\hat{M}_\epsilon^{(n)}$ is a martingale containing all jumps of $\hat{M}^{(n)}$ whose sizes are bigger than ϵ in absolute value. Its predictable quadratic variation is $\langle \hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)} \rangle_t = \lambda_n t E({Z^{(n)}}^2 I_{\{|Z^{(n)}|>\epsilon\}})$, and it converges to 0, because $\lambda_n (Z^{(n)})^2$ is integrable and $P(|\sqrt{\lambda_n} Z| > \epsilon\sqrt{\lambda_n})$ goes to 0 as n goes to ∞.

By Rebolledo's theorem,

$$\left(\frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}}, \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i^{(n)}, \log S_t^{(n)} - \left(\mu - \frac{1}{2}\sigma^2\right) t \right) \xrightarrow{\mathcal{D}} (\tilde{W}_t, W_t^1, \sigma B_t),$$

as processes, where $\tilde{W}$ is independent of W^1 and B, $\langle W^1, B\rangle_t = -\text{sgn}(\mu)t$, and they are all Brownian motions under $\mathbf{P}$. Therefore, $W^1 = -\text{sgn}(\mu)B$ and

$$\left(\log \frac{d\mathbf{P}^{*(n)}}{d\mathbf{P}^{(n)}}|_{\mathcal{F}_t^{(n)}}, \log S_t^{(n)} \right) \xrightarrow{\mathcal{D}} \left(-\frac{\mu}{\sigma} B_t + \eta \tilde{W}_t - \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \eta^2\right) t, \log S_t \right).$$

However, for $d\mathbf{P}^{*(n)}/d\mathbf{P}^{(n)}$ to converge to $d\mathbf{P}^*/d\mathbf{P}$, we need $\eta = 0$. When $\eta = 0$, we have

$$\log \tilde{Y}_t^{(n)} \xrightarrow{\mathcal{D}} -\frac{\mu}{\sigma} B_t - \frac{\mu^2}{2\sigma^2} t.$$

Proof of Theorem 4.2

We will need the following assumptions to prove the theorem.

ASSUMPTION A.1. *For a process,* $\tilde{Z}^{(n)}$, *satisfying* $\min(S_{t-}^{(n)}, S_t^{(n)}) \leq \tilde{Z}_t^{(n)} \leq \max(S_{t-}^{(n)}, S_t^{(n)})$,

$$\int_0^T E^* \left(C_S^{(4)}(\tilde{Z}_u^{(n)}, u)(S_u^{(n)})^4 \right) du = O(1), \tag{A.3}$$

$$\int_0^T \int_0^t E^* \left(C_S^{(v)}(S_u^{(n)}, u)(S_u^{(n)})^v \right) dudt = O(1) \tag{A.4}$$

for $v = 2, 3$ *and* 4,

$$\int_0^T \int_0^t E^* \left(C_S^{(v)}(\tilde{Z}_u^{(n)}, u)(\tilde{Z}_u^{(n)})^{v-3}(S_u^{(n)})^3 \right) dudt = O(1) \tag{A.5}$$

for $v = 3, 4$ *and* 5, *and*

$$\int_0^T \int_0^t E^* \left(\left(C_S^{(v)}(\tilde{Z}_u^{(n)}, u)(\tilde{Z}_u^{(n)})^{v-2} - C_S^{(v)}(S_u^{(n)}, u)(S_u^{(n)})^{v-2} \right) (S_u^{(n)})^2 \right) dudt = o(1) \qquad \text{(A.6)}$$

for $v = 3, 4$ *and* 5.

By Itô's formula and Taylor expansion, since $S^{(n)}$ is a pure jump process, we get

$$\begin{aligned} C(S_t^{(n)}, t) = C(S_0, 0) + \int_0^t C_t(S_{u-}^{(n)}, u)du + \sum_{u \le t} \Big\{ C_S(S_{u-}^{(n)}, u)\Delta S_u^{(n)} \\ + \frac{1}{2} C_{SS}(S_{u-}^{(n)}, u)(\Delta S_u^{(n)})^2 + \frac{1}{6} C_{SSS}(S_{u-}^{(n)}, u)(\Delta S_u^{(n)})^3 \\ + \frac{1}{24} C_S^{(4)}(\tilde{Z}_u^{(n)}, u)(\Delta S_u^{(n)})^4 \Big\}, \end{aligned}$$

where $\tilde{Z}^{(n)}$ is a process satisfying $\min(S_{t-}^{(n)}, S_t^{(n)}) \le \tilde{Z}_t^{(n)} \le \max(S_{t-}^{(n)}, S_t^{(n)})$. $C_t(x, t)$ is the first derivative of $C(x, t)$ with respect to t. $C(S_t^{(n)}, t)$ is also written as

$$\begin{aligned} C(S_t^{(n)}, t) = C(S_0, 0) + \int_0^t C_t(S_{u-}^{(n)}, u)du + \int_0^t C_S(S_{u-}^{(n)}, u)dS_u^{(n)} \\ + \int_0^t \frac{1}{2} C_{SS}(S_{u-}^{(n)}, u)d[S^{(n)}, S^{(n)}]_u \\ + \int_0^t \frac{1}{6} C_{SSS}(S_{u-}^{(n)}, u)d[S^{(n)}, S^{(n)}, S^{(n)}]_u \\ + \int_0^t \frac{1}{24} C_S^{(4)}(\tilde{Z}_u^{(n)}, u)d[S^{(n)}, S^{(n)}, S^{(n)}, S^{(n)}]_u, \end{aligned} \qquad \text{(A.7)}$$

where $[S^{(n)}, \ldots, S^{(n)}]^v$ is the vth order optional variation of $S^{(n)}$. It is defined as $[S^{(n)}, \ldots, S^{(n)}]^v = \lim_{\max|t_{i+1}-t_i| \to 0} \sum (S_{i+1}^{(n)} - S_i^{(n)})^v$. Since $S^{(n)}$ is a pure jump process,

$$\left[S^{(n)}, \ldots, S^{(n)} \right]_t^v = \sum_{i=1}^{N_t^{(n)}} (\Delta S_{\tau_i^{(n)}}^{(n)})^v = \sum_{i=1}^{N_t^{(n)}} (S_{\tau_{i-}^{(n)}}^{(n)})^v (\exp(Z_i^{(n)}) - 1)^v,$$

where $\tau_i^{(n)}$ is the time of the ith jump of $S^{(n)}$. By the uniqueness of Doob-Meyer decomposition,

$$\langle S^{(n)}, \ldots, S^{(n)} \rangle_t^{*^v} = E^*(\exp(Z^{(n)}) - 1)^v \int_0^t (S_{u-}^{(n)})^v \lambda_n du.$$

$\langle S^{(n)}, \ldots, S^{(n)}\rangle_t^{*v}$ denotes the compensator of $[S^{(n)}, \ldots, S^{(n)}]^v$ under $\mathbf{P}^{*(n)}$. Using (A.7) and the fact that $[S^{(n)}, \ldots, S^{(n)}]^v - \langle S^{(n)}, \ldots, S^{(n)}\rangle_t^{*v}$ is a martingale under $\mathbf{P}^{*(n)}$,

$$
\begin{aligned}
E^*C\left(S_T^{(n)}, T\right) &= C(S_0, 0) + \int_0^T E^*C_t(S_{u-}^{(n)}, u)du \\
&+E^*\int_0^T E^*(e^{Z^{(n)}}-1)\lambda_n S_{u-}^{(n)} C_S(S_{u-}^{(n)}, u)du \\
&+E^*\int_0^T \frac{1}{2}E^*(e^{Z^{(n)}}-1)^2\lambda_n (S_{u-}^{(n)})^2 C_{SS}(S_{u-}^{(n)}, u)du \\
&+E^*\int_0^T \frac{1}{6}E^*(e^{Z^{(n)}}-1)^3\lambda_n (S_{u-}^{(n)})^3 C_{SSS}(S_{u-}^{(n)}, u)du \\
&+E^*\int_0^T \frac{1}{24}E^*(e^{Z^{(n)}}-1)^4\lambda_n (S_{u-}^{(n)})^4 C_S^{(4)}(\tilde{Z}_u^{(n)}, u)du.
\end{aligned}
$$

Applying Black-Scholes PDE, we get

$$
\begin{aligned}
E^*C(S_T^{(n)}, T) &= C(S_0, 0) + E^*(e^{Z^{(n)}}-1)\lambda_n E^*\int_0^T S_u^{(n)} C_S(S_u^{(n)}, u)du \\
&+\frac{1}{2}(E^*(e^{Z^{(n)}}-1)^2\lambda_n - \sigma^2)E^*\int_0^T (S_u^{(n)})^2 C_{SS}(S_u^{(n)}, u)du \\
&+\frac{1}{6}E^*(e^{Z^{(n)}}-1)^3\lambda_n E^*\int_0^T (S_u^{(n)})^3 C_{SSS}(S_u^{(n)}, u)du \\
&+\frac{1}{24}E^*(e^{Z^{(n)}}-1)^4\lambda_n E^*\int_0^T (S_u^{(n)})^4 C_S^{(4)}(\tilde{Z}_u^{(n)}, u)du.
\end{aligned}
$$

Then, it becomes to

$$
\begin{aligned}
E^*C(S_T^{(n)}, T) &= C(S_0, 0) + \frac{1}{2}\left\{\frac{k_3 + \sqrt{\lambda_n}E(Q^2 \log g_n(Q))}{\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right)\right\} \\
&\times E^*\int_0^T (S_u^{(n)})^2 C_{SS}(S_u^{(n)}, u)du \\
&+\frac{1}{6}\left\{\frac{k_3}{\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right)\right\} E^*\int_0^T (S_u^{(n)})^3 C_{SSS}(S_u^{(n)}, u)du \\
&+\frac{1}{24}o\left(\lambda_n^{-\frac{1}{2}}\right) E^*\int_0^T (S_u^{(n)})^4 C_S^{(4)}(\tilde{Z}_u^{(n)}, u)du.
\end{aligned}
$$

And by (A.3),

$$
E^*C(S_T^{(n)}, T) = C(S_0, 0) + \left\{\frac{k_3 + \sqrt{\lambda_n}E(Q^2 \log g_n(Q))}{2\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right)\right\}
$$

$$\times E^* \int_0^T (S_u^{(n)})^2 C_{SS}(S_u^{(n)}, u)du + \left\{ \frac{k_3}{6\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right) \right\}$$
$$\times E^* \int_0^T (S_u^{(n)})^3 C_{SSS}(S_u^{(n)}, u)du + o\left(\lambda_n^{-\frac{1}{2}}\right).$$

Now, we look at the term $E^* \int_0^T (S_u^{(n)})^2 C_{SS}(S_u^{(n)}, u)du$. Define $K(S_t^{(n)}, t) = C_{SS}(S_t^{(n)}, t)(S_t^{(n)})^2$. It is easy to check that $\{K(S_t^{(n)}, t)\}$ also satisfies the Black-Scholes PDE. Then similarly to $C(S_t^{(n)}, t)$, we get

$$E^* K(S_t^{(n)}, t) = K(S_0, 0) + O\left(\lambda_n^{-\frac{1}{2}}\right) \times E^* \int_0^t (S_u^{(n)})^2 K_{SS}(S_u^{(n)}, u)du$$
$$+ O\left(\lambda_n^{-\frac{1}{2}}\right) \times E^* \int_0^t (S_u^{(n)})^3 K_{SSS}(\hat{Z}_u^{(n)}, u)du,$$

where $\hat{Z}^{(n)}$ is a process satisfying $\min(S_{t-}^{(n)}, S_t^{(n)}) \leq \hat{Z}_t^{(n)} \leq \max(S_{t-}^{(n)}, S_t^{(n)})$. Note that two terms of $O(\lambda_n^{-1/2})$ in the above formula do not depend on t. Thus,

$$\int_0^T E^* \left\{ (S_t^{(n)})^2 C_{SS}(S_t^{(n)}, t) \right\} dt$$
$$= \int_0^T E^* K(S_t^{(n)}, t)dt$$
$$= K(S_0, 0)T + O\left(\lambda_n^{-\frac{1}{2}}\right) \times \int_0^T \int_0^t E^*(S_u^{(n)})^2 K_{SS}(S_u^{(n)}, u)dudt$$
$$+ O\left(\lambda_n^{-\frac{1}{2}}\right) \times \int_0^T \int_0^t E^*(S_u^{(n)})^3 K_{SSS}(\hat{Z}_u^{(n)}, u)dudt.$$

We can deal with $E^* \int_0^T (S_u^{(n)})^3 C_{SSS}(S_u^{(n)}, u)du$ in a similar way. Define $L(S_t^{(n)}, t) = C_{SSS}(S_t^{(n)}, t)(S_t^{(n)})^3$. Since $\{L(S_t^{(n)}, t)\}$ also satisfies the Black-Scholes PDE, we get

$$E^* L(S_t^{(n)}, t) = L(S_0, 0) + \frac{1}{2} \left\{ E^*(e^{Z^{(n)}} - 1)^2 \lambda_n - \sigma^2 \right\} E^* \int_0^t (S_u^{(n)})^2 L_{SS}(S_u^{(n)}, u)du$$
$$+ \frac{1}{2} E^*(e^{Z^{(n)}} - 1)^2 \lambda_n E^* \int_0^t (S_u^{(n)})^2 \left\{ L_{SS}(\breve{Z}_u^{(n)}, u) - L_{SS}(S_u^{(n)}, u) \right\} du$$
$$= L(S_0, 0) + O\left(\lambda_n^{-\frac{1}{2}}\right) \times E^* \int_0^t (S_u^{(n)})^2 L_{SS}(S_u^{(n)}, u)du$$
$$+ O(1) \times E^* \int_0^t (S_u^{(n)})^2 \left\{ L_{SS}(\breve{Z}_u^{(n)}, u) - L_{SS}(S_u^{(n)}, u) \right\} du,$$

where $\breve{Z}^{(n)}$ is a process satisfying $\min(S_{t-}^{(n)}, S_t^{(n)}) \le \breve{Z}_t^{(n)} \le \max(S_{t-}^{(n)}, S_t^{(n)})$. Note that two terms of $O(\lambda_n^{-1/2})$ and $O(1)$ in the above formula do not depend on t. Thus,

$$\begin{aligned}
&\int_0^T E^* \left\{ (S_t^{(n)})^3 C_{SSS}(S_t^{(n)}, t) \right\} dt \\
&= L(S_0, 0)T + O\left(\lambda_n^{-\frac{1}{2}}\right) \times \int_0^T \int_0^t E^* \left\{ (S_u^{(n)})^2 L_{SS}(S_u^{(n)}, u) \right\} dudt \\
&\quad + O(1) \times \int_0^T \int_0^t E^* (S_u^{(n)})^2 \left\{ L_{SS}(\breve{Z}_u^{(n)}, u) - L_{SS}(S_u^{(n)}, u) \right\} dudt.
\end{aligned}$$

By (A.4), (A.5) and (A.6), therefore,

$$\begin{aligned}
&E^* C(S_T^{(n)}, T) \\
&= C(S_0, 0) + \left\{ \frac{k_3 + \sqrt{\lambda_n} E(Q^2 \log g_n(Q))}{2\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right) \right\} \left\{ K(S_0, 0)T + O\left(\lambda_n^{-\frac{1}{2}}\right) \right\} \\
&\quad + \left\{ \frac{k_3}{6\sqrt{\lambda_n}} + o\left(\lambda_n^{-\frac{1}{2}}\right) \right\} \left\{ L(S_0, 0)T + O\left(\lambda_n^{-\frac{1}{2}}\right) + o(1) \right\} + o\left(\lambda_n^{-\frac{1}{2}}\right) \\
&= C(S_0, 0) + \frac{k_3 + \sqrt{\lambda_n} E(Q^2 \log g_n(Q))}{2\sqrt{\lambda_n}} C_{SS}(S_0, 0) S_0^2 T \\
&\quad + \frac{k_3}{6\sqrt{\lambda_n}} C_{SSS}(S_0, 0) S_0^3 T + o\left(\lambda_n^{-\frac{1}{2}}\right) \\
&= C(S_0, 0) + \frac{k_3 T}{\sqrt{\lambda_n}} \left\{ \frac{1}{2} C_{SS}(S_0, 0) S_0^2 + \frac{1}{6} C_{SSS}(S_0, 0) S_0^3 \right\} \\
&\quad + \frac{T}{2\sqrt{\lambda_n}} \left\{ \sqrt{\lambda_n} E(Q^2 \log g_n(Q)) \right\} C_{SS}(S_0, 0) S_0^2 + o\left(\lambda_n^{-\frac{1}{2}}\right).
\end{aligned}$$

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