The Fuzzy Jacobson Radical of a k-Semiring

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Abstract

We define and study the fuzzy Jacobson radical of a $k-$semiring. Also it is shown that the Jacobson radical of the quotient semiring $R/\text{FJR}(R)$ of a $k-$ semiring by the fuzzy Jacobson radical $\text{FJR}(R)$ is semisimple. And the algebraic properties of the fuzzy ideals $\text{FJR}(R)$ and $\text{FJR}(S)$ under a homomorphism from R onto S are also discussed.

Key words : $k-$ semiring, $k-$ideal, fuzzy ideal, fuzzy cosets of a fuzzy ideal, fuzzy maximal(semiprime, prime) ideal, quotient semiring, fuzzy Jacobson radical.

1. Introduction

Chun, Kim and Kim [2] constructed an extension of a $k-$semiring and studied a $k-$ideal of a $k-$semiring. The first author et al.[3] constructed the quotient semiring of a $k-$semiring by a $k-$ideal. Liu [14] introduced and studied the notion of fuzzy ideal of a ring. Following Liu, Mukherjee and Sen [17] defined and examined fuzzy prime ideals of a ring. Kumbhojkar and Bapat [6,7] defined studied the ring R/J of the cosets of the fuzzy ideal J.

Kumar [8]-[12] extended the concept of fuzzy ideal to fuzzy semiprimary (semiprime, primary, prime, maximal) ideals in a ring. Also Malik and Mordeson [15] gave the necessary and sufficient conditions for a fuzzy subring or a fuzzy ideal A of a commutative ring R to be extended to one A^e of a commutative ring S containing R as a subring.

In particular, Kuraoki and Kuroki [13] defined fuzzy quotient rings and gave homomorphism theorems and isomorphism theorem as to fuzzy ideals.

Kim and Park [4] defined and studied the notion of the $k-$fuzzy ideal in a semiring, and they also introduced and studied the quotient semiring R/A of a $k-$semiring R by a $k-$fuzzy ideal A.

Kim [5] defined and investigated a fuzzy maximal ideal of a $k-$semiring and also characterized the quotient $k-$semiring R/A of a $k-$semiring R by a fuzzy maximal ideal A.

Furthermore, Kumar [10] defined and studied the fuzzy Jacobson radical $\text{FJR}(R)$ and the fuzzy prime radical $\text{FPR}(R)$ of a ring R.

The purpose of this paper is to define and study the fuzzy Jacobson radical $\text{FJR}(R)$ of a $k-$ semiring R. In particular, we show that the Jacobson radical of the quotient semiring $R/\text{FJR}(R)$ of a $k-$semiring R by the fuzzy Jacobson radical $\text{FJR}(R)$ is semisimple, and we also discuss the algebraic properties of the fuzzy ideals $\text{FJR}(R)$ and $\text{FJR}(S)$ under a homomorphism from a $k-$semiring R onto a $k-$semiring S.

2. Preliminaries

In this section, we review some definitions and some results which will be used in the later sections.

Definition 2.1. ([2]). A set R together with associative binary operations called addition and multiplication (denoted by + and $\cdot$ respectively) will be called a semiring provided:

(1) addition is a commutative operation,

(2) there exists $0\in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x\in R$,

and

(3) multiplication distributes over addition both from the left and the right.

Definition 2.2. ([2]). A semiring R will be called a $k-$semiring if for any $a, b\in R$ there exists a unique element c in R such that either $b = a + c$ or $a = b + c$ but not both.

Definition 2.3. ([3]). A non-empty subset I of a semiring R is called a subsemiring if I is itself a semiring with respect to the binary operations defined in R. A subsemiring I is called an ideal of R if $r\in R$ and $a\in I$ imply $ar\in I$ and $ra\in I$.

Definition 2.4. ([3]). An ideal I of a semiring R is called a $k-$ideal if $r + a\in I$ implies $r\in I$ for each $r\in R$ and each $a\in I$.

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Let R be a $k-$semiring. Let R' be a set of the same cardinality with $R-\{0\}$ such that $R \cap R' = \emptyset$ and let denote the image of $a \in R-\{0\}$ under a given bijection by a'. Let $\oplus$ and $\odot$ denote addition and multiplication respectively on a set $\bar{R} = R \cup R'$ as follows:

$$a \oplus b = \begin{cases} a+b & \text{if } a, b \in R \\ (x+y)' & \text{if } a = x', b = y' \in R' \\ c & \text{if } a \in R, b = y' \in R', a = y + c \\ c' & \text{if } a \in R, b = y' \in R', a + c = y, \end{cases}$$

where c is the unique element in R such that either $a = y + c$ or $a + c = y$ but not both, and

$$a \odot b = \begin{cases} ab & \text{if } a, b \in R \\ xy & \text{if } a = x', b = y' \in R' \\ (ay)' & \text{if } a \in R, b = y' \in R' \\ (xb)' & \text{if } a = x' \in R', b \in R, \end{cases}$$

It can be shown that these operations are well defined and thus if R is a $k-$semiring, then $(\bar{R}, \oplus, \odot)$ is a ring, called the extension ring of R.

Remark 2.5. Let $\ominus a$ denote the additive inverse of any element $a \in R$ and write $a \oplus (\ominus b)$ simply as $a \ominus b$. Then it is clear that $a' = \ominus a$ and $a = \ominus a'$ for all $a \in R$. Note that if R is a $k-$semiring with identity, then $\bar{R}$ is a ring with identity.

Theorem 2.6. ([2]). Let R be a $k-$semiring, I an ideal, and $I' = \{a' \in R' | a \in I\}$. Then I is a $k-$ideal of R if and only if $\bar{I} = I \cup I'$ is an ideal of the extension ring $\bar{R}$, called the extension ideal of I.

Note that if R is a $k-$semiring and $\bar{R}$ is the extension ring of R, then each ideal of $\bar{R}$ is the extension ideal of a $k-$ideal of R and each $k-$ideal of R is the intersection of its extension ideal and R (see [2]).

Let R be a $k-$semiring and $\bar{R}$ its extension ring. Let I be a $k-$ideal of R and $\bar{I}$ its extension ideal of $\bar{R}$. Define a relation $a \equiv b$ by $a \oplus b' \in \bar{I}$, where $a, b \in R$. Then this relation is an equivalence relation on R. Let $a \oplus I$ be the equivalence class containing $a \in R$ determined by $\equiv$. Let $R/I = \{a \oplus I | a \in R\}$ be the set of all equivalence classes determined by $\equiv$. Then $R/I = \{a \oplus I | a \in R\}$ is a $k-$semiring under the two operations $(a \oplus I) \oplus (b \oplus I) = (a+b) \oplus I$ and $(a \oplus I) \odot (b \oplus I) = (ab) \oplus I$ (see [3]).

Definition 2.7. ([3]). A mapping f from a k-semiring R into a k-semiring S is called a homomorphism if $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Theorem 2.8. ([3]). Let $f : R \rightarrow S$ be a k-semiring homomorphism. Let $\bar{R}$ and $\bar{S}$ be the extension rings of R and S respectively. Define a map $\bar{f} : \bar{R} \rightarrow \bar{S}$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in R \\ f(x')' & \text{if } x \in R' \end{cases}$$

Then $\bar{f}$ is a ring homomorphism, called an extension ring homomorphism of f.

Definition 2.9. ([2]). A $k-$ ideal I of a $k-$semiring R is maximal provided that $I \neq R$ and whenever J is a $k-$ideal of R with $I \subset J \subsetneq R$ then $I = J$.

Theorem 2.10. ([2]). Let $\bar{R}$ be the extension ring of a commutative $k-$ semiring with identity, I a $k-$ideal of R and $\bar{I}$ the extension ideal of I in $\bar{R}$. Then I is a maximal $k-$ideal of R iff $\bar{I}$ is a maximal ideal of $\bar{R}$.

Definition 2.11. Let R be a $k-$semiring and $\bar{R}$ the extension ring of R. The intersection of all maximal ideals of R is called the Jacobson radical of R, denoted by $JR(R)$.

Theorem 2.12. Let R be a $k-$semiring and $\bar{R}$ the extension ring of R. Then $JR(R) = JR(\bar{R}) \cap R$.

Proof.

$$\begin{aligned} JR(R) &= \cap\{M_i \mid M_i \text{ is a maximal k-ideal of } R\} \\ &= \cap\{\bar{M}_i \cap R \mid \bar{M}_i \text{ is a maximal ideal of } \bar{R}\} \\ &= \cap\{\bar{M}_i \mid \bar{M}_i \text{ is a maximal ideal of } \bar{R}\} \cap R \\ &= JR(\bar{R}) \cap R. \end{aligned}$$

□

Definition 2.13. Let R be a $k-$semiring. If $JR(R)=\{0\}$, then R is said to be a semisimple $k-$semiring.

3. Fuzzy ideals of a $k-$semiring

In this section, we review some definitions and some properties of the fuzzy ideals of commutative $k-$ semirings with identity. Throughout this paper unless otherwise all semirings are commutative $k-$semirings with identity.

Definition 3.1. ([4]). A fuzzy ideal of a semiring R is a function $A : R \rightarrow [0,1]$ satisfying the following conditions:
(1) $A(x+y) \geq \min\{A(x), A(y)\}$
(2) $A(xy) \geq \max\{A(x), A(y)\}$ for all $x, y \in R$

Theorem 3.2. ([4]). Let A be a fuzzy ideal of a semiring R. Then $A(x) \leq A(0)$ for all $x \in R$.

Definition 3.3. ([4]). Let A be a fuzzy subset of a semiring R. Then the set $A_t = \{x \in R | A(x) \geq t\} (t \in [0,1])$ is called the level subset of R with respect to A.

Theorem 3.4. ([4]). Let A be a fuzzy ideal of a semiring R. Then the level subset $A_t (t \leq A(0))$ is the ideal of R.

In general, It is not true that if A is a fuzzy ideal of a semiring R, then $A_t (t \leq A(0))$ is $k-$ideal of R, for we have the following example.

Example 3.5. ([4]). Let $R = Z^*$, the set of nonnegative integers and let I=(2,3) be an ideal of R generated by 2 and 3. Define a fuzzy subset A of R by

$$A(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then A is a fuzzy ideal but $A_R = I$ is not a k−ideal of R.

Definition 3.6. ([15]). Let $f : R \to S$ be a homomorphism of semirings and B a fuzzy subset of S. We define a fuzzy subset $f^{-1}B$ of R by $f^{-1}B(x) = B(f(x))$ for all $x \in R$.

Definition 3.7. ([19]). Let $f : R \to S$ be a homomorphism of semirings and A a fuzzy subset of R. We define a fuzzy subset $f(A)$ of S by

$$f(A)(y) = \begin{cases} \text{Sup}\{A(t)|t \in R, f(t) = y\} & \text{if} f^{-1}(y) \neq \emptyset \\ 0 & \text{if} f^{-1}(y) = \emptyset \end{cases}$$

Definition 3.8. ([14]). A fuzzy ideal of a ring R is a function $A : R \to [0, 1]$ satisfying the following axioms

(1) $A(x + y) \geq \min\{A(x), A(y)\}$

(2) $A(xy) \geq \max\{A(x), A(y)\}$

(3) $A(-x) = A(x)$

Let R be a commutative k−semiring, $\bar{R}$ its extension ring. If A is a fuzzy ideal of R such that all its level subsets are k−ideals of R, then $R = \bigcup_{t \in \text{Im}A} A_t$, $\bar{R} = \bigcup_{t \in \text{Im}A} \bar{A}_t$, and $s > t$ if and only if $A_s \subset A_t$ if and only if $\bar{A}_s \subset \bar{A}_t$. Thus we have the following theorem.

Theorem 3.9. ([4]). Let R be a commutative k−semiring, $\bar{R}$ its extension ring. Let A be a fuzzy ideal of R such that all its level subsets are k−ideals of R. Define the fuzzy subset $\bar{A}$ of $\bar{R}$ by for all $x \in \bar{R}$, $\bar{A}(x) = \sup\{t | x \in \bar{A}_t, t \in \text{Im}A\}$. Then $\bar{A}$ is a fuzzy ideal of $\bar{R}$.

Theorem 3.10. ([4]). Let A be as in Theorem 3-9. Then $\bar{A}$ is an extension of A.

Definition 3.11. ([4]). Let A be as in Theorem 3-9 and let $\bar{A}$ be the extension ideal of A The fuzzy subset $x + A : R \to [0, 1]$ defined by $(x + A)(z) = \bar{A}(z \oplus x')$ is called a coset of the fuzzy ideal A.

Theorem 3.12. ([4]). Let R, $\bar{R}$, A and $\bar{A}$ be as in Theorem 3-9. Then $x + A = y + A$ $(x, y \in R)$ if and only if $\bar{A}(x \oplus y') = A(0)$.

Theorem 3.13. ([4]). Let A be as in Theorem 3-9 and $\bar{A}$ the extension of A. If $x + A = u + A$ and $y + A = v + A (x, y, u, v \in R)$, then

(1) $x + y + A = u + v + A$

(2) $xy + A = uv + A$

Theorem 3-13 allows us to define two binary operation "+" and "·" on the set R/A of cosets of the fuzzy ideal A as follows:

$$(x + A) + (y + A) = x + y + A$$

and

$$(x + A) \cdot (y + A) = xy + A$$

It is easy to show that R/A is a k−semiring under these well-defined binary operations with additive identity A and multiplicative identity 1+A. In this case, the semiring R/A is called the factor semiring or the quotient semiring of R by A.

Theorem 3.14. Let R, $\bar{R}$, A and $\bar{A}$ be as in Theorem 3-9. Then $\overline{R/A} \cong \bar{R}/\bar{A}$.

Proof. It is clear. □

Definition 3.15. ([12]). A fuzzy ideal A of R is called a fuzzy prime if $\forall a, b \in R$, either $A(ab) = A(a)$ or else $A(ab) = A(b)$.

Definition 3.16. ([12]). A fuzzy ideal A of R is called a fuzzy semiprime if $A(a^m) = A(a), \forall a \in R$ and $\forall m \in \mathbb{Z}_+$.

Definition 3.17. ([5]). Let A be a fuzzy ideal of a k−semiring R such that all its level subsets are k−ideals of R. A fuzzy ideal A of R is called a fuzzy maximal if $(i) A(0) = 1; (ii) A(e_R) < A(0)$; and (iii) whenever $A(b) < A(0)$ for some $b \in R$, then $\bar{A}(e_R \oplus (rb)') = A(0)$ for some $r \in R$, where e_R is identity of R.

Let $R = Z^*$, the set of nonnegative integers. Define a fuzzy subset α of R by

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in (2) \\ s & \text{if } x \notin (2) \text{ for } s \in [0, 1) \end{cases}$$

Then α is a fuzzy maximal ideal of R.

Kumar[10] defined the fuzzy maximal ideal of a ring as follow;

Definition 3.18. ([10]). A fuzzy ideal A of a ring R is called fuzzy maximal if $(i) A(0) = 1; (ii) A(e_R) < A(0)$; and (iii) whenever $A(b) < A(0)$ for some $b \in R$, then $A(e_R - rb) = A(0)$ for some $r \in R$.

In the following theorem, we have the relation between the fuzzy maximal ideal of a k−semiring and the fuzzy maximal ideal of a ring.

Theorem 3.19. ([5]). Let A be as in Definition 3-17, $\bar{A}$ its extension and $\bar{R}$ the extension ring of R. Then A is a fuzzy maximal ideal of R iff $\bar{A}$ is a fuzzy maximal ideal of $\bar{R}$.

Theorem 3.20. ([5]). Let $f : R \longrightarrow S$ be an epimorphism of k−semirings and B a fuzzy ideal of S. Then B is a fuzzy maximal ideal of S iff $f^{-1}B$ is a fuzzy maximal ideal of R.

4. The fuzzy Jacobson radical of a $k-$semiring

In this section, we define the fuzzy Jacobson radical FJR(R) of a $k-$semiring R and have some properties of the quotient ring $R/$FJR(R) of R by the fuzzy Jacobson radical FJR(R), and obtain some algebraic properties of FJR(R) and FJR(S) under a homomorphism from a $k-$semiring R onto a $k-$semiring S.

Kumar[10] defined the fuzzy Jacobson radical FJR(R) of a ring R as follows : FJR(R) $=$ $\cap\{\theta \mid \theta$ is a fuzzy maximal ideal of $R\}$. Similarly, we define a fuzzy Jacobson radical of a $k-$semiring.

Definition 4.1. Let R be a $k-$semiring. The intersection of all fuzzy maximal ideals of R is called the fuzzy Jacobson radical of R, denoted by FJR(R).

Theorem 4.2. Let α and β be fuzzy ideals of a $k-$semiring R such that all its level subsets are $k-$ideals of R. Then $\alpha\cap\beta$ is a fuzzy ideal of R such that all its level subsets are $k-$ideals of R.

Proof. Let $(\alpha\cap\beta)(x+y)\geq t$ and $(\alpha\cap\beta)(y)\geq t$. for each $t\in[0,1]$. Then $\alpha(x+y)\geq t$ and $\beta(x+y)\geq t$. But $(\alpha\cap\beta)(y)\geq t$. Thus $\alpha(y)\geq t$ and $\beta(y)\geq t$. Since α and β are fuzzy ideals of R such that all its level subsets are $k-$ideals of R, we have $\alpha(x)\geq t$ and $\beta(x)\geq t$. Hence $(\alpha\cap\beta)(x)=\min\{\alpha(x),\beta(x)\}\geq t$. This completes the proof.

□

Theorem 4.3. Let α and β be fuzzy ideals of a $k-$semiring R such that all its level subsets are $k-$ideals of R. Then $\overline{\alpha\cap\beta}=\bar{\alpha}\cap\bar{\beta}$.

Proof. Let x be any element of $\bar{R}$. If $x\in R$, then

$$\begin{aligned}(\bar{\alpha}\cap\bar{\beta})(x) &= \min\{\bar{\alpha}(x),\bar{\beta}(x)\}\\ &= \min\{\alpha(x),\beta(x)\}\\ &= (\alpha\cap\beta)(x)\\ &= \overline{\alpha\cap\beta}(x).\end{aligned}$$

If $x\in R'$ and let $x=y'(y\in R)$. Then

$$\begin{aligned}(\bar{\alpha}\cap\bar{\beta})(x) &= (\bar{\alpha}\cap\bar{\beta})(y')\\ &= \min\{\bar{\alpha}(y'),\bar{\beta}(y')\}\\ &= \min\{\bar{\alpha}(y),\bar{\beta}(y)\}\\ &= (\alpha\cap\beta)(y)\\ &= \overline{\alpha\cap\beta}(y')\\ &= \overline{\alpha\cap\beta}(x).\end{aligned}$$

Hence $\overline{\alpha\cap\beta}(x)=(\bar{\alpha}\cap\bar{\beta})(x)$ for all $x\in\bar{R}$. □

Theorem 4.4. Let R be a $k-$semiring and $\bar{R}$ the extension ring of R. If A be a fuzzy maximal ideal of $\bar{R}$, then there exists a fuzzy maximal ideal α of R such that $\bar{\alpha}=A$.

Proof. Let α be the restriction $A_{|R}$ of A to R. For each $t\in[0,1]$, let $x+y\in\alpha_t$, and $y\in\alpha_t$, then $\alpha(x+y)\geq t$ and $\alpha(y)\geq t$. So $A(x+y)\geq t$ and $A(y)\geq t$, which implies that $x+y\in A_t$ and $y\in A_t$. Since A_t is an ideal of $\bar{R}$, $x\in A_t$. Thus $A(x)\geq t$ and thus $\alpha(x)\geq t$. So $x\in\alpha_t$. Hence α_t is a $k-$ideal of R for all $t\in[0,1]$ and hence $\bar{\alpha}=A$. On the other hand, by Theorem 3-19, α is a fuzzy maximal ideal of R. This completes the proof. □

Lemma 4.5. Let R be a $k-$semiring and $\bar{R}$ the extension ring of R. Let FMI(R) be the collection of all fuzzy maximal ideals of R and FMI$(\bar{R})$ the collection of all fuzzy maximal ideals of $\bar{R}$. Then a mapping f : FMI(R) $\longrightarrow$ FMI$(\bar{R})$ defined by $f(\alpha)=\bar{\alpha}$ is bijective.

Proof. Let $f(\alpha)=f(\beta)$. Then $\bar{\alpha}=\bar{\beta}$ and $\alpha=\beta$. Thus f is one-one. Let A be any element of FMI$(\bar{R})$. Then by Theorem 4-4, there exists an element $\alpha\in$ FMI(R) such that $\bar{\alpha}=A$. Thus $f(\alpha)=\bar{\alpha}=A$. Hence f is onto. This completes the proof. □

Theorem 4.6. Let R be a $k-$semiring and $\bar{R}$ the extension ring of R. Then $\overline{\text{FJR}(R)}=\text{FJR}(\bar{R})$.

Proof. By Theorem 4-3 and Lemma 4-5,

$$\begin{aligned}\overline{\text{FJR}(R)} &= \overline{\cap\{\alpha\mid\alpha \text{ is a fuzzy maximal ideal of } R\}}\\ &= \cap\{\bar{\alpha}\mid\bar{\alpha} \text{ is a fuzzy maximal ideal of } \bar{R}\}\\ &= \text{FJR}(\bar{R}).\end{aligned}$$

□

Theorem 4.7. ([10]). Let R be a commutative ring with identity and let $\mu=$ FJR(R). Then R/μ is semisimple.

Similarly, we have the following theorem.

Theorem 4.8. Let R be a $k-$semiring and let μ $=$ FJR(R). Then R/μ is semisimple.

Proof.

$$\begin{aligned}JR(R/\mu) &= JR(\overline{R/\mu})\cap R/\mu && \text{by Theorem [2-12]}\\ &= JR(\bar{R}/\bar{\mu})\cap R/\mu && \text{by Theorem [3-14]}\\ &= \{0\}\cap R/\mu && \text{by Theorem[4-7]}\end{aligned}$$

□

Lemma 4.9. ([10]). If μ is any fuzzy ideal of a commutative ring with identity R, then $\mu(x-y)=\mu(0)\Longleftrightarrow x+\mu=y+\mu$ for any $x,y\in R$.

Theorem 4.10. ([10]). Let R be a commutative ring with identity and let $\theta = \text{FJR}(R)$. Then $a \in R$ is invertible iff $a + \theta$ is invertible.

Theorem 4.11. Let R be a $k-$ semiring with $e \neq 0$ and let $\mu = \text{FJR}(R)$. Then a is invertible in R iff $a+\mu$ is invertible in R/μ.

Proof. Suppose that a is invertible in R. Then there exists b in R such that $ab = e$. Thus $ab + \mu = (a + \mu)(b + \mu) = e+\mu$, so that $a+\mu$ is invertible in R/μ. Conversely suppose that $a + \mu$ is invertible in R/μ. Then there exists $b + \mu \in R/\mu$ such that $(a+\mu)(b+\mu) = (ab+\mu) = (e+\mu)$. Thus $\bar{\mu}(ab \oplus e') = \mu(0) = \bar{\mu}(0)$, and thus $(ab + \bar{\mu}) = (e + \bar{\mu})$ by Lemma 4-9, which implies that $a + \bar{\mu}$ is invertible in $\bar{R}/\bar{\mu}$. Hence a is invertible in $\bar{R}$ by Theorem 4-10 and hence there exists $c \in \bar{R}$ such that $ac = e$. If $c \in R'$, where $R' = \{x' | x \in R\}$, then $ac \in R'$ and $e \in R$. Since $R \cap R' = \{0\}$, $e = 0$. This is impossible. Therefore $c \in R$, which completes the proof. □

Theorem 4.12. Let α be a fuzzy maximal ideal of a $k-$semiring R. Then α is a fuzzy semiprime ideal of R.

Proof. If α is a fuzzy maximal ideal of a $k-$semiring R, then by Theorem 3-19, $\bar{\alpha}$ is a fuzzy maximal ideal of the extension ring $\bar{R}$. Thus $\bar{\alpha}$ is a fuzzy prime ideal of $\bar{R}$ and thus α is a fuzzy prime ideal of R. Hence α is a fuzzy semiprime ideal of R. □

Theorem 4.13. If α and β be fuzzy semiprime ideals of R, then $\alpha \cap \beta$ is a fuzzy semiprime ideal of R.

Proof. For all $x \in R$ and all $n \in \mathbb{Z}_+$, we have $(\alpha \cap \beta)(x^n) = \min\{\alpha(x^n), \beta(x^n)\} = \min\{\alpha(x), \beta(x)\} = (\alpha \cap \beta)(x)$. So $\alpha \cap \beta$ is fuzzy semiprime ideal of R. □

From Theorem 4-12 and Theorem 4-13, we have the following.

Corollary 4.14. If R is a $k-$semiring, then $\text{FJR}(R)$ is a fuzzy semiprime ideal of R.

Lemma 4.15. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S. Let β_1 and β_2 be fuzzy ideals of S. Then $\varphi^{-1}(\beta_1 \cap \beta_2) = \varphi^{-1}\beta_1 \cap \varphi^{-1}\beta_2$.

Proof. For every $x \in R$, we have

$$\begin{aligned}
\varphi^{-1}(\beta_1 \cap \beta_2)(x) &= (\beta_1 \cap \beta_2)(\varphi(x)) \\
&= \min\{\beta_1\varphi(x), \beta_2\varphi(x)\} \\
&= \min\{\varphi^{-1}\beta_1(x), \varphi^{-1}\beta_2(x)\} \\
&= (\varphi^{-1}\beta_1 \cap \varphi^{-1}\beta_2)(x).
\end{aligned}$$

This completes the proof. □

Lemma 4.16. Let α and β be fuzzy ideals of a $k-$semiring R. Then $(\alpha \cap \beta)_t = \alpha_t \cap \beta_t$ for all $t \in [0, 1]$.

Lemma 4.17. Let α and β be fuzzy ideals of a $k-$semiring R. If α_t and β_t are $k-$ ideals of R for all $t \in [0, 1]$, then $\alpha_t \cap \beta_t$ is a $k-$ ideal of R.

Theorem 4.18. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S. Then $\varphi^{-1}(\text{FJR}(S)) \supseteq \text{FJR}(R)$.

Proof.

$$\begin{aligned}
\varphi^{-1}(\text{FJR}(S)) &= \varphi^{-1}(\cap\{\beta \mid \beta \text{ is a fuzzy maximal ideal of } S\}) \\
&= \cap\{\varphi^{-1}(\beta) \mid \beta \text{ is a fuzzy maximal ideal of } S\} \\
&\supseteq \text{FJR}(R).
\end{aligned}$$

□

From Theorem 4-18, we have the following.

Corollary 4.19. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S. Then $\varphi(\text{FJR}(R)) \subseteq \text{FJR}(S)$.

Proof. By Theorem 4-18, we have $\varphi(\text{FJR}(R)) \subseteq \varphi(\varphi^{-1}(\text{FJR}(S))) = \text{FJR}(S)$. □

Definition 4.20. ([8]). Let R and S be any sets and let $f : R \to S$ be a function. A fuzzy subset A of R is called $f-$invariant if $f(x) = f(y)$ implies $A(x) = A(y)$, where $x, y \in R$.

Lemma 4.21. ([5]). Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S and α an $\varphi-$invariant fuzzy ideal of R such that all its level subsets are $k-$ideals of R. Then α is a fuzzy maximal ideal of R iff $\varphi(\alpha)$ is a fuzzy maximal ideal of S.

Lemma 4.22. ([5]). Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S and let α be an $\varphi-$invariant fuzzy ideal of R such that all its level subsets are $k-$ideals of R. Then $\bar{\alpha}$ is $\bar{\varphi}-$invariant.

Theorem 4.23. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S, $\bar{\varphi}$ the extension of φ, $\text{FMI}(R)$ as in Lemma 4-5 and let $\mu = \text{FJR}(R)$. If μ is $\varphi-$invariant, then every element of $\text{FMI}(R)$ is $\varphi-$invariant.

Proof. Let $\varphi(x) = \varphi(y)$ $(x, y \in R)$. Then $\bar{\varphi}(x) = \bar{\varphi}(y)$. Since μ is $\varphi-$invariant, by Lemma 4-22, $\bar{\mu}$ is $\bar{\varphi}-$invariant. Thus $\bar{\mu}(x) = \bar{\mu}(y)$, so that $\bar{\alpha}(x - y) \doteq \bar{\alpha}(0)$ for all $\alpha \in \text{FMI}(R)$. Thus $\bar{\alpha}(x) = \bar{\alpha}(y)$ and thus $\alpha(x) = \alpha(y)$ for all $\alpha \in \text{FMI}(R)$. This completes the proof. □

From Lemma 4-21 and Theorem 4-23, we have the following.

Theorem 4.24. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S and $\text{FMI}(R)$ as in Lemma 4-5, and let $\mu = \text{FJR}(R)$ be $\varphi-$invariant. Then if $\alpha \in \text{FMI}(R)$, then $\varphi(\alpha) \in \text{FMI}(S)$.

Lemma 4.25. Let φ, μ and FMI(R) be as Theorem 4-24. Then f : FMI$(R) \longrightarrow$ FMI(S) defined by $f(\alpha) = \varphi(\alpha)$ is bijective.

Proof. By Theorem 4-24, f is well defined. Let $f(\alpha) = f(\beta)$. Then $\varphi(\alpha) = \varphi(\beta)$. Since μ is $\varphi-$invariant, by Theorem 4-23, α and β are $\varphi-$invariant, so that $\alpha = \beta$. Thus f is one-one. Let β be any element of FMI(S). Then by Theorem 3-20, there exists $\alpha = \varphi^{-1}(\beta) \in$ FMI(R) such that $f(\alpha) = \varphi(\alpha) = \varphi(\varphi^{-1}(\beta)) = \beta$. Thus f is onto. This completes the proof. □

By Lemma 4-25, we have FMI$(R) = \{\varphi^{-1}(\beta) \mid \beta \in$ FMI$(S)\}$. So we obtain the following Theorem.

Theorem 4.26. Let φ, μ and FMI(R) be as Theorem 4-24 and $\nu =$ FJR(S). Then $\varphi^{-1}(\nu) = \mu$.

Proof. $\varphi^{-1}(\nu) = \varphi^{-1}(\cap\{\beta \mid \beta \in \text{FMI}(S)\}) = \cap\{\varphi^{-1}(\beta) \mid \beta \in \text{FMI}(S)\} = \mu$. □

Corollary 4.27. Let φ, μ and FMI(R) be as Theorem 4-24 and $\nu =$ FJR(S). Then $\varphi(\mu) = \nu$.

Proof. By Theorem 4-26 $\varphi(\mu) = \varphi(\varphi^{-1}(\nu)) = \nu$. □

Theorem 4.28. Let $\varphi : R \longrightarrow S$ be an epimorphism from a $k-$semiring R onto a $k-$semiring S and let $\mu =$ FJR(R). Then

(1) $\varphi(\alpha \cap \beta) \subseteq \varphi(\alpha) \cap \varphi(\beta)$ for any fuzzy ideals α and β of R.

(2) If μ is $\varphi-$invariant, then $\varphi(\alpha \cap \beta) = \varphi(\alpha) \cap \varphi(\beta)$ for any fuzzy ideals α and β of R.

Proof. (1) Straightforward. (2) Let y be any element of S. Then there exists x in R such that $\varphi(x) = y$. Thus $(\varphi(\alpha) \cap \varphi(\beta))(y) = \min\{\varphi(\alpha)(y), \varphi(\beta)(y)\} = \min\{\alpha(x), \beta(x)\} = (\alpha\cap\beta)(x) = \varphi(\alpha\cap\beta)(y)$ by Theorem 4-23. This completes the proof. □

From Theorem 4-6, Theorem 4-26 and Corollary 4-27, we have the following theorem.

Theorem 4.29. Let φ, μ and FMI(R) be as Theorem 4-24 and $\nu =$ FJR(S). Let $\bar{R}$ and $\bar{S}$ be the extension rings of R and S respectively. Then

(1) $\overline{\varphi^{-1}(\nu)} =$ FJR$(\bar{R})$.

(2) $\overline{\varphi(\mu)} =$ FJR$(\bar{S})$.

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