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ON SIMILARITY INVARIANTS OF EP MATRICES

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Abstract. We describe the class of invertible matrices T such that TAT^{-1} is EPr, for a given EPr matrix A of order n. Necessary and sufficient condition is determined for TAT^{-1} to be EP for an arbitrary matrix A of order n.

1. Introduction

All matrices considered here are matrices over the complex field $\mathbb{C}$. In general, the class of EPr matrices is not similarity invariant under $GL(n, \mathbb{C})$, the group of invertible $n \times n$ matrices. This means that, for $A \in EPr$ and $T \in GL(n, \mathbb{C})$, TAT^{-1} need not be EPr. Hence the following questions arise: (i) Given an EPr matrix A of order n, $(r < n)$ for which invertible T, TAT^{-1} is EPr? (ii) Given an arbitrary matrix A of order n, is TAT^{-1} EP for some $T \in GL(n, \mathbb{C})$?

In this paper, for a given EPr matrix A, we characterize the set of all invertible T such that TAT^{-1} is EPr. Specifically, we describe the class $EPr(A) = \{T \in GL(n, \mathbb{C}) : TAT^{-1} \text{ is } EPr\}$. For a matrix A of order n, A^* denotes conjugate transpose of A. In [4] Donald W. Robinson has given the description of the class $C(A) = \{T \in GL(n, \mathbb{C}) : (TAT^{-1})^\dagger = TA^\dagger T^{-1}\}$, for a given square matrix A, where $A^\dagger$ is the Moore–Penrose inverse of A, the unique solution of the equations $AXA = A, XAX = X, (AX)^* = AX$ and $(XA)^* = XA$. It is shown that for a given EPr matrix $A, EPr(A) = C(A)$ and the structure of each element of $EPr(A)$ is determined. Further a necessary

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and sufficient condition is obtained for a square matrix to be range Hermitiable by a $T \in GL(n, \mathbb{C})$.

A matrix A is called EP or range-Hermitian if $R(A) = R(A^*)$, where $R(X)$ denotes the range space of X. An EP matrix of rank r is called EPr matrix. We shall assume familiarity with the basic theory of Moore–Penrose inverse as given in [1]. In the sequel we shall make use of the following known results.

THEOREM 1.1. *Let A be an $n \times n$ matrix and $T \in GL(n, \mathbb{C})$. Then A is EP if and only if T^*AT is EP.*

THEOREM 1.2. *Let A be an $n \times n$ matrix. Then*
(i) A is an EPr matrix if and only if there is a unitary matrix U and a nonsingular $r \times r$ matrix D such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$$

(ii) A is EP if and only if $A^\dagger$ is a polynomial in A [1, p. 173]
(iii) A is EP if and only if $AA^\dagger = A^\dagger A$.

THEOREM 1.3. *Let N be an $n \times n$ Hermitian matrix and A be any $n \times n$ matrix, then $R(NA) \subseteq R(A)$ if and only if $NAA^\dagger = AA^\dagger N$.*

2. Main Results

THEOREM 2.1. *Let A be an EPr matrix and $T \in GL(n, \mathbb{C})$. Then $T \in EPr(A)$ if and only if T^*T commutes with $AA^\dagger$.*

Proof. Assume that $T \in EPr(A)$. Since A is an EPr matrix, $R(A) = R(A^*)$ and $T \in EPr(A)$, implies that TAT^{-1} is EPr. By Theorem 1.1, T^*TA is EPr. Hence, $R(T^*TA) = R(A^*T^*T) = R(A^*) = R(A)$. This implies that $R(T^*TA) \subseteq R(A)$. Hence by Theorem 1.3, $T^*TAA^\dagger = AA^\dagger T^*T$. Each step is reversible and so the converse holds. □

COROLLARY 2.2. *If A and B are two EPr matrices such that $R(A) = R(B)$, then $EPr(A) = EPr(B) = EPr(AB)$.*

Proof. Since $R(A) = R(B), AA^\dagger = BB^\dagger$, hence by Theorem 2.1, $EPr(A) = EPr(B)$. Further A and B are EPr matrices such that $R(A) = R(B)$, by Theorem 3 of [2] it follows that AB is EPr matrix and hence $R(AB){=}R(A) = R(B)$ and $EPr(AB) = EPr(A) = EPr(B)$. □

REMARK 2.3. We note that for a given EPr matrix A, the set $B_A = \{B \mid B \text{ an } EPr \text{ matrix with } R(B) = R(A)\}$ characterized in [2], has the property that $EPr(A) = EPr(X)$ for each element X in B_A.

THEOREM 2.4. *Let A be an EPr matrix and $T \in EPr(A)$. Then T^*T is unitarily similar to a block Hermitian matrix*

$$\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix},$$

where K is of order $r \times r$ and L is of order $(n-r)\times(n-r)$.

Proof. Let A be an EPr matrix. Then by Theorem 1.2 there exist a unitary matrix U and a nonsingular matrix D of order r such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^* \quad \text{and hence } A^\dagger = U \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Let $T \in EPr(A)$. Then by Theorem 2.1,
$T^*TAA^\dagger = AA^\dagger T^*T$. That is,

$$T^*TU \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*T^*T.$$

Pre and post-multiplication by U^* and U respectively, yields

$$U^*T^*TU \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*T^*TU.$$

This shows that $(TU)^*(TU)$ commutes with $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Hence

$$(TU)^*(TU) = \begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix},$$

where K and L are Hermitian matrices of order r and $n-r$ respectively. $\square$

REMARK 2.5. In the above theorem if the unitary matrix U corresponding to the unitarily similarity of T^*T is that of unitary matrix associated with A, then the converse holds.

COROLLARY 2.6. *Let A be an EPr matrix. Then $EPr(A) = C(A)$.*

Proof. By Lemma 1 of [4], $T \in C(A)$ if and only if T^*T commutes with both $AA^\dagger$ and $A^\dagger A$. In particular if A is EP, then $T \in C(A)$ if and only if T^*T commutes with $AA^\dagger$. Hence by Theorem 2.1, for a given EPr matrix $A, T \in EPr(A)$ if and only if $T \in C(A)$ hence $EPr(A) = C(A)$. $\square$

REMARK 2.7. If A is EP, then UAU^{-1} is EP for every $U \in SU(n, \mathbb{C})$, the group of invertible scalar multiples of unitary matrix. Hence $SU(n, \mathbb{C}) \subseteq EPr(A) \subseteq GL(n, \mathbb{C})$.

3. Range Hermitizable Matrices

DEFINITION 3.1. *A square matrix A of order n is said to be range -Hermitizable, if there exists a $T \in GL(n, \mathbb{C})$ such that TAT^{-1} is range-Hermitian.*

It is clear that every range-Hermitian matrix is range-Hermitizable. Thus the class of range-Hermitizable matrices is a generalization of the class of range-Hermitian matrices.

THEOREM 3.2. *A matrix A is range-Hermitizable by $T \in GL(n, \mathbb{C})$ if and only if $R(A^*) = R(T^*TA)$.*

Proof. This can be proved by the similar argument given in Theorem 2.1 and hence omitted. $\square$

COROLLARY 3.3. *A is range-Hermitizable by a $T \in GL(n, \mathbb{C})$ if and only if it is range-Hermitizable by a $P \in HP(n, \mathbb{C})$, the set of Hermitian positive definite matrices.*

Proof. Let A be range-Hermitizable by $T \in GL(n, \mathbb{C})$. Then $B = TAT^{-1}$ is EP. Now $A = T^{-1}BT = (T^{-1}BT^{-1*})(T^*T) = CQ$. Since B is EP, by Theorem 1.1, $T^{-1}BT^{-1*} = C$ is EP. Again $A = CQ$ implies that $Q^{1/2}AQ^{-1/2} = Q^{1/2}CQ^{1/2}$. As C is EP, by Theorem 1.1, $Q^{1/2}CQ^{1/2}$ is also EP. Thus $Q^{1/2}AQ^{-1/2} = PAP^{-1}$ is EP, where $Q^{1/2} = P > 0$, and hence A is range-Hermitizable by $P \in HP(n, \mathbb{C})$. The converse is clear. □

REMARK 3.4. In general, A is range-Hermitizable need not imply that A is EP. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and } T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Then

$$TAT^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is EP. Thus A is range-Hermitizable by T but A is not EP.

The following theorem gives a necessary and sufficient condition for a range-Hermitizable matrix to be range-Hermitian.

THEOREM 3.5. *Let A be range-Hermitizable by $T \in GL(n, \mathbb{C})$. Then A is EP if and only if $T^*TAA^\dagger = AA^\dagger T^*T$.*

Proof. Let $T \in GL(n, \mathbb{C})$ be such that TAT^{-1} is EP. The result automatically holds for any $T \in SU(n, \mathbb{C})$. For, if $T = \alpha U$ for some non-zero scalar α and unitary U, then $TAT^{-1} = |\alpha|^2 UAU^*$ is EP. Therefore A is EP and $T^*T = |\alpha|^2 I$ commutes with $AA^\dagger$. Hence let us assume that $T \notin SU(n, \mathbb{C})$ which implies that $T^*T \neq \alpha I$ for any complex scalar α. Now TAT^{-1} is EP and by Theorem 3.2 we get that $R(A^*) = R(T^*TA)$.

We have that

$$\begin{aligned} A \text{ is } EP &\Leftrightarrow R(A) = R(A^*) \\ &\Leftrightarrow R(A) = R(T^*TA) \\ &\Leftrightarrow T^*TAA^\dagger = AA^\dagger T^*T, \text{ by Theorem 1.3.} \end{aligned}$$

□

Now Theorems 2.1, 3.2 and 3.5 can be combined as follows.

THEOREM 3.6. *Let A be any $n \times n$ matrix and $T \in GL(n, \mathbb{C})$. Then the following are equivalent:*

(i) *A is EP and $T \in C(A)$;*

(ii) *A is EP and $T \in EPr(A)$;*

(iii) *A is range-Hermitizable by T and $T^*TAA^\dagger = AA^\dagger T^*T$;*

(iv) *$R(A^*) = R(T^*TA)$ and $T^*TAA^\dagger = AA^\dagger T^*T$.*

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