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ORTHOGONAL GENERALIZED $(\sigma,\tau)-$DERIVATIONS OF SEMIPRIME $\Gamma-$RINGS

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Abstract. The purpose of this paper is to show some results of M. Bresar and J. Vukman concerning orthogonal derivations of semiprime rings in [3] for $(\sigma,\tau)-$derivations and generalized $(\sigma,\tau)-$derivations in $\Gamma-$rings.

1. Introduction

The notation of a $\Gamma-$ring,where Γ is an additive abelian group, was introduced by Nobusawa in [6]. Barnes [7] then weakened slightly the defining conditions for Nobusawa's $\Gamma-$rings. Following Barnes we say that an additive abelian group M is a $\Gamma-$ring if the following conditions are satisfied for all $a,b,c \in M, \alpha,\beta \in \Gamma$:

(1) $a\alpha b \in M$.

(2) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$.

(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

M is said to be semiprime if $x\alpha M\alpha x = 0$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ holds for all $x,y \in M, \alpha \in \Gamma$. There has been a great deal of work on prime and semiprime $\Gamma-$rings admitting derivations. Recently, in [2], M. Bresar defined a generalized derivation. The concept of generalized derivations cover both the concept of a derivation. Many authors have investigated the properties of prime and semiprime rings with

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generalized derivations. It is natural to look for comparable results on $\Gamma-$rings.

On the other hand, orthogonal derivations has been introduced by M. Bresar and J. Vukman in [3]. They have proved some results concerning orthogonal derivations of semiprime rings, which are related to a theorem of Posner [1] for the product of derivations on a prime ring. M. Soytürk proved some these results for $\Gamma-$rings in [4]. In [5], N. Argaç, et al. introduced the notion of ortogonality for a pair $(D, d), (G, g)$ of generalized derivations on semiprime rings and they gave several necessary and sufficient conditions of (D, d) and (G, g) to be orthogonal. In this paper, our aim is to extend their results to orthogonal $(\sigma, \tau)-$derivations and generalized $(\sigma, \tau)-$derivations in $\Gamma-$rings.

Throughout we assume that M is $2-$torsion free semiprime $\Gamma-$ring, σ, τ automorphisms of M, d, g are $(\sigma, \tau)-$derivations of M such that $g\tau = \tau g, d\tau = \tau d, \sigma g = g\sigma, \sigma d = d\sigma$. We denote a generalized $(\sigma, \tau)-$derivation $D : M \to M$ determined by a $(\sigma, \tau)-$derivation d of M by (D, d) and let generalized $(\sigma, \tau)-$derivations (D, d) and (G, g) such that $G\tau = \tau G, D\tau = \tau D, \sigma G = G\sigma, \sigma D = D\sigma$.

2. Results

DEFINITION 1. *Let M be a $\Gamma-$ ring and $\sigma, \tau : M \to M, \Gamma-$automorphisms of M. An additive mapping $d : M \to M$ is called a $(\sigma, \tau)-$derivation of M if*

$$d(x\alpha y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \text{ for all } x, y \in M, \alpha \in \Gamma.$$

LEMMA 1. [4, Lemma 3.4.1] *Let M be a $2-$torsion free semiprime $\Gamma-$ring and $a, b \in M$. Then the following conditions are equivalent.*

(i) $a\Gamma x\Gamma b = 0$ *for all* $x \in M$.
(ii) $b\Gamma x\Gamma a = 0$ *for all* $x \in M$.
(iii) $a\alpha x\beta b + b\alpha x\beta a = 0$ *for all* $x \in M$ and $\alpha, \beta \in \Gamma$.
If one of these conditions is fulfilled, then $a\Gamma b = b\Gamma a = 0$ *too.*

LEMMA 2. [4, Lemma 3.4.2] *Let M be a semiprime $\Gamma-$ring. Suppose that additive mappings f and h of M into itself satisfy*

$$f(x)\Gamma M h(x) = 0$$

for all $x \in M$. Then $f(x)\Gamma M\Gamma h(y) = 0$ for all $x, y \in M$.

LEMMA 3. *Let M be a $2-$torsion free semiprime $\Gamma-$ring. $(\sigma,\tau)-$derivations d and g of M are orthogonal if and only if*

$$d(x)\Gamma g(y) + g(x)\Gamma d(y) = 0$$

for all $x, y \in M$.

Proof. Suppose that $d(x)\beta g(y) + g(x)\gamma d(y) = 0$ for all $x, y \in M$ and $\beta, \gamma \in \Gamma$. Substituting $y\alpha x$ for y in this equation, we get

$$\begin{aligned}
(2.1)\qquad 0 &= d(x)\beta g(y\alpha x) + g(x)\gamma d(y\alpha x)\\
(2.2)\qquad &= (d(x)\beta g(y) + g(x)\gamma d(y))\alpha\sigma(x) + d(x)\beta\tau(y)\alpha g(x)\\
&\quad + g(x)\gamma\tau(y)\alpha d(x)\\
&= d(x)\beta\tau(y)\alpha g(x) + g(x)\gamma\tau(y)\alpha d(x).
\end{aligned}$$

Substituting $\gamma + \gamma'$ for γ in (2.1), we get

$$g(x)\gamma'\tau(y)\alpha d(x) = 0 \text{ for all } x \in M \text{ and } \alpha, \gamma' \in \Gamma$$

Since τ is an automorphism of M, we have $g(x)\Gamma M\Gamma d(x) = 0$ for all $x \in M$. Hence $g(x)\Gamma M\Gamma d(y) = 0$ for all $x, y \in M$ by Lemma 2.

Substituting $\beta + \beta'$ for β in (2.1), we get

$$d(x)\beta'\tau(y)\alpha g(x) = 0 \text{ for all } x \in M \text{ and } \alpha, \beta' \in \Gamma$$

Since τ is an automorphism of M, we have $d(x)\Gamma M\Gamma g(x) = 0$ for all $x \in M$. Hence $d(x)\Gamma M\Gamma g(y) = 0$ for all $x, y \in M$ by Lemma 2.

Using Lemma 1, we see that d and g are orthogonal.

Conversely, if d and g are orthogonal, then $d(x)\Gamma M\Gamma g(y) = 0 = g(y)\Gamma M\Gamma d(x)$ for all $x, y \in M$. Using Lemma 1, we get $d(x)\Gamma g(y) + g(x)\Gamma d(y) = 0$ for all $x, y \in M$. □

THEOREM 1. *Let M be a $2-$torsion free semiprime $\Gamma-$ring. $(\sigma,\tau)-$derivations d and g of M are orthogonal if and only if one of the following conditions holds.*

(i) $dg = 0$.
(ii) $dg + gd = 0$.
(iii) dg *is a* $(\sigma^2, \tau^2)-$derivation of M.
(iv) There exists $a, b \in M$ and $\alpha, \beta \in \Gamma$ such that $dg\,(x) = a\alpha x + x\beta b$ for all $x \in M$.

Proof. (i)$\Rightarrow$"d and g are orthogonal": Suppose that $dg = 0$. Hence

$$0 = dg(x\alpha y) = dg(x)\alpha\sigma^2(y) + \tau(g(x))\alpha d(\sigma(y)) + d(\tau(x))\alpha\sigma(g(y)) + \tau^2(x)\alpha dg(y)$$

and so

$$\tau(g(x))\alpha d(\sigma(y)) + d(\tau(x))\alpha\sigma(g(y)) = 0, \quad \text{for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Using $g\tau = \tau g, d\tau = \tau d, \sigma g = g\sigma, \sigma d = d\sigma$ and σ, τ are automorphisms of M, we get

$$g(x)\alpha d(y) + d(x)\alpha g(y) = 0, \quad \text{for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Hence d and g are orthogonal by Lemma 3.

"d and g are orthogonal"$\Rightarrow$(i): We have $d(x)\alpha y\beta g(z) = 0$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$0 = d(d(x)\alpha y\beta g(z)) = d^2(x)\alpha\sigma(y)\beta\sigma(g(z)) + \tau(d(x))\alpha d(y)\beta\sigma(g(z)) + \tau(d(x))\alpha\tau(y)\beta dg(z).$$

Since d and g are orthogonal, σ, τ are automorphisms of M, we get

$$d(x)\alpha y\beta dg(z) = 0, \quad \text{for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

Writing $g(z)$ by x in this relation, we get

$$dg(z)\alpha y\beta dg(z) = 0 \ \text{ for all } y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

Since M is semiprime, we obtain that $dg = 0$.

(ii)$\Longrightarrow$"d and g are orthogonal": Suppose that $dg + gd = 0$. Then we have

$$0 = (dg + gd)\,(x\alpha y) = dg\,(x)\,\alpha\sigma^2\,(y) + \tau\,(g\,(x))\,\alpha d\,(\sigma\,(y)) + d\,(\tau\,(x))\,\alpha\sigma\,(g\,(y)) + \tau^2(x)\alpha dg(y) + gd(x)\alpha\sigma^2(y) + \tau(d(x))g(\sigma(y)) + g(\tau(x))\sigma(d(y)) + \tau^2(x)\alpha gd(y)$$

and so

$$2(g(\tau(x))\alpha d(\sigma(y))+d(\tau(x))\alpha g(\sigma(y))=0,$$

for all $x,y\in M$ and $\alpha\in\Gamma$. Since σ,τ are automorphisms of semiprime $2-$torsion free $\Gamma-$ring of M, we conclude that $g(x)\alpha d(y)+d(x)\alpha g(y)=0$, for all $x,y\in M$ and $\alpha\in\Gamma$. Thus d and g are orthogonal by Lemma 3.

"d and g are orthogonal"⇒(ii): We have proved that any orthogonal $(\sigma,\tau)-$derivations satisfy $dg=0$ and by Lemma 2 $gd=0$. Thus (ii) holds as well.

(iii)⇒"d and g are orthogonal": By a direct computation we verify the following identity:

$$dg(x\alpha y)=dg(x)\alpha\sigma^2(y)+\tau(g(x))\alpha d(\sigma(y))$$
$$+d(\tau(x))\alpha\sigma(g(y))+\tau^2(x)\alpha dg(y).$$

Since dg *is a* $(\sigma^2,\tau^2)-$*derivation of* M*, we have*

$$dg(x\alpha y)=dg(x)\alpha\sigma^2(y)+\tau^2(x)\alpha dg(y)\text{ for all }x,y\in M\text{ and }\alpha\in\Gamma.$$

Comparing this two expression of $dg(x\alpha y)$, we obtain

$$\tau(g(x))\alpha d(\sigma(y))+d(\tau(x))\alpha\sigma(g(y))=0$$

and so

$$g(x)\alpha d(y)+d(x)\alpha g(y)=0,\quad\text{for all }x,y\in M\text{ and }\alpha\in\Gamma.$$

By Lemma 3, d and g are orthogonal.

"d and g are orthogonal"⇒(iii): We have proved that any orthogonal derivations satisfy $dg=0$. Thus dg *is a* $(\sigma^2,\tau^2)-$*derivation of* M.

(iv)⇒"d and g are orthogonal": Suppose that there exists $a,b\in M$, $\alpha,\beta\in\Gamma$ such that $dg(x)=a\alpha x+x\beta b$, for all $x\in M$. Then we have

$$(2.3)\quad dg\left(x\gamma y\right)=a\alpha\left(x\gamma y\right)+\left(x\gamma y\right)\beta b\text{ for all }x,y\in M\text{ and }\beta,\gamma\in\Gamma.$$

and

$$(2.4)\quad dg\left(x\gamma y\right)=dg\left(x\right)\gamma\sigma^2\left(y\right)+\tau\left(g\left(x\right)\right)\gamma d\left(\sigma\left(y\right)\right)$$
$$+d\left(\tau\left(x\right)\right)\gamma\sigma\left(g\left(y\right)\right)+\tau^2(x)\alpha dg(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$. Using $dg(x) = a\alpha x + x\beta b$ and $dg(y) = a\alpha y + y\beta b$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$ in (2.4), we get

$$dg(x\gamma y) = a\alpha x\gamma\sigma^2(y) + x\beta b\gamma\sigma^2(y) + \tau(g(x))\gamma d(\sigma(y)) + d(\tau(x))\gamma\sigma(g(y)) + \tau^2(x)\gamma a\alpha y + \tau^2(x)\gamma y\beta b$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since σ and τ are automorphisms of M, we get

$$dg(x\gamma y) = a\alpha x\gamma y + x\beta b\gamma y + \tau(g(x))\gamma d(\sigma(y)) + d(\tau(x))\gamma\sigma(g(y)) + x\gamma a\alpha y + x\gamma y\beta b \text{ for all } x, y \in M \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Using (2.3), we get

$$x\beta b\gamma y + g(\tau(x))\gamma d(\sigma(y)) + d(\tau(x))\gamma g(\sigma(y)) + x\gamma a\alpha y = 0 \tag{2.5}$$

for all $x, y \in M$ and $\beta, \gamma \in \Gamma$. Substituting $y\delta x$ for y in (2.5), we get

$$0 = x\beta b\gamma y\delta x + \tau(g(x))\gamma d(\sigma(y))\delta\sigma^2(x) + g(\tau(x))\gamma\tau(\sigma(y))\delta d(\sigma(x)) + d(\tau(x))\gamma g(\sigma(y))\delta\sigma^2(x) + d(\tau(x))\gamma\tau(\sigma(y))\delta g(\sigma(x)) + x\gamma a\alpha y\delta x$$

for all $x, y \in M$ and $\delta, \beta, \gamma \in \Gamma$. Since σ is an automorphism of M, we get

$$0 = x\beta b\gamma y\delta x + g(\tau(x))\gamma d(\sigma(y))\delta x + g(\tau(x))\gamma\tau(\sigma(y))\delta d(\sigma(x)) + d(\tau(x))\gamma g(\sigma(y))\delta x + d(\tau(x))\gamma\tau(\sigma(y))\delta g(\sigma(x)) + x\gamma a\alpha y\delta x$$

for all $x, y \in M$ and $\delta, \beta, \gamma \in \Gamma$. Using (2.5) in last relation, we get

$$g(\tau(x))\gamma\tau(\sigma(y))\delta d(\sigma(x)) + d(\tau(x))\gamma\tau(\sigma(y))\delta g(\sigma(x)) = 0$$

for all $x, y, z \in M$ and $\gamma, \delta \in \Gamma$.

Since σ and τ are automorphisms of M, we get

$$g(x)\gamma y\delta d(z) + d(x)\gamma y\delta g(z) = 0 \text{ for all } x, y, z \in M \text{ and } \gamma, \delta \in \Gamma.$$

Hence, $g(x)\Gamma M\Gamma d(z) + d(x)\Gamma M\Gamma g(z) = 0$ for all $x, z \in M$ and so d and g are orthogonal.

"d and g are orthogonal"$\Rightarrow$(iv): We have proved that any orthogonal derivations satisfy $dg = 0$. Since $0 = dg(x) = 0\alpha x + x\beta 0$ for $0 \in M$, $\alpha, \beta \in \Gamma$ and for all $x \in M$, we see (iv). □

COROLLARY 1. *Let M be a prime Γ−ring of characteristic not two. If (σ,τ)−derivations d and g of M satisfy one of the conditions in Theorem 1, then either $d=0$ or $g=0$.*

COROLLARY 2. *Let M be 2−torsion free semiprime Γ−ring. If d is (σ,τ)−derivation of M such that d^2 is (σ^2,τ^2) −derivation then $d=0$.*

COROLLARY 3. *Let M be 2−torsion free semiprime Γ−ring and d be (σ,τ)−derivation of M. If there exists $a,b\in M$ and $\alpha,\beta\in\Gamma$ such that $d^2(x)=a\alpha x+x\beta b$ for all $x\in M$ then $d=0$.*

LEMMA 4. *Let M be a 2−torsion free semiprime Γ−ring. If (σ,τ)-generalized derivations (D,d) and (G,g) of M are orthogonal, then the following relations hold.*

(i) $D(x)\Gamma G(y)=G(x)\Gamma D(y)=0$, hence $D(x)\Gamma G(y)+G(x)\Gamma D(y)=0$, for all $x,y\in M$.
(ii) d and G are orthogonal and $d(x)\Gamma G(y)=G(y)\Gamma d(x)=0$, for all $x,y\in M$.
(iii) g and D are orthogonal and $g(x)\Gamma D(y)=D(y)\Gamma g(x)=0$, for all $x,y\in M$.
(iv) d and g are orthogonal.
(v) $dG=Gd=0$, $gD=Dg=0$ and $DG=GD=0$.

Proof. (i) By the hypothesis $D(x)\Gamma M\Gamma G(y)=0$, for all $x,y\in M$. Hence we get $D(x)\Gamma G(y)=0=G(x)\Gamma D(y)$, for all $x,y\in M$ by Lemma 1. Thus $D(x)\Gamma G(y)+G(x)\Gamma D(y)=0$, *for all* $x,y\in M$.

(ii) Since $D(x)\Gamma G(y)=0$ and $D(x)\Gamma M\Gamma G(y)=0$, for all $x,y\in M$, we have

$$0=D(r\beta x)\alpha G(y)=(D(r)\beta\sigma(x)+\tau(r)\beta d(x))\alpha G(y)$$
$$=\tau(r)\beta d(x)\alpha G(y).$$

By the semiprimenessly of M, then

$$d(x)\alpha G(y)=0,\quad \text{for all } x,y\in M \text{ and } \alpha\in\Gamma. \tag{2.6}$$

Writing $x\beta r$ by x in (2.6) , we get

$$0=d(x\beta r)\alpha G(y)=(d(x)\beta\sigma(r)+\tau(x)\beta d(r))\alpha G(y)$$

and so

$$d(x)\beta\sigma(r)\alpha G(y) = 0, \quad \text{for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma.$$

Since σ is an automorphism of M, we have $d(x)\Gamma M\Gamma G(y) = 0$, for all $x, y \in M$. Therefore by Lemma 1, we obtain that $G(y)\Gamma d(x) = 0$, for all $x, y \in M$, which shows (ii).

(iii) Using the same arguments in the proof of (ii), we prove (iii).

(iv) Since $D(x)\Gamma G(y) = 0$, for all $x, y \in M$, we have

$$\begin{aligned} 0 &= D(x\beta z)\alpha G(y\gamma w) \\ &= (D(x)\beta\sigma(z) + \tau(x)\beta d(z))\alpha(G(y)\gamma\sigma(w) + \tau(y)\gamma g(w)) \\ &= D(x)\beta\sigma(z)\alpha G(y)\gamma\sigma(w) + D(x)\beta\sigma(z)\alpha\tau(y)\gamma g(w) \\ &\quad + \tau(x)\beta d(z)\alpha G(y)\gamma\sigma(w) + \tau(x)\beta d(z)\alpha\tau(y)\gamma g(w). \end{aligned}$$

Using (ii) and (iii), we arrive at

$$\tau(x)\beta d(z)\alpha\tau(y)\gamma g(w) = 0, \quad \text{for all } x, y, z, w \in M \text{ and } \beta, \alpha, \gamma \in \Gamma.$$

Since τ is an automorphism of M, we see that

$$d(z)\alpha y\gamma g(w) = 0, \quad \text{for all } y, z, w \in M \text{ and } \alpha, \gamma \in \Gamma.$$

Thus, d and g are orthogonal.

(v) We know that d and G are orthogonal by (ii). Hence

$$0 = G(d(x)\alpha z\beta G(y)) = Gd(x)\alpha\sigma(z)\beta\sigma(G(y)) + \tau(d(x))\alpha g(zG(y)).$$

Using $d\tau = \tau d, G\sigma = \sigma G$ and d and g are orthogonal, we obtain that

$$Gd(x)\alpha\sigma(z)\beta G(\sigma(y)) = 0$$

and so

$$Gd(x)\alpha z\beta G(y) = 0, \quad \text{for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

Replacing y by $d(x)$ in the above relation, we get $Gd = 0$ by the semipriness of R.

Similarly, since each of $d(G(x)\alpha z\beta d(y)) = 0, D(g(x)\alpha z\beta D(y)) = 0, g(D(x)\alpha z\beta g(y)) = 0$ and $G(D(x)\alpha z\beta G(y)) = 0$ holds for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, we have $dG = Dg = gD = DG = GD = 0$, respectively. □

THEOREM 2. *Let M be a $2-$torsion free semiprime Γ-ring. (σ,τ)-generalized derivations (D,d) and (G,g) of M are orthogonal if and only if one of the following conditions holds.*

(i)a) $D(x)\Gamma G(y)+G(x)\Gamma D(y)=0,$ *for all* $x,y\in M.$
b) $d(x)\Gamma G(y)+g(x)\Gamma D(y)=0,$ *for all* $x,y\in M.$
(ii) $D(x)\Gamma G(y)=d(x)\Gamma G(y)=0,$ *for all* $x,y\in M.$
(iii) $D(x)\Gamma G(y)=0,$ *for all* $x,y\in M$ *and* $dG=dg=0.$

Proof. (i)⇒"(D,d) and (G,g) are orthogonal": Replacing $x\alpha z$ by x in (a), we get

$$\begin{aligned}0&=D(x\alpha z)\beta G(y)+G(x\alpha z)\beta D(y)\\&=D(x)\alpha\sigma(z)\beta G(y)+G(x)\alpha\sigma(z)\beta D(y)\\&\quad+\tau(x)\alpha(d(z)\beta G(y)+g(z)\beta D(y)).\end{aligned}$$

Using the hypothesis (b) and σ is an automorphism of M, we see that

$$D(x)\alpha z\beta G(y)+G(x)\alpha z\beta D(y)=0,\quad \text{for all } x,y,z\in M \text{ and } \alpha,\beta\in\Gamma.$$

Thus (D,d) and (G,g) are orthogonal by Lemma 1.

"(D,d) and (G,g) are orthogonal"⇒(i) is proved by Lemma 4.

(ii)⇒"(D,d) and (G,g) are orthogonal": Since $D(x)\Gamma G(y)=0$ and $d(x)\Gamma G(y)=0,$ for all $x,y\in M$, we have

$$\begin{aligned}0=D(x\alpha z)\beta G(y)&=D(x)\alpha\sigma(z)\beta G(y)+\tau(x)\alpha d(z)\beta G(y)\\&=D(x)\alpha\sigma(z)\beta G(y)\end{aligned}$$

and so $D(x)\Gamma M\Gamma G(y)=0,$ for all $x,y\in M$. We get the result by Lemma 1.

"(D,d) and (G,g) are orthogonal"⇒(ii) is proved by Lemma 4.

(iii)⇒"(D,d) and (G,g) are orthogonal": Assume that $D(x)\Gamma G(y)=0,$ for all $x,y\in M$ and $dG=dg=0$. Thus, we have

$$\begin{aligned}0=dG(x\alpha y)&=dG(x)\alpha\sigma^2(y)+\tau(G(x))\alpha d(\sigma(y))\\&\quad+d(\tau(x))\alpha\sigma(g(y))+\tau^2(x)\alpha dg(y).\end{aligned}$$

Using $d\tau=\tau d,\sigma g=g\sigma,\sigma d=d\sigma,G\tau=\tau G$ and σ,τ are automorphisms of M, we obtain that

$$G(x)\alpha d(y)+d(x)\alpha g(y)=0,\quad \text{for all } x,y\in M \text{ and } \alpha\in\Gamma.$$

We know that d and g are orthogonal by Theorem 1 (i). Hence, we arrive at

$$G(x)\alpha d(y) = 0, \quad \text{for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

If we take $x\beta z$ instead of x in the last equation and using d and g are orthogonal, we conclude that

$$G(x)\beta\sigma(z)\alpha d(y) = 0, \quad \text{for all } x, y \in M \text{ and } \beta, \alpha \in \Gamma.$$

By Lemma 1, we have $d(y)\Gamma G(x) = 0$, for all $x, y \in M$, which satisfies (ii). Then (ii) follows (D, d) and (G, g) are orthogonal.

"(D, d) and (G, g) are orthogonal"⇒(iii): By Lemma 4, $D(x)\Gamma G(y) = 0$ and $dG = 0$. Therefore, d and g are orthogonal by Theorem 1, and so, $dg = 0$. □

REFERENCES

[1] E. C. Posner, "Derivations in prime rings", Proc. Amer. Soc. 8, 1957, 1093-1100

[2] M. Bresar, "On the distance of the composition of two derivations to the generalized derivations", Glasgow Math. J., 33 (1991), 89-93.

[3] M. Bresar and J. Vukman, "Orthogonal derivation and extension of a theorem of Posner", Radovi Matematicki, 5 (1989), 237-246.

[4] M. Soytürk, "Some Generalizations in prime ring with derivation", Ph. D. Thesis, Cumhuriyet Univ. Graduate School of Natural and Applied Sci. Dep. of Math., 1994.

[5] N. Argaç, N. Atsushi and E. Albaş, "On orthogonal generalized derivations of semiprime rings", Turk J. Math., 28 (2004), 185-194.

[6] N. Nobusawa, "On a generalization of the ring theory", Osaka J. Math., (1964), 81-89.

[7] W. E. Barnes, "On the $\Gamma-$rings of Nobusawa", Pasific J. Math., 18 (1966), 411-422.

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