# Fuzzy Linear Parameter Varying Modeling and Control of an Anti-Air Missile

**Ali Reza Mehrabian and Seyed Vahid Hashemi**

**Abstract:** An analytical framework for fuzzy modeling and control of nonlinear systems using a set of linear models is presented. Fuzzy clustering is applied on the aerodynamic coefficients of a missile to obtain an optimal number of rules in a Tagaki-Sugeno fuzzy rule-set. Next, the obtained membership functions and rule-sets are applied to a set of linear optimal controllers towards extraction of a global controller. Reported simulations demonstrate the performance, stability, and robustness of the controller.

**Keywords:** Flight control, fuzzy systems, gain scheduling, linear parameter-varying systems.

## 1. INTRODUCTION

Gain-scheduling (GS) is a well-established engineering approach to controller design for time varying and/or non-linear plants [1]. The two main advantages of GS controllers is that this approach lets the designer of the control algorithm apply well-established linear control theories in her design; in addition, in GS control method, the controller's parameters, which are being designed off-line, can change very quickly in response to alteration in the plant's dynamics. The main drawback of GS is that it naturally obliterates nonlinear, time-varying nature of the plant, since the global controller is found by gain interpolation over the operation region. This causes loss of performance and in some cases, loss of stability of the closed-loop system. Another challenge of GS design method is that the procedure for finding appropriate operating points of the nonlinear system is a tedious ad-hoc procedure, which depends on the designer's experience and knowledge from the system as well as the closed-loop system's performance [2,3]. However, mentioned weaknesses in classic GS can be changed into advantages by combining GS with Tagaki-Sugeno (TS) fuzzy systems.

This study presents a fuzzy-clustering-based procedure for nonlinear control-oriented identification of a nonlinear anti-air missile [4] by combining TS fuzzy systems and GS design procedure. To be precise, fuzzy clustering is applied to the nonlinear aerodynamic coefficients in the equations of motion of the flight vehicle in order to obtain an optimal set of fuzzy rules that can do approximation over the operating envelope of the system. Next, by combining the acquired fuzzy rule-set and linear-parameter-varying (LPV) modeling approach, an approximate fuzzy model of the nonlinear system is obtained. The obtained fuzzy rule-set includes a set of linear continuous dynamic systems (i.e., transfer functions) in its consequent, where a soft interpolation is applied between them by denominating membership of the measured input signal to the antecedent of the rule set. In this way, there is a gain-scheduled model of the nonlinear model in hand, which is then employed for fuzzy controller design by combining with a set of linear time invariant (LTI) optimal state-vector feedback controllers. The fuzzy GS controller is designed based on the equilibrium points and the scheduling procedure, which are obtained by finding the optimal number of clusters and membership functions of the fuzzy rule set, respectively.

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## 2. THE MISSILE MODEL AND DESIGN REQUIREMENTS

Considering a rigid airframe, the missile model in the longitudinal plane is obtained as [2,3]:

$$\dot{\alpha} = K_\alpha M[c_n(\alpha, M)\alpha + d_n(\alpha, M)\delta]\cos\alpha + q, \quad (1)$$

$$\dot{q} = K_q M^2[c_m(\alpha, M)\alpha + d_m(\alpha, M)\delta], \quad (2)$$

$$a_N = K_z M^2[c_n(\alpha, M)\alpha + d_n(\alpha, M)\delta], \quad (3)$$

$$\dot{M} = \frac{1}{v_s}\left[-|a_N|\sin|\alpha| + A_x M^2\cos\alpha\right], \quad (4)$$

where the states are angle of attack $\alpha$, pitch rate $q$, and Mach number, $M$. The system's input is $\delta$ and its output is normal acceleration, $a_N$; $v_s$ is the speed of sound. Note that the actuator has a second order dynamics with natural frequency of $\omega_a$ and damping ration of $\zeta$ [2].

The numerical values of the system coefficients: $K_\alpha, K_q, K_z, A_x, \omega_a, \zeta$, and aerodynamic coefficients: $c_n, c_m, d_n$, and $d_m$, can be found elsewhere [2,3]. Using Taylor linearization, a quasi-LPV (QLPV) description of the nonlinear missile is obtained as [3]:

$$\frac{d}{dt}x^o = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix} x^o + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \delta, \quad (5)$$

$$a_N = \begin{bmatrix} c_1 & 0 \end{bmatrix} x^o + d\delta, \quad (6)$$

where $x^o = [\alpha, q]^T$; the numerical values of the coefficient of the linearized plant can be found in [3]. The operating range (flight envelope) of the missile is specified by $[\alpha(t), M(t)]$ such that $-30^\circ \le \alpha \le 30^\circ$ and $2 \le M \le 4$. Note that plant linearizations at constant operating points in the range considered in the flight envelope, exhibit *non-minimum phase zeros* and for small angle of attack exhibit *positive real-part eigenvalues* (unstable open-loop system) [2,3].

Consider the missile flying at the altitude of 6100m, the performance goals for the closed-loop system are as follows [2,3]:

1) Maintain stability over the operating range;
2) Track step commands in $a_N^{com}$, with time constant no grater than 0.35sec, maximum overshoot no greater that 10~20 per cent, and steady-state error no greater than 1 per cent;
3) Maximum tail deflection rate for 1$g$ step command in $a_N^{com}$ should not exceed 25deg/sec.

## 3. FUZZY MULTI-MODEL IDENTIFICATION OF THE MISSILE

A fuzzy clustering-based procedure is applied for the selection of operating points and the design of interpolation mechanism, which are the two main difficulties in the classic gain-scheduling control system design as suggested in [5]. As the first step, fuzzy Gustafson-Kessel (GK) clustering [6] is applied on the aerodynamic coefficients $c_n, c_m$ (available from either look-up tables or fitted polynomials); this is due to the fact that the aerodynamic coefficients play the key role in the instability of the missile airframe and have considerable contribution on the nonlinear behavior of the system. To generate data points for clustering, a grid is applied to the operating envelope that has a step of 0.1 for $M$ and 0.01(rad) for $\alpha$; next, values of the aerodynamics coefficients are evaluated for the grid point data set, which are employed for fuzzy clustering. Finally, membership functions are obtained by projecting the clusters on to the grid point set variables $(M, \alpha)$.

### 3.1. Clustering validation

In this paper two practical approaches are applied to determine the appropriate number of clusters that are EVM [5] and XB [7]. The EVM approach is more significant than XB since it is a measure of the squared difference between the outputs of the TS fuzzy model and the corresponding true values in the data set. Clustering validation results are depicted in Fig. 1, which shows an obvious relationship between the accuracy and the complexity of the models. The model with higher complexity has more accuracy; however, both indices show local minima for $c = 6, 9, 11$. Models with 9 and 11 clusters have lower error, in modeling, than the model with 6 clusters. On the other hand, the model with 11 clusters does not show a considerable performance improvement comparing with the model with 9 clusters; thus, a set of nine rules are employed (according to nine clusters) for modeling the nonlinear system; eight clusters are for stable region and one cluster represents the unstable region. The clustered flight envelope is presented in Fig. 2.

Since membership functions are obtained by orthogonal projection of the clusters, the approximation of fuzzy partition is more accurate in regions with clusters laid parallel to one of vertical or horizontal axes. Hence, for unstable region, where the orientation of this cluster is nearly diagonal, the membership function is modified and linear membership functions are applied for this area, which adds three membership functions.

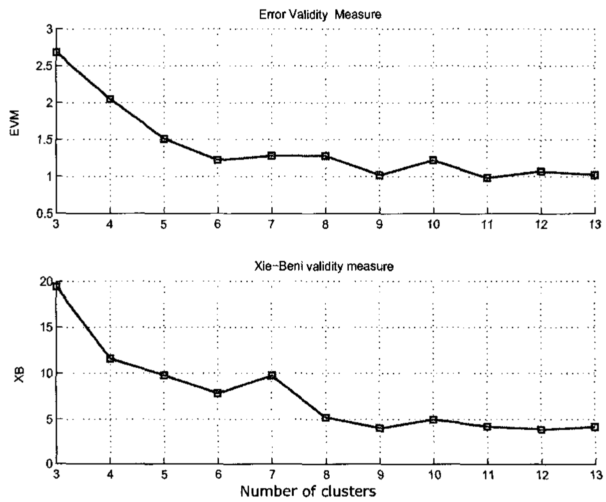


Fig. 1. Performance of the validity measures as a function of the number of clusters.

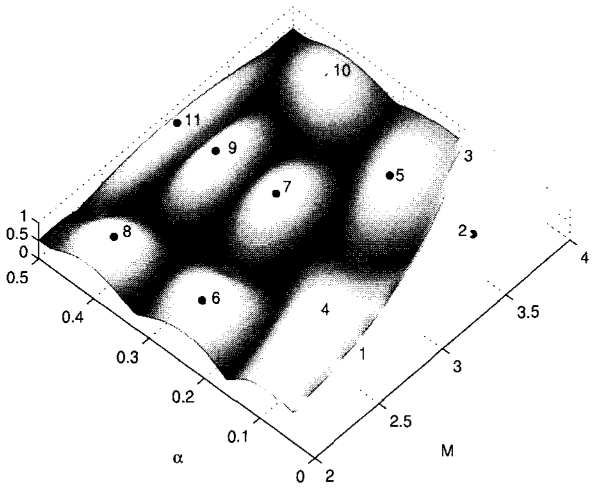


Fig. 2. Results of fuzzy clustering of the flight envelope.

3.2. Development of the fuzzy multi-model for the missile

The multi-dimensional fuzzy partition is approximated by the Cartesian product of one-dimensional membership functions defined in the operating envelope, using TS fuzzy model with the following rule-base:

$$\begin{aligned} &Rule_i: \ IF \ u_1 = \Delta_{i1} \ AND \ u_2 = \Delta_{i2} \\ &THEN \Theta = \Theta_i, \forall i = 1,2,\ldots,c, \end{aligned} \tag{7}$$

where $\Theta_i = [a_{11}, a_{21}, b_1, b_2, c_1, d]_i$; $u_1$ and $u_2$ represent scheduling variables $(M, \alpha)$; $\Delta_{ij}$ denotes the fuzzy set used for input $u_i$ in rule $i$, and the number of clusters is represented by $c$ [8].

## 4. FUZZY LINEAR PARAMETER VARYING CONTROL OF THE MISSILE

4.1. Development of the closed-loop dynamics

Consider the linear plant, a QLPV model for longitudinal dynamics of the missile, equations (5), (6); a typical control problem is to design an autopilot that accepts a normal acceleration command, from the guidance system, and produces the output normal acceleration to track the command signal. The subject of interest is to track an acceleration command, therefore, it is preferred to use the acceleration error as a state variable instead of angle of attack. It is easy to show that the single fifth-order state-space equation defining the linear closed-loop system is [3]:

$$\frac{d}{dt} x^c = A^c x^c + B^c \delta_c + E a_N^{com}, \tag{8}$$

where $x^c = [z, q, \delta, \dot{\delta}, \int z]^T$ and $z$ is the tracking error. The state-space matrices are:

$$A^c = \begin{bmatrix} a_{11} & -c_1 & (c_1 b_1 - a_{11} d) & -d & 0 \\ \frac{-a_{21}}{c_1} & 0 & b_2 - \frac{a_{21} d}{c_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\omega_a^2 & -2\zeta\omega_a & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B^c = \begin{bmatrix} 0 & 0 & 0 & \omega_a^2 & 0 \end{bmatrix}^T,$$

$$E = \begin{bmatrix} -a_{11} & \frac{a_{21}}{c_1} & 0 & 0 & 0 \end{bmatrix}^T.$$

4.2. Controller design at each operating point

In this paper linear quadratic tracking (LQT) approach is used for controller design. LQT controller picks up $\delta_c(t)$ so that the cost function shown in (9) is minimized:

$$J_0 = \int_0^\infty \left\{ [y^c - r]^T Q [y^c - r] + \delta_c^2 R \right\} dt, \tag{9}$$

where $Q$ and $R$ are the weight matrices and $y^c = \left[ z, \int z, \dot{\delta} \right]^T$. The reference signal is set to be a constant vector: $r = [0, \varepsilon, 0]^T$.

It is desired that the closed-loop system have zero tracking error but the integral of the tracking error should not converge to zero, otherwise the system will have a zero time constant and the control effort, $\delta_c$, will reach infinity. Thus, a constant value is assigned to the tracking error in order to satisfy the required time constant and prevent generation of an infinite command signal by the controller. The actuator rate is also included in the performance index to limit its maximum value as required in the performance objectives. Determining the best possible control signal is a trade off between the system's performance and the control effort. $Q$ and $R$ are the weights which have influence on the mentioned trade off. The weight matrices as well as the nonzero element in the reference signal, $\varepsilon$, are chosen by trail and error at each design point; however, there are three restrictions: (1) $R \neq 0$, otherwise the solution will include large components in the controller gains; (2) the $Q$ must also be nonnegative definite; and (3) $\varepsilon > 0$, otherwise the optimization problem will not have a solution.

The control signal has the following form:

$$\delta_c = -K x^c + K_{ff} a_N^{com}, \tag{10}$$

where $K$ and $K_{ff}$ are state-vector feedback and feedforward gains, respectively. The procedure for finding the feedback and feedforward gains is straight forward, which can be found in the literature [9].

4.3. Fuzzy gain-scheduled controller

Construction of the fuzzy linear parameter varying controller is not difficult, when the controller structure and gains (developed in the Section 4.2) and the fuzzy interpolation mechanism (developed in the Section 3) are in hand. The fuzzy gain-scheduler is a TS fuzzy model with a rule-base given as:

$$\begin{aligned} Rule_i: \ & IF \ \ u_1 = \Delta_{i1} \ \ AND \ \ u_2 = \Delta_{i2} \\ & THEN \ \ \Xi = \Xi_i, \forall i = 1,2,\ldots,c, \end{aligned} \tag{11}$$

where $\Xi = [K, K_{ff}]$. The only difference between the fuzzy gain-scheduled controller and the developed fuzzy QLPV model, is in the output of the fuzzy rule-set.

## 5. SIMULATION STUDIES

In Fig. 3 families of step responses for perturbations in the aerodynamic coefficients $c_n$, $c_m$, $d_n$ and $d_m$ are depicted. The performance, stability, and the robustness of the system is obvious. Tail deflection rate for the designed controller is depicted in Fig. 4 for $1g$ step input with three values of initial Mach number: $M_0 = 2,3,4$; In each case the specified rate limit of 25deg/sec is satisfied.

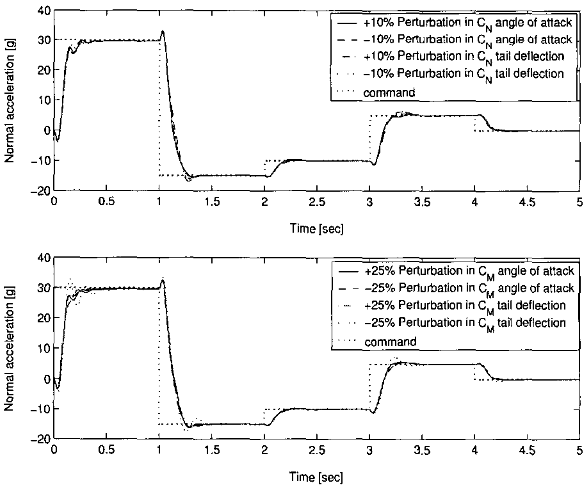


Fig. 3. Closed-loop system's response to a sequence of step commands in acceleration with perturbations in aerodynamic coefficients.

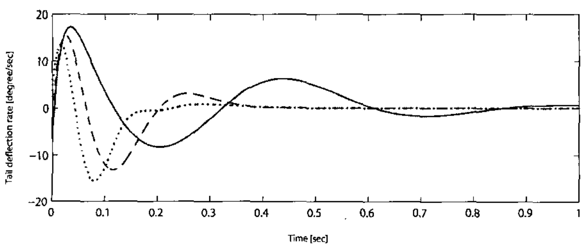


Fig. 4. Tail-deflection rate for different Mach numbers; Mach 2: solid line (–); Mach 3: dashed line (- -); Mach 4: dotted line (..).

## 6. CONCLUSIONS

The problem of nonlinear fuzzy linear parameter varying (LPV) and control of a nonlinear unstable flight vehicle is studied in this paper. It is shown that the fuzzy systems are able to represent proper model for the dynamic system based on the granular data provided from the aerodynamic coefficients. It is believed that the proposed procedure can solve the two main drawbacks of the classic gain-scheduling controller design approach, which are (1) finding the proper set points for designing of linear time-invariant (LTI) controllers and (2) obtaining the appropriate scheduling procedure between the design points. The studied benchmark is different is several ways from the previously studied application [5]. First, the studied system is unstable and nonminimum phase in some parts of the flight envelope. Second, for the first time, fuzzy sets are employed for the development of the fuzzy model using LPV model of a nonlinear system in this paper.

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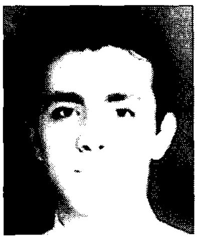

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