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UPPER BOUNDS FOR BIVARIATE BONFERRONI-TYPE INEQUALITIES USING CONSECUTIVE EVENTS

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ABSTRACT. Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, denote the numbers of those A_i's and B_j's which occur. We establish new bivariate Bonferroni-type inequalities using consecutive events and deduce a known result.

1. Introduction

Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, denote the numbers of those A_i's and B_j's which occur. Put $S_{0,0} = 1$ and, for integers r and t, set

$$S_{r,t} = \sum\sum P(A_{i_1}A_{i_2}\cdots A_{i_r}B_{j_1}B_{j_2}\ \cdots B_{j_t}), \tag{1}$$

where the summation is over all subscripts satisfying $1 \le i_1 < i_2 < \cdots < i_r \le m$ and $1 \le j_1 < j_2 < \cdots < j_t \le n$, $0 \le r \le m$ and $0 \le t \le n$ (we abbreviate $A \cap B$ as AB and an empty intersection is the sample space). We can easily prove that $S_{r,t}$ at (1) is the binomial moment of the vector (X, Y) and then write the moment form

$$S_{r,t} = E\left[\binom{X}{r}\binom{Y}{t}\right].$$

We are interested in bivariate Bonferron-type inequalities which mean bound by linear combinations of the binomial moment $S_{r,t}$. In particular, we want to establish upper bound of $y_{1,1} = P(X_m \ge 1, Y_n \ge 1)$ which appears in many problems in statistics.

Galambos and Xu [3] proved that

$$y_{1,1} = P(\cup_{i=1}^m A_i, \cup_{j=1}^n B_j) \le S_{1,1} - \frac{2}{m}S_{2,1} - \frac{2}{n}S_{1,2} + \frac{4}{mn}S_{2,2},$$

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which insists the best upper bound among all upper bounds of the form $d_1S_{1,1}+d_2S_{2,1}+d_3S_{1,2}+d_4S_{2,2}$.

The classical lower bound for bivariate probability of degree two is

$$S_{1,1}-S_{1,2}-S_{2,1}\leq P(X_m\geq 1, Y_n\geq 1)$$

and our idea is to reduce the number of terms in binomial moments $S_{1,2}$ and $S_{2,1}$ in order to get an upper bound. For a related idea, see the graph-dependent models of Renyi [5] and Galambos [2].

In this direction, we establish new bivariate Bonferroni-type inequalities using consecutive events and deduce a known result.

Theorem 1. *For integers $m,n\geq 2$ and $1\leq i\leq m,\ \ 1\leq j\leq n$, then*

$$\begin{aligned} y_{1,1} = P(X_m\geq 1, Y_n\geq 1) &\leq S_{1,1}-\sum_{i=1}^{m-1}P(A_iA_{i+1}B_k) \\ &-\sum_{j=1}^{n-1}P(A_kB_jB_{j+1})-\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}P(A_iA_{i+1}B_jB_{j+1}). \end{aligned}\tag{2}$$

Taking the averages over $i=1,\dots,m,\ \ j=1,\dots,n$ of (2), we get Corollary 1.

Corollary 1.

$$y_{1,1}\leq S_{1,1}-\frac{2}{mn}S_{2,1}-\frac{2}{mn}S_{1,2}-\frac{4}{mn}S_{2,2}$$

Theorem 2. *For integers $m,n\geq 2$ and $1\leq i\leq m,\ \ 1\leq j\leq n$, then*

$$\begin{aligned} y_{1,1}\leq &S_{1,1}-\sum_{i=1}^{m-1}\sum_{j=1}^{n}P(A_iA_{i+1}B_j)-\sum_{i=1}^{m}\sum_{j=1}^{n-1}P(A_iB_jB_{j+1}) \\ &+\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}P(A_iA_{i+1}B_jB_{j+1}). \end{aligned}\tag{3}$$

Taking the averages over $i=1,2,\dots,m,\ j=1,2,\dots,n$ of (3), we get the following bivariate Bonferroni-type inequality.

Corollary 2.

$$y_{1,1}\leq S_{1,1}-\frac{2}{m}S_{2,1}-\frac{2}{n}S_{1,2}+\frac{4}{mn}S_{2,2}.$$

Theorem 3. *For integers $m, n \geq 2$ and $1 \leq i \leq m, 1 \leq j \leq n$, then*
(4)

$$\begin{aligned} y_{1,1} \leq S_{1,1} &- \sum_{i=1}^{m-1} P(A_iA_{i+1}B_k) - \sum_{i=1}^{m-2} P(A_iA_{i+2}B_k) - \sum_{j=1}^{n-1} P(A_kB_jB_{j+1}) \\ &- \sum_{j=1}^{n-2} P(A_kB_jB_{j+2}) - \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P(A_iA_{i+1}B_jB_{j+1}) \\ &+ \sum_{i=1}^{m-2} P(A_iA_{i+1}A_{i+2}B_k) + \sum_{j=1}^{n-2} P(A_kB_jB_{j+1}B_{j+2}). \end{aligned}$$

Taking the averages over $i = 1, \ldots, m,\ j = 1, \ldots, n$ of (4), we get Corollary 3.

Corollary 3.

$$\begin{aligned} y_{1,1} \leq S_{1,1} &- \frac{(2m-3)}{\binom{m}{2}n} S_{2,1} - \frac{(2n-3)}{\binom{n}{2}m} S_{1,2} - \frac{(m-1)(n-1)}{\binom{m}{2}\binom{n}{2}} S_{2,2} \\ &+ \frac{(m-2)}{\binom{m}{3}n} S_{3,1} + \frac{(n-2)}{\binom{n}{3}m} S_{1,3}. \end{aligned}$$

Theorem 4. *For integers $m, n \geq 2$ and $1 \leq i \leq m, 1 \leq j \leq n$, then*
(5)

$$\begin{aligned} y_{1,1} \leq S_{1,1} &- \sum_{i=1}^{m} \sum_{1\leq j<k\leq j+2}^{n-2} P(A_iB_jB_k) - \sum_{1\leq i<l\leq i+2}^{m-2} \sum_{j=1}^{n} P(A_iA_lB_j) \\ &+ \sum_{1\leq i<l\leq i+2}^{m-2} \sum_{1\leq j<k\leq j+2}^{n-2} P(A_iA_lB_jB_k) + \sum_{i=1}^{m}\sum_{j=1}^{n-2} P(A_iB_jB_{j+1}B_{j+2}) \\ &+ \sum_{i=1}^{m-2}\sum_{j=1}^{n} P(A_iA_{i+1}A_{i+2}B_j) - \sum_{1\leq i<l\leq i+2}^{m-2} \sum_{j=1}^{n-2} P(A_iA_lB_jB_{j+1}B_{j+2}) \\ &- \sum_{i=1}^{m-2} \sum_{1\leq j<k\leq j+2}^{n-2} P(A_iA_{i+1}A_{i+2}B_jB_k) \\ &+ \sum_{i=1}^{m-2}\sum_{j=1}^{n-2} P(A_iA_{i+1}A_{i+2}B_jB_{j+1}B_{j+2}). \end{aligned}$$

Taking the averages over $i = 1, \ldots, m,\ j = 1, \ldots, n$ of (5), we get Corollary 4.

Corollary 4.

$$y_{1,1} \le S_{1,1} - \frac{m(2n-3)}{\binom{m}{1}\binom{n}{2}}S_{1,2} - \frac{(2m-3)n}{\binom{m}{2}\binom{n}{1}}S_{2,1} + \frac{(2m-3)(2n-3)}{\binom{m}{2}\binom{n}{2}}S_{2,2}$$
$$+ \frac{m(n-2)}{\binom{m}{1}\binom{n}{3}}S_{1,3} + \frac{(m-2)n}{\binom{m}{3}\binom{n}{1}}S_{3,1} - \frac{(2m-3)n}{\binom{m}{2}\binom{n}{3}}S_{2,3}$$
$$- \frac{(m-2)(2n-3)}{\binom{m}{3}\binom{n}{2}}S_{3,2} + \frac{(m-2)(n-2)}{\binom{m}{3}\binom{n}{3}}S_{3,3}.$$

2. Proofs

Proof of Theorem 1. We use the method of indicators. Let

$$I(X \ge 1, Y \ge 1) = \begin{cases} 1, & \text{if } X \ge 1 \text{ and } Y \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

By using binomial moments and indicators, the right hand side of (2) becomes

$$E\Big[XY - \sum_{i=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) - \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}) - \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1})\Big]. \tag{6}$$

Then $E[I(X \ge 1, Y \ge 1)] = P(X \ge 1, Y \ge 1)$, it suffices to show that

$$I(X \ge 1)I(Y \ge 1) \le XY - \Big[\sum_{i=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) + \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}) + \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1})\Big]. \tag{7}$$

Note that both sides of (7) are zero if either X or Y equals zero, hence, in proving (7) we may assume that $X \ge 1$ and $Y \ge 1$, in which the left hand side of (7) is identically one. Thus, we have to prove that

(8) $u(X,Y) =$ the right hand side of (7) ≥ 1 for $1 \le X \le m\,,\ 1 \le Y \le n.$

We distinguish three cases:

(i) The case $X = 1$, $Y = 1$; that is, there are only two events A_i and B_j occur. Then this case is evident, having one on both sides of (8).

(ii) The case $X = 1$, $Y = q$ or $X = p$, $Y = 1$ for $2 \le p \le m,\ 2 \le q \le n,$; that is, there are the events that exactly one $A_i(B_j)$ and at least two more $B_j{}'s(A_i{}'s)$ occur. Then

$$u(1,q) = 1 \cdot q - (q-1) = 1 \text{ and } u(p,1) = p \cdot 1 - (p-1) = 1.$$

Hence, we get (8).

(iii) The case $X = p$, $Y = q$ for $2 \le p \le m$, $2 \le q \le n$; that is, there are the events that at least two more $A_i{}'s$ and $B_j{}'s$ occur. Then

$$u(p,q) = p \cdot q - \{(p-1) + (q-1) + (p-1) \cdot (q-1)\} = 1$$

Hence, we get (8). This completes the proof. □

Proof of Theorem 2. We can prove (3) by the same way of proof of Theorem 1. □

Proof of Theorem 3. We can prove (4) by the same way of proof of Theorem 1. □

Proof of Theorem 4. We use Bonferroni-type inequality of Lee [4], that is,

$$P(\bigcup_{i=1}^{m} A_i) \le \sum_{i=1}^{m} P(A_i) - \sum_{i<j\le i+2}^{m-2} P(A_iA_j) + \sum_{i=1}^{m-2} P(A_iA_{i+1}A_{i+2}).$$

We consider two univariate Bonferroni-type inequalities.

$$(9) \qquad P(\bigcup_{i=1}^{m} A_i) \le \sum_{i=1}^{m} P(A_i) - \sum_{i<l\le i+2}^{m-2} P(A_iA_l) + \sum_{i=1}^{m-2} P(A_iA_{i+1}A_{i+2}),$$

$$(10) \qquad P(\bigcup_{j=1}^{n} B_i) \le \sum_{j=1}^{n} P(B_i) - \sum_{j<k\le j+2}^{n-2} P(B_iB_k) + \sum_{j=1}^{n-2} P(B_jB_{j+1}B_{j+2}).$$

Turning to indicators, (12) and (13) become

$$(11) \quad I(X \ge 1) \le \sum_{i=1}^{m} I(A_i) - \sum_{i<l\le i+2}^{m-2} I(A_i)I(A_l) + \sum_{i=1}^{m-2} I(A_i)I(A_{i+1})I(A_{i+2}),$$

$$(12) \quad I(Y \ge 1) \le \sum_{j=1}^{n} I(B_j) - \sum_{j<k\le j+2}^{n-2} I(B_j)I(B_k) + \sum_{j=1}^{n-2} I(B_j)I(B_{j+1})I(B_{j+2}).$$

By multiplying (11) and (12) and taking expectations, we get Theorem 4. □

3. Numerical examples

Example 3-1. Let a machine consist of two pieces of equipments A and B. Let X_i be the time to failure of the i-th component of equipment A and let Y_j be the time to failure of the j-th component of equipment B. Assume that each X_i and each Y_j are unit exponential variates, that is, for each i, j,

$$P(X_i < x) = 1 - e^{-x}, \quad x > 0 \quad \text{and} \quad P(Y_j < y) = 1 - e^{-y}, \quad y > 0.$$

Consider a group A of ten components and a group B of five components. Let $X_1, X_2, \ldots, X_{10}$ be independent and identically distributed random variables and let $Y_1, Y_2, \ldots, Y_5$ be independent and identically distributed random variables. We assume the structure is such that each X_i is completely dependent on each Y_j and it has probability zero that at least one component of equipment $A(B)$ fails within $x(y)$ period of time and all components of equipment $B(A)$ fail after $y(x)$ period of time, that is, for each $1 \leq i \leq 10, 1 \leq j \leq 5$,

$$P(\bigcup_{i=1}^{10}(X_i < x), \bigcap_{j=1}^{5}(Y_j \geq y)) = P(\bigcap_{i=1}^{10}(X_i \geq x), \bigcup_{j=1}^{5}(Y_j < y)) = 0.$$

We also specify the bivariate distributions and the trivariate distributions of the combination of X_i and Y_j. For simplicity, let us use the same bivariate and trivariate distributions for all dependent components. Let, for $1 \leq i \leq 10, \quad 1 \leq j \leq 5$,

$$\begin{aligned}
&P(X_i < x, Y_j < y) = (1-e^{-x})(1-e^{-y})(1-\frac{1}{2}e^{-x-y}),\\
&P(X_{i_1} < x, X_{i_2} < x, Y_j < y) = (1-e^{-x})^2(1-e^{-y})(1-\frac{1}{3}e^{-2x-y}),\\
&P(X_i < x, Y_{j_1} < y, Y_{j_2} < y) = (1-e^{-x})(1-e^{-y})^2(1-\frac{1}{3}e^{-x-2y}),\\
&P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_j < y) = (1-e^{-x})^3(1-e^{-y})(1-\frac{1}{4}e^{-3x-y}),\\
&P(X_{i_1} < x, X_{i_2} < x, Y_{j_1} < y, Y_{j_2} < y) = (1-e^{-x})^2(1-e^{-y})^2(1-\frac{1}{4}e^{-2x-2y}),\\
&P(X_{i_1} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y) = (1-e^{-x})(1-e^{-y})^3(1-\frac{1}{4}e^{-x-3y}),\\
&P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_{j_1} < y, Y_{j_2} < y)\\
&\qquad\qquad = (1-e^{-x})^3(1-e^{-y})^2(1-\frac{1}{5}e^{-3x-2y}),\\
&P(X_{i_1} < x, X_{i_2} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y)\\
&\qquad\qquad = (1-e^{-x})^2(1-e^{-y})^3(1-\frac{1}{5}e^{-2x-3y}),\\
&P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y)\\
&\qquad\qquad = (1-e^{-x})^3(1-e^{-y})^3(1-\frac{1}{6}e^{-3x-3y}).
\end{aligned}$$

No further assumption is made. We would like to estimate $P(W_X \geq x, W_Y \geq y)$, where $W_X = \min(X_1, X_2, \ldots, X_{10})$ and $W_Y = \min(Y_1, Y_2, \ldots, Y_5)$. Here, of course, the events $A_i = (X_i < x)$ and $B_j = (Y_j < y)$ and thus $(V_{10} = 0, U_5 = 0) = (W_X \geq x, W_Y \geq y)$. We can now compute the following probability. For a numerical calculation, let us choose $x = 0.1$ and $y = 0.2$. Let V_{10} be the number of those $A_i = (X_i < 0.1)$ which occur and let U_5 be the number of those $B_j = (Y_j < 0.2)$ which occur.

$$S_{1,1} = \binom{10}{1}\binom{5}{1}(1 - e^{-0.1})(1 - e^{-0.2})^2(1 - \frac{1}{2}e^{-0.3}) = 0.54301,$$

$$\sum_{i=1}^{9} P(A_iA_{i+1}B_k) = 9(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3}e^{-0.4}) = 0.011472,$$

$$\sum_{j=1}^{4} P(A_kB_jB_{j+1}) = 4(1 - e^{-0.1})(1 - e^{-0.2})^2(1 - \frac{1}{3}e^{-0.5}) = 0.009979,$$

$$\sum_{i=1}^{9}\sum_{j=1}^{4} P(A_iA_{i+1}B_jB_{j+1}) = 36(1 - e^{-0.1})^2(1 - e^{-0.1})^2(1 - \frac{1}{4}e^{-0.6}) = 0.009242,$$

$$\sum_{i=1}^{9}\sum_{j=1}^{5} P(A_iA_{i+1}B_j) = 45(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3}e^{-0.4}) = 0.057362,$$

$$\sum_{i=1}^{10}\sum_{j=1}^{4} P(A_iB_jB_{j+1}) = 40(1 - e^{-0.1})(1 - e^{-0.2})^2(1 - \frac{1}{3}e^{-0.5}) = 0.099787,$$

$$\sum_{i=1}^{8} P(A_iA_{i+2}B_k) = 8(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3}e^{-0.4}) = 0.010198,$$

$$\sum_{j=1}^{3} P(A_kB_jB_{j+2}) = 3(1 - e^{-0.1})(1 - e^{-0.2})^2(1 - \frac{1}{3}e^{-0.5}) = 0.007484,$$

$$\sum_{i=1}^{8} P(A_iA_{i+1}A_{i+2}B_k) = 8(1 - e^{-0.1})^3(1 - e^{-0.2})(1 - \frac{1}{4}e^{-0.5}) = 0.001060,$$

$$\sum_{j=1}^{3} P(A_kB_jB_{j+1}B_{j+2}) = 3(1 - e^{-0.1})(1 - e^{-0.2})^3(1 - \frac{1}{4}e^{-0.7}) = 0.001489,$$

$$\sum_{i=1}^{10}\sum_{1\le j<k\le j+2}^{3} P(A_iB_jB_k) = 70(1 - e^{-0.1})(1 - e^{-0.2})^2(1 - \frac{1}{3}e^{-0.5}) = 0.174627,$$

$$\sum_{1\le i<l\le i+2}^{8}\sum_{j=1}^{5} P(A_iA_lB_j) = 85(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3}e^{-0.4}) = 0.108350,$$

$$\sum_{1\le i<l\le i+2}^{8}\sum_{1\le j<k\le j+2}^{3} P(A_iA_lB_jB_k) = 119(1 - e^{-0.1})^2(1 - e^{-0.2})^2(1 - \frac{1}{4}e^{-0.6}) = 0.030550,$$

$$\sum_{i=1}^{10}\sum_{j=1}^{3} P(A_iB_jB_{j+1}B_{j+2}) = 30(1 - e^{-0.1})(1 - e^{-0.2})^3(1 - \frac{1}{4}e^{-0.7}) = 0.014893,$$

$$\sum_{i=1}^{8}\sum_{j=1}^{5} P(A_iA_{i+1}A_{i+2}B_j) = 40(1-e^{-0.1})^3(1-e^{-0.2})(1-\frac{1}{4}e^{-0.5}) = 0.005301,$$

$$\sum_{1\le i<l\le i+2}^{8}\sum_{j=1}^{3} P(A_iA_lB_jB_{j+1}B_{j+2}) = 51(1-e^{-0.1})^2(1-e^{-0.2})^3(1-\frac{1}{5}e^{-0.8}) = 0.002504,$$

$$\sum_{i=1}^{8}\sum_{1\le j<k\le j+2}^{3} P(A_iA_{i+1}A_{i+2}B_jB_k) = 56(1-e^{-0.1})^3(1-e^{-0.2})^2(1-\frac{1}{5}e^{-0.7}) = 0.001428,$$

$$\sum_{i=1}^{8}\sum_{j=1}^{3} P(A_iA_{i+1}A_{i+2}B_jB_{j+1}B_{j+2}) = 24(1-e^{-0.1})^3(1-e^{-0.2})^3(1-\frac{1}{6}e^{-0.9}) = 0.000115.$$

Now, we can get the upper bounds of $P(V_{10} \ge 1, U_5 \ge 1)$. Since $P(W_X \ge 0.1, W_Y \ge 0.2) = 1 - P(V_{10} \ge 1, U_5 \ge 1)$ by our earlier assumption on dependence, we get the following lower bounds of $P(W_X \ge 0.1, W_Y \ge 0.2)$.

Lower bounds for $P(W_X \ge 0.1, W_Y \ge 0.2)$

inequality	upper bound for $y_{1,1}$	lower bound
(2)	0.512317	0.487683
(3)	0.395104	0.604896
(4)	0.497185	0.502815
(5)	0.306961	0.693039

In the above table, we see that (5) is the best upper bound for $y_{1,1}$.

Example 3-2. Consider a numerical example in the paper of Chen and Seneta [1]. Let $C_1, \ldots, C_6$ be events with specified probabilities (see table 1 of [1]). Let $C_1 = A_1,\ C_2 = A_2,\ C_3 = A_3,\ C_4 = B_1,\ C_5 = B_2,\ C_6 = B_3$. Then $S_{1,1} = 1.259, S_{2,1} = 0.225, S_{1,2} = 0.37, S_{2,2} = 0.055, S_{1,3} = S_{2,3} = S_{3,1} = S_{3,2} = S_{3,3} = 0$. The upper bound by Chen and Seneta[1] is following

$$\begin{aligned}
&P(m_n \ge a_1, m_N \ge a_2)\\
&\le S_{a_1,a_2} - \left(\frac{a_1+1}{n-a_1} - \binom{n}{a_1+1}^{-1}\right) S_{a_1+1,a_2}\\
&- \left(\frac{a_2+1}{N-a_2} - \binom{N}{a_2+1}^{-1}\right) S_{a_1,a_2+1}\\
&+ \left(\frac{a_1+1}{n-a_1} - \binom{n}{a_1+1}^{-1}\right)\left(\frac{a_2+1}{N-a_2} - \binom{N}{a_2+1}^{-1}\right) S_{a_1+1,a_2+1}.
\end{aligned}$$

This yields $y_{1,1} \leq 0.887$ (see table 2 of [1]). But Corollary 4 gives $y_{1,1} \leq 0.719$.

References

[1] T. Chen and E. Seneta, *Multivariate identities, permutation and Bonferroni upper bounds*, Combinatorics, Probability and Computing **4** (1995), 331–342.

[2] J. Galambos, *On the seive methods in probability theory I*, Studia Sci. Math., Hungar. **1** (1966), 39–50.

[3] J. Galambos and Y. Xu, *Some optimal bivariate Bonferroni-type bounds*, Proc. Amer. Math. Soc. **117** (1993), 523–528.

[4] M. Y. Lee, *Bonferroni-type inequalities*, Aequationes. Math. **44** (1992), 220–225.

[5] A. Renyi, *A general method for proving theorems in probability theory and some of its applications. Original Hungarian. Translated into English in : Selected Papers of A. Renyi*, Akademical Kiado, Budapest. **2** (1961), 297–302.

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