Commun. Korean Math. Soc. **22** (2007), No. 2, pp. 297–303

ON SPACES IN WHICH COMPACT-LIKE SETS ARE CLOSED, AND RELATED SPACES

Woo Chorl Hong

Abstract. In this paper, we study on C-closed spaces, SC-closed spaces and related spaces. We show that a sequentially compact SC-closed space is sequential and as corollaries obtain that a sequentially compact space with unique sequential limits is sequential if and only if it is C-closed [7, 1.19 Proposition] and every sequentially compact SC-closed space is C-closed.

We also show that a countably compact WAP and C-closed space is sequential and obtain that a countably compact (or compact or sequentially compact) WAP-space with unique sequential limits is sequential if and only if it is C-closed as a corollary.

Finally we prove that a weakly discretely generated AP-space is C-closed. We then obtain that every countably compact (or compact or sequentially compact) weakly discretely generated AP-space is Fréchet-Urysohn with unique sequential limits, for weakly discretely generated AP-spaces, unique sequential limits $\equiv$ KC $\equiv$ C-closed $\equiv$ SC-closed, and every continuous surjective function from a countably compact (or compact or sequentially compact) space onto a weakly discretely generated AP-space is closed as corollaries.

1. Introduction

All spaces considered here are always assumed to be infinite and T_1. Our terminology is standard and follows [1] and [4].

A topological space X is *KC* [11] (*C-closed* [7], *SC-closed*) if every compact (resp. countably compact, sequentially compact) subset of X is closed. A space X has *unique sequential limits* [5] if every sequence of points of X may converge to at most one limit.

It is well-known that every space with unique sequential limits is T_1 and KC is a separation property and it lies between Hausdorff and unique sequential limits. However, the properties C-closed and SC-closed are quite different from KC.

Received September 2, 2006.

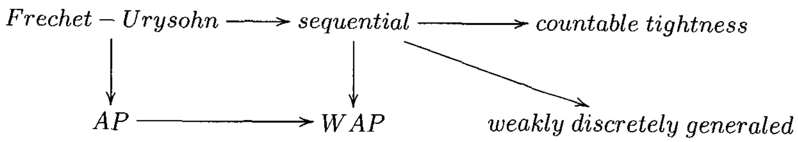
2000 *Mathematics Subject Classification.* 54A20, 54A25, 54D20, 54D55, 54D99.

Key words and phrases. KC, C-closed, SC-closed, Fréchet-Urysohn, sequential, AP, WAP, weakly discretely generated.

This paper was supported by Pusan National University Research Grant.

A topological space X is *Fréchet-Urysohn* [1, 6] (also called *Fréchet* [5, 8]) if for each subset A of X and each $x \in \overline{A}$ there exists a sequence of points of A which converges to x. A space X is *sequential* [5] if for each non-closed subset A of X there exists a sequence of points of A which converges to some $x \in \overline{A} - A$. A space X is *AP* (standing for Approximation by Points) [2] (also called *Whyburn* [8]) if for each non-closed subset A of X and each $x \in \overline{A}$ there exists a subset B of A such that $\overline{B} - B = \{x\}$; equivalently, $\overline{B} = B \cup \{x\}$. A space X is *WAP* (standing for Weak Approximation by Points) [2] (also called *weakly Whyburn* [8]) if for each non-closed subset A of X there exist $x \in \overline{A} - A$ and a subset B of A such that $\overline{B} = B \cup \{x\}$. A space X is *weakly discretely generated* [3] if for each non-closed subset A of X there exist $x \in \overline{A} - A$ and a subset D of A such that D is discrete and $x \in \overline{D}$. A space X has *countable tightness* [3] if for each subset A of X and each $x \in \overline{A}$ there exists a countable subset C of A such that $x \in \overline{C}$.

By definitions, one easily know that for spaces with unique sequential limits, the following diagram exhibits the general relationships among the properties mentioned above which are generalizations of first countability. (see [1, 2, 3, 5, 6, 7, 8, 10])



Notice that a Fréchet-Urysohn T_1-space need not be AP and a sequential T_1-space also need not be WAP as well as weakly discretely generated. Let $\mathbb{R}$ be the set of real numbers with the finite complement topology. Then the space $\mathbb{R}$ is T_1, sequential and Fréchet-Urysohn, but it is neither unique sequential limits, WAP, AP, nor weakly discretely generated.

In this paper, we study on C-closed spaces, SC-closed spaces and related spaces. We show that every sequentially compact SC-closed space is sequential and as corollaries obtain that a sequentially compact space with unique sequential limits is sequential if and only if it is C-closed [7, 1.19 Proposition] and every sequentially compact SC-closed space is C-closed.

We also show that every countably compact WAP and C-closed space is sequential and obtain that a countably compact (or compact or sequentially compact) WAP-space with unique sequential limits is sequential if and only if it is C-closed as a corollary.

Finally we prove that every weakly discretely generated AP-space is C-closed. Note that sequentiality and being weakly discretely generated AP are independent even if Hausdorff. We then obtain that every weakly discretely generated AP-space is KC as well as SC-closed, every countably compact (or compact or sequentially compact) weakly discretely generated AP-space is Fréchet-Urysohn with unique sequential limits, for weakly discretely generated

AP-spaces, unique sequential limits $\equiv$ KC $\equiv$ C-closed $\equiv$ SC-closed, and every continuous surjective function from a countably compact (or compact or sequentially compact) space onto a weakly discretely generated AP-space is closed as corollaries.

2. Results

We begin by showing some basic properties of C-closed spaces and SC-closed spaces.

Proposition 2.1. *Every C-closed space as well as every SC-closed space has unique sequential limits.*

Proof. Since a sequentially compact space is countably compact, every C-closed space is SC-closed. Hence it is enough to show that every SC-closed space has unique sequential limits. Suppose on the contrary that there exists a convergent sequence (x_n) of points of an SC-closed space X which converges to distinct two points x and y in X. Then clearly $\{x_n : n \in \mathbb{N}\} \cup \{x\}$, where $\mathbb{N}$ denotes the set of natural numbers and $\{x_n : n \in \mathbb{N}\}$ is the range of the sequence (x_n), is a non-closed subset of X. By SC-closedness of X, $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is not sequentially compact and hence there exists a sequence (y_n) of points of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ which does not have any convergent subsequence in $\{x_n : n \in \mathbb{N}\} \cup \{x\}$. Clearly, $\{y_n : n \in \mathbb{N}\}$ is infinite. Thus there exists a subsequence $(y_{\phi(n)})$ of (y_n), where ϕ is an increasing function from $\mathbb{N}$ into $\mathbb{N}$ itself, such that $(y_{\phi(n)})$ is also a subsequence of (x_n). It follows that the subsequence $(y_{\phi(n)})$ converges to x, which is a contradiction. $\square$

Proposition 2.2. *Let X be a sequential space. Then the following statements are equivalent:*

(1) *X has unique sequential limits.*
(2) *X is KC.*
(3) *X is C-closed.*
(4) *X is SC-closed.*

Proof. In [5, 5.4 and 5.5 Propositions], S. P. Franklin showed that every sequential space with unique sequential limits is C-closed and (1) $\equiv$ (3) $\equiv$ (4). Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Thus it holds. $\square$

Obviously, every Hausdorff space is KC. However, a countably compact (sequentially compact) Hausdorff space need not be C-closed (resp. SC-closed) in general as shown by the following remarkable example.

Example 2.3. The space of ordinals $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, is a Hausdorff compact and sequentially compact (and hence countably compact) space [9, p.70, 43(14)]. Clearly, the subspace $[0, \omega_1)$ of X is countably compact as well as sequentially compact and it is not closed in X. Thus X is neither C-closed nor SC-closed. Moreover, the space X does not have countable tightness and hence not sequential. For, $\omega_1 \in \overline{[0,\omega_1)}$, but there does not exist a countable subset A of $[0,\omega_1)$ such that $\omega_1 \in \overline{A}$.

We show that SC-closedness is a sufficient condition for a sequentially compact space to be sequential.

Theorem 2.4. *Every sequentially compact SC-closed space is sequential.*

Proof. Let A be a non-closed subset of a sequentially compact SC-closed space X. Then by SC-closedness of X, A is not sequentially compact and hence there exists a sequence (x_n) of points of A such that it does not have any convergent subsequence in A. Since X is sequentially compact and since every closed subspace of a sequentially compact space is sequentially compact, there exists a convergent subsequence $(x_{\phi(n)})$ of (x_n) which converges to some $x \in \overline{A}$. By the fact that (x_n) is a sequence of points of A and it does not have any convergent subsequence in A, it follows that $x \in \overline{A} - A$. Hence we have that there exist $x \in \overline{A} - A$ and a sequence $(x_{\phi(n)})$ of points of A such that $(x_{\phi(n)})$ converges to x. Therefore, X is sequential. □

From Theorem 2.4 and Propositions 2.1 and 2.2, we obtain immediately the following corollaries and hence we omit the proofs.

Corollary 2.5. *A sequentially compact space with unique sequential limits is sequential if and only if it is SC-closed.*

Corollary 2.6. ([7, 1.19 Proposition]) *A sequentially compact space with unique sequential limits is sequential if and only if it is C-closed.*

Corollary 2.7. *Every sequentially compact SC-closed space is C-closed.*

We also show that C-closedness is a sufficient condition for a countably compact WAP-space to be sequential.

Theorem 2.8. *Every countably compact WAP and C-closed space is sequential.*

Proof. Let A be a non-closed subset of a countably compact WAP and C-closed space X. Then since X is WAP, there exist $x \in \overline{A} - A$ and a subset B of A such that $\overline{B} = B \cup \{x\}$. Clearly, $\overline{B} - \{x\} := B$ is not closed. By C-closedness of X, B is not countably compact and hence there exists a sequence (x_n) of points of B such that it does not have any accumulation point in B. Since X is countably compact, $\overline{B}$ is countably compact and so the sequence (x_n) has an accumulation point in $\overline{B}$. It follows that the point x is a unique accumulation point of (x_n) in $\overline{B}$ and hence in X. We now claim that (x_n) converges to x. Suppose on the contrary that it is not. Then there exists an open neighborhood U of x in X such that (x_n) is not eventually in U. Hence we can construct a subsequence $(x_{\phi(n)})$ of (x_n) such that $x_{\phi(n)} \notin U$ for all $n \in \mathbb{N}$. Clearly, x is not an accumulation point of $(x_{\phi(n)})$. Since $(x_{\phi(n)})$ is a subsequence of (x_n) and x is a unique accumulation point of (x_n) in $\overline{B}$, it follows that $(x_{\phi(n)})$ does not have any accumulation point in $\overline{B}$. This is a contradiction to the fact that

$(x_{\phi(n)})$ is a sequence of points of a countably compact subspace $\overline{B}$. Thus we have that there exist $x \in \overline{A} - A$ and a sequence (x_n) of points of A such that (x_n) converges to x. Therefore, X is sequential. □

From Theorem 2.8 and Proposition 2.2, we have directly the following corollary and hence we omit the proof.

Corollary 2.9. *A countably compact (or compact or sequentially compact) WAP-space with unique sequential limits is sequential if and only if it is C-closed.*

Remark 2.10. In [10, 2.2 Theorem], V. V. Tkachuk and I. V. Yaschenko proved that every countably compact Hausdorff AP-space is Fréchet-Urysohn. However, it is well-known that a countably compact (or compact or sequentially compact) WAP-space need not be sequential even if Hausdorff.

The space of ordinals $X = [0, \omega_1]$, in Example 2.3, is scattered (i.e., any subspace of X has an isolated point.) [9, 43(13)] and in [10, 2.7 Theorem], it is proved that every scattered space is WAP. Hence we know that the space X is WAP and thus it is a Hausdorff compact and sequentially compact WAP-space. But, in Example 2.3, X is not sequential.

Finally we give a sufficient property, which is independent to sequentiality and is a generalization of Fréchet-Urysohn property, for a space to be C-closed.

Theorem 2.11. *Every weakly discretely generated AP-space is C-closed.*

Proof. Suppose that there exists a non-closed countably compact subset A of a weakly discretely generated AP-space X. Then since X is weakly discretely generated and A is not closed, there exist $x \in \overline{A} - A$ and a subset D of A such that D is discrete and $x \in \overline{D}$. Clearly, D is not closed. And since X is AP, there exists a subset E of D such that $\overline{E} = E \cup \{x\}$. By countable compactness of A, every sequence of points of A has an accumulation point in A. Let (x_n) be a sequence of distinct points of E. Then the sequence (x_n) has some accumulation point $y \in A$ and clearly $y \in \overline{E}$. Since E is discrete, it is obvious that each point of E cannot be an accumulation point of (x_n); that is, $y \notin E$. Since $\overline{E} = E \cup \{x\}$, we have that $x = y$. Thus $x \in A$. This is a contradiction. □

Note that a sequential space with unique sequential limits is weakly discretely generated WAP with countable tightness. From the following examples, however, we see that a sequential space need not be AP and a weakly discretely generated AP-space need not be sequential in general, even if Hausdorff.

Example 2.12. Let X be the set containing of pairwise distinct objects of the following three types: points x_{mn} where $m, n \in \mathbb{N}$, points y_n where $n \in \mathbb{N}$ and a point z. We set $V_k(y_n) = \{y_n\} \cup \{x_{mn} : m \geqq k\}$ and let γ denote the set of subsets W of X such that $z \in W$ and there exists a positive integer p such that $V_1(y_n) - W$ is finite and $y_n \in W$ for all $n \geqq p$. The collection

$$\{\{x_{mn}\} : m, n \in \mathbb{N}\} \cup \gamma \cup \{V_k(y_n) : k, n \in \mathbb{N}\}$$

is a base for a topology on X. Then the space X with the topology generated by the base is Hausdorff sequential [1, p.13, Example 13]. But, it is not AP. For, we have easily that $z \in \overline{\{x_{mn} : m, n \in \mathbb{N}\}}$ and for each subset A of $\{x_{mn} : m, n \in \mathbb{N}\}$ with $z \in \overline{A}$, $\{y_n : n \in \mathbb{N}\} \cap \overline{A}$ is infinite. Hence there does not exist a subset A of $\{x_{mn} : m, n \in \mathbb{N}\}$ such that $\overline{A} = A \cup \{z\}$.

Example 2.13. Let $X = \{(0,0)\} \cup (\mathbb{N} \times \mathbb{N})$. We define a topology τ on X by for each $(m,n) \in X - \{(0,0)\}$, $\{(m,n)\} \in \tau$ and $(0,0) \in U \in \tau$ if and only if for all but a finite number of integers m, the sets $\{n \in \mathbb{N} : (m,n) \notin U\}$ are each finite. Thus each point $(m,n) \in X - \{(0,0)\}$ is isolated and each open neighborhood of $(0,0)$ contains all but a finite number of points in each of all but a finite number of columns (see Arens-Fort space in [9]). Then it is clear that the space X is Hausdorff weakly discretely generated with countable tightness and there is a unique non-isolated point $(0,0)$ in X. Note that any space with a unique non-isolated point is AP [10, 2.1 Proposition (10)]. Thus X is AP. But, it is not sequential. For, $(0,0) \in \overline{\mathbb{N} \times \mathbb{N}}$, but there does not exist any convergent sequence of points of $\mathbb{N} \times \mathbb{N}$ [9, 26(3)].

By Proposition 2.1, Theorems 2.8 and 2.11, we have immediately the following corollaries and hence we omit the proofs.

Corollary 2.14. *Every weakly discretely generated AP-space is KC as well as SC-closed.*

Note that a sequential AP-space is Fréchet-Urysohn.

Corollary 2.15. *Every countably compact(or compact or sequentially compact) weakly discretely generated AP-space is Fréchet-Urysohn with unique sequential limits.*

Corollary 2.16. *Let X be a weakly discretely generated AP-space. Then the following statements are equivalent:*
(1) *X has unique sequential limits.*
(2) *X is KC.*
(3) *X is C-closed.*
(4) *X is SC-closed.*

Note that a first countable space with unique sequential limits is Hausdorff.

Corollary 2.17. *Let X be a first countable space. Then the following statements are equivalent:*
(1) *X has unique sequential limits.*
(2) *X is Hausdorff.*
(3) *X is KC.*
(4) *X is C-closed.*
(5) *X is SC-closed.*

Corollary 2.18. *Every continuous surjective function $f : X \to Y$ from a countably compact (or compact or sequentially compact) space X onto a weakly discretely generated AP-space Y is closed. Moreover, if f is bijective then it is a homeomorphism.*

Remark 2.19. (1) In [11, Theorem 2], A. Wilansky showed that for first countable spaces, Hausdorff $\equiv$ KC $\equiv$ unique sequential limits. It is a part of Corollary 2.17.

(2) By Corollary 2.15, the range space Y in Corollary 2.18 is a countably compact Fréchet-Urysohn space with unique sequential limits. But, it need not be Hausdorff in general. In [5, 6.2 Example], it is showed that there is a compact Fréchet-Urysohn space with unique sequential limits which is not Hausdorff.

(3) Recall that a function is *perfect* [4] if it is continuous closed surjective and the inverse image of every singleton is compact. In Corollary 2.18, if X is compact then f is perfect.

Acknowledgement. The author wish to thank the referee for his/her kind detailed comments and suggestions in improving the exposition of the paper.

References

[1] A. V. Arhangel'skii and L. S. Pontryagin(eds.), *General Topology I*, Encyclopaedia of Mathematical Sciences, vol. 17, Springer-Verlage, Berlin, 1990.

[2] A. Bella and I. V. Yaschenko, *On AP and WAP spaces*, Comment. Math. Univ. Carolinae **40** (1999), no. 3, 531–536.

[3] A. Dow, M. G. Tkachenko, V. V. Tkachuk and R. G. Wilson, *Topologies generated by discrete subspaces*, Glasnik Matematicki **37** (2002), no. 57, 187–210.

[4] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1970.

[5] S. P. Franklin, *Spaces in which sequences suffice II*, Fund. Math. **61** (1967), 51–56.

[6] W. C. Hong, *A Theorem on countably Fréchet-Urysohn spaces*, Kyungpook Math. J. **43** (2003), no. 3, 425–431.

[7] M. Ismail and P. Nyikos, *On spaces in which countably compact sets are closed, and hereditary properties*, Topology Appl. **11** (1980), 281–292.

[8] J. Penlant, M. G. Tkachenko, V. V. Tkachuk, and R. G. Wilson, *Pseudocompact Whyburn spaces need not be Fréchet*, Proc. Amer. Math. Soc. **131** (2002), no. 10, 3257–3265.

[9] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*, Springer-Verlag, Berlin, 1978.

[10] V. V. Tkachuk and I. V. Yaschenko, *Almost closed sets and topologies they determine*, Comment. Math. Univ. Carolinae **42** (2001), no. 2, 395–405.

[11] A. Wilansky, *Between T_1 and T_2*, Amer. Math. Monthly **74** (1967), 261–266.

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: wchong@pusan.ac.kr