Commun. Korean Math. Soc. **22** (2007), No. 2, pp. 247–258

ON A SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper a new class of system of generalized nonlinear variational inequalities involving strongly monotone, relaxed cocoercive and relaxed generalized monotone mappings in Hilbert spaces is introduced and studied. Based on the projection method, an equivalence between the system of generalized nonlinear variational inequalities and the fixed point problem is established, which is used to suggest some new iterative algorithms for computing approximate solutions of the system of generalized nonlinear variational inequalities. A few sufficient conditions which ensure the existence and uniqueness of solution of the system of generalized nonlinear variational inequalities are given, and the convergence analysis of iterative sequences generated by the algorithms are also discussed.

1. Introduction and preliminaries

It is well known that the variational inequality theory has been extended and generalized in many different directions to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. For details, we refer the reader to [1, 3-9] and the references therein.

Recently, some interesting and important problems related to variational inequalities and complementarity problems have been studied by many authors. For example, Verma [5-8] studied the approximation-solvability of a few kinds of systems of variational inequalities in Euclidean spaces and Hilbert spaces, respectively. Wu, Liu, Shim and Kang [9] also introduced a more general class of systems of variational inequalities than that in Verma [8] and extended the corresponding results of other authors in this field.

Motivated and inspired by the above works, in this paper, we introduce and investigate a new system of generalized nonlinear variational inequalities dealing with strongly monotone, relaxed cocoercive and relaxed generalized monotone mappings in Hilbert spaces. We prove the existence and uniqueness

Received October 4, 2006.

2000 Mathematics Subject Classification. 47J20, 49J40.

Key words and phrases. system of generalized nonlinear variational inequalities, iterative algorithms, solution, convergence.

of solution for the system of generalized nonlinear variational inequalities, construct some new algorithms for computing approximate solutions of the system of generalized nonlinear variational inequalities, and discuss the convergence of iterative sequences generated by the algorithms. The results presented in this paper improve, extend and unify many known results in the literature.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $A, B, C, D : H \to H$ be any mappings, $M, N : H \times H \to H$ nonlinear mappings, and K a nonempty closed convex subset of H. Let α and β be positive constants, and f and g be arbitrary elements in H. Consider the following problem:

Determine elements $x, y \in K$ such that

$$(1.1)\qquad \begin{cases} \langle \alpha(M(A(y), B(y)) - f) + x - y, u - x\rangle \geq 0, \\ \langle \beta(N(C(x), D(x)) - g) + y - x, u - y\rangle \geq 0, \quad \forall u \in K, \end{cases}$$

which is known as the *system of generalized nonlinear variational inequalities.*

If $A = C, B = D$, and $M(x, y) = N(x, y) = x - y$ for all $x, y \in K$, then the problem (1.1) is equivalent to finding $x, y \in K$ such that

$$(1.2)\qquad \begin{cases} \langle \alpha(A(y) - B(y) - f) + x - y, u - x\rangle \geq 0, \\ \langle \beta(A(x) - B(x) - g) + y - x, u - y\rangle \geq 0, \quad \forall u \in K, \end{cases}$$

which is introduced and studied by Wu, Liu, Shim and Kang [9].

If $A = C, f = g = 0$, and $M(x, y) = N(x, y) = x$ for all $x, y \in K$, then the problem (1.1) reduces to the following one: find $x, y \in K$ such that

$$(1.3)\qquad \begin{cases} \langle \alpha A(y) + x - y, u - x\rangle \geq 0, \\ \langle \beta A(x) + y - x, u - y\rangle \geq 0, \quad \forall u \in K, \end{cases}$$

which is called the *system of nonlinear variational inequalities,* see Verma [8].

For suitable and appropriate choices of elements f, g and the mappings M, N, A, B, C, D, we can get various new and previously known systems of variational inequalities as special cases of the system of generalized nonlinear variational inequalities (1.1).

Now we recall and introduce the following results and concepts.

Lemma 1.1. *For a given $z \in H$, the element $u \in K$ satisfies the following inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K(z)$, where P_K is the projection of H into K.

Furthermore, P_K is nonexpansive, that is,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 1.2 ([2]). *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ be sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subseteq [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} c_n < +\infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Definition 1.1. Let $T : H \to H$ be a mapping. T is said to be *t-Lipschitz continuous* if there exists a constant $t > 0$ satisfying

$$\|Tx - Ty\| \leq t\|x - y\|, \quad \forall x, y \in H.$$

Definition 1.2. Let $T : H \to H$ and $M : H \times H \to H$ be two nonlinear mappings. M is said to be

(1) *t-Lipschitz continuous* in the first argument if there exists a constant $t > 0$ satisfying

$$\|M(x,u) - M(y,u)\| \leq t\|x - y\|, \quad \forall x, y, u \in H;$$

(2) *t-strongly monotone* with respect to T in the first argument if there exists a constant $t > 0$ satisfying

$$\langle M(T(x),u) - M(T(y),u), x - y\rangle \geq t\|x - y\|^2, \quad \forall x, y, u \in H;$$

(3) *t-relaxed generalized monotone* with respect to T in the first argument if there exists a constant $t \in (0,1)$ satisfying

$$\begin{aligned} &\langle M(T(x),u) - M(T(y),u), x - y\rangle \\ \geq\ & -t\|M(T(x),u) - M(T(y),u)\| \cdot \|x - y\|, \quad \forall x, y, u \in H; \end{aligned}$$

(4) *(h,r)-relaxed cocoercive* with respect to T in the first argument if there exist constants $h > 0$ and $r > 0$ satisfying

$$\begin{aligned} &\langle M(T(x),u) - M(T(y),u), x - y\rangle \\ \geq\ & h\|x - y\|^2 - r\|T(x) - T(y)\|^2, \quad \forall x, y, u \in H. \end{aligned}$$

Similarly we can define the Lipschitz continuity of M in the second argument.

2. Main results

Lemma 2.1. *Let α and β be positive constants, and f and g be arbitrary elements in H. Then the following statements are equivalent:*

(a) *the system of generalized nonlinear variational inequalities* (1.1) *has a solution* $(x,y) \in K \times K$;

(b) *there exists* $(x,y) \in K \times K$ *satisfying*

$$x = P_K(y - \alpha(M(A(y), B(y)) - f)) \tag{2.1}$$

and

$$y = P_K(x - \beta(N(C(x), D(x)) - g)); \tag{2.2}$$

(c) *the mappings F and $G : H \to H$ defined by*

$$\begin{aligned} F(u) = P_K\{&P_K\big[u - \beta(N(C(u), D(u)) - g)\big] \\ &- \alpha\big[M\big(A(P_K(u - \beta(N(C(u), D(u)) - g))), \\ &B(P_K(u - \beta(N(C(u), D(u)) - g)))\big) - f\big]\}, \quad \forall u \in H, \end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
G(u) = P_K\{&P_K[u - \alpha(M(A(u),B(u)) - f)] \\
&- \beta[N(C(P_K(u - \alpha(M(A(u),B(u)) - f))), \\
&D(P_K(u - \alpha(M(A(u),B(u)) - f)))) - g]\}, \quad \forall u \in H
\end{aligned}
\tag{2.4}
$$

have fixed points $x, y \in H$, respectively, satisfying (2.1) *or* (2.2).

Proof. By Lemma 1.1, (a) is equivalent to (b) clearly. Suppose that (b) holds. It follows from (2.1) and (2.2) that

$$
\begin{aligned}
x &= P_K(y - \alpha(M(A(y),B(y)) - f)) \\
&= P_K\{P_K[x - \beta(N(C(x),D(x)) - g)] \\
&\quad - \alpha[M(A(P_K(x - \beta(N(C(x),D(x)) - g))), \\
&\quad B(P_K(x - \beta(N(C(x),D(x)) - g)))) - f]\} = F(x)
\end{aligned}
$$

and

$$
\begin{aligned}
y &= P_K(x - \beta(N(C(x),D(x)) - g)) \\
&= P_K\{P_K[y - \alpha(M(A(y),B(y)) - f)] \\
&\quad - \beta[N(C(P_K(y - \alpha(M(A(y),B(y)) - f))), \\
&\quad D(P_K(y - \alpha(M(A(y),B(y)) - f)))) - g]\} = G(y),
\end{aligned}
$$

that is, (c) holds.

Conversely, if (c) holds, without lose of generalization, suppose that the fixed points x and y of F and G satisfy (2.1). From (2.4), we derive that

$$
\begin{aligned}
y &= G(y) \\
&= P_K\{P_K[y - \alpha(M(A(y),B(y)) - f)] \\
&\quad - \beta[N(C(P_K(y - \alpha(M(A(y),B(y)) - f))), \\
&\quad D(P_K(y - \alpha(M(A(y),B(y)) - f)))) - g]\} \\
&= P_K(x - \beta(N(C(x),D(x)) - g)).
\end{aligned}
$$

Therefore, (b) holds. This completes the proof. □

Remark 2.1. Lemma 1.3 of Verma [8] and Lemma 2.1 of Wu, Liu, Shim and Kang [9] are special cases of Lemma 2.1.

Based on Lemma 2.1, we suggest the following general iterative algorithms for the system of generalized nonlinear variational inequalities (1.1).

Algorithm 2.1. *For any elements $x_0, y_0 \in H$, compute sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ by the following iterative procedures*

$$
\begin{aligned}
s_n &= (1 - c_n)x_n + c_n F(x_n) + p_n, \\
x_{n+1} &= (1 - a_n)x_n + a_n F(s_n) + u_n, \\
t_n &= (1 - d_n)x_n + d_n G(y_n) + q_n, \\
y_{n+1} &= (1 - b_n)y_n + b_n G(t_n) + v_n, \quad \forall n \geq 0,
\end{aligned}
\tag{2.5}
$$

where F *and* G *are defined by* (2.3) *and* (2.4), *respectively,* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}$ *and* $\{d_n\}_{n\geq 0}$ *are any sequences in* $[0,1]$ *and* $\{p_n\}_{n\geq 0}, \{q_n\}_{n\geq 0}, \{u_n\}_{n\geq 0}$ *and* $\{v_n\}_{n\geq 0}$ *are arbitrary sequences in* H *satisfying*

$$\sum_{n=0}^{\infty} \min\{a_n, b_n\} = +\infty, \quad \sum_{n=0}^{\infty} \max\{\|u_n\|, \|v_n\|\} < +\infty, \tag{2.6}$$
$$\text{and} \quad \lim_{n\to\infty} \max\{\|p_n\|, \|q_n\|\} = 0.$$

Algorithm 2.2. *For any elements* $x_0, y_0 \in H$, *compute sequences* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *by the following iterative procedures*

$$\begin{aligned} y_n &= P_K(x_n - \beta(N(C(x_n), D(x_n)) - g)) + u_n, \\ x_{n+1} &= (1 - a_n)x_n \\ &\quad + a_n P_K(y_n - \alpha(M(A(y_n), B(y_n)) - f)) + v_n, \quad \forall n \geq 0, \end{aligned} \tag{2.7}$$

where $\{a_n\}_{n\geq 0}$ *is any sequences in* $[0,1]$ *and* $\{u_n\}_{n\geq 0}$ *and* $\{v_n\}_{n\geq 0}$ *are arbitrary sequences in* H *satisfying*

$$\sum_{n=0}^{\infty} a_n = +\infty, \quad \lim_{n\to\infty} \|u_n\| = 0, \quad \text{and} \quad \sum_{n=0}^{\infty} \|v_n\| < +\infty. \tag{2.8}$$

If $\|p_n\| = \|q_n\| = \|u_n\| = \|v_n\| = 0$ for all $n \geq 0$, Algorithms 2.1 and 2.2 reduce to the following iterative algorithms, respectively.

Algorithm 2.3. *For any elements* $x_0, y_0 \in H$, *compute sequences* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *by the following iterative procedures*

$$\begin{aligned} s_n &= (1 - c_n)x_n + c_n F(x_n), \\ x_{n+1} &= (1 - a_n)x_n + a_n F(s_n), \\ t_n &= (1 - d_n)x_n + d_n G(y_n), \\ y_{n+1} &= (1 - b_n)y_n + b_n G(t_n), \quad \forall n \geq 0, \end{aligned} \tag{2.9}$$

where $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ *and* $\{d_n\}_{n\geq 0}$ *are any sequences in* $[0,1]$ *satisfying* $\sum_{n=0}^{\infty} \min\{a_n, b_n\} = +\infty$.

Algorithm 2.4. *For any elements* $x_0, y_0 \in H$, *compute sequences* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *by the following iterative procedures*

$$\begin{aligned} y_n &= P_K(x_n - \beta(N(C(x_n), D(x_n)) - g)), \\ x_{n+1} &= (1 - a_n)x_n \\ &\quad + a_n P_K(y_n - \alpha(M(A(y_n), B(y_n)) - f)), \quad \forall n \geq 0, \end{aligned} \tag{2.10}$$

where $\{a_n\}_{n\geq 0}$ *is any sequence in* $[0,1]$ *satisfying* $\sum_{n=0}^{\infty} a_n = +\infty$.

Theorem 2.1. *Let* $A, B, C, D : H \to H$ *be Lipschitz continuous with constants* a, b, c, d, *respectively. Let* $M : H \times H \to H$ *be* m_1*-Lipschitz continuous in the first argument,* m_2*-Lipschitz continuous in the second argument,* p*-strongly monotone with respect to* A *in the first argument, and* q*-relaxed generalized* *monotone with respect to B in the second argument. Let $N : H \times H \to H$ be n_1-Lipschitz continuous in the first argument, n_2-Lipschitz continuous in the second argument, and (h, r)-relaxed cocoercive with respect to C in the first argument. If there exist positive constants α and β satisfying*

$$\begin{aligned}\theta &= \sqrt{1 - 2\alpha(p - bm_2q) + \alpha^2(am_1 + bm_2)^2} \\ &\quad \times \left(\sqrt{1 - 2\beta(h - c^2r) + \beta^2c^2n_1^2} + \beta dn_2\right) < 1,\end{aligned} \tag{2.11}$$

then for any given $f, g \in H$, the system of generalized nonlinear variational inequalities (1.1) *has a unique solution $(x, y) \in H \times H$.*

Proof. It follows from the hypothesis that

$$\begin{aligned}
&\|F(u) - F(v)\|^2 \\
&= \|P_K\{P_K[u - \beta(N(C(u), D(u)) - g)] \\
&\quad - \alpha[M(A(P_K(u - \beta(N(C(u), D(u)) - g))), \\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))) - f]\} \\
&\quad - P_K\{P_K[v - \beta(N(C(v), D(v)) - g)] \\
&\quad - \alpha[M(A(P_K(v - \beta(N(C(v), D(v)) - g))), \\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g)))) - f]\}\|^2 \\
&\le \|P_K[u - \beta(N(C(u), D(u)) - g)] \\
&\quad - P_K[v - \beta(N(C(v), D(v)) - g)] \\
&\quad - \alpha\{M(A(P_K(u - \beta(N(C(u), D(u)) - g))), \\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))) \\
&\quad - M(A(P_K(v - \beta(N(C(v), D(v)) - g))), \\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g))))\}\|^2 \\
&= \|P_K[u - \beta(N(C(u), D(u)) - g)] \\
&\quad - P_K[v - \beta(N(C(v), D(v)) - g)]\|^2 \\
&\quad - 2\alpha\langle P_K[u - \beta(N(C(u), D(u)) - g)] \\
&\quad - P_K[v - \beta(N(C(v), D(v)) - g)], \\
&\quad M(A(P_K(u - \beta(N(C(u), D(u)) - g))), \\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))) \\
&\quad - M(A(P_K(v - \beta(N(C(v), D(v)) - g))), \\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g))))\rangle \\
&\quad + \alpha^2\|M(A(P_K(u - \beta(N(C(u), D(u)) - g))), \\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g))))
\end{aligned}$$

$$
\begin{aligned}
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g)))\big)\|^2\\
&= \|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|^2\\
&\quad - 2\alpha\big\langle P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big],\\
&\quad M\big(A(P_K(u - \beta(N(C(u), D(u)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\\
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\big\rangle\\
&\quad - 2\alpha\big\langle P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big],\\
&\quad M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\\
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g)))\big)\big\rangle\\
&\quad + \alpha^2\|M\big(A(P_K(u - \beta(N(C(u), D(u)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\\
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\\
&\quad + M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)\\
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g)))\big)\|^2\\
&\le \|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|^2\\
&\quad - 2\alpha p\|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|^2\\
&\quad + 2\alpha q\|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|\\
&\quad \times \|M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(u - \beta(N(C(u), D(u)) - g)))\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - M\big(A(P_K(v - \beta(N(C(v), D(v)) - g))),\\
&\quad B(P_K(v - \beta(N(C(v), D(v)) - g))))\|\\
&\quad + \alpha^2(am_1 + bm_2)^2 \|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \|P_K\big[u - \beta(N(C(u), D(u)) - g)\big]\\
&\quad - P_K\big[v - \beta(N(C(v), D(v)) - g)\big]\|^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \|u - v - \beta(N(C(u), D(u)) - N(C(v), D(v)))\|^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \big(\|u - v - \beta(N(C(u), D(u)) - N(C(v), D(u)))\|\\
&\quad + \beta\|N(C(v), D(u)) - N(C(v), D(v))\|\big)^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \big[\big(\|u - v\|^2 - 2\beta\langle u - v, N(C(u), D(u)) - N(C(v), D(u))\rangle\\
&\quad + \beta^2\|N(C(u), D(u)) - N(C(v), D(u))\|^2\big)^{\frac{1}{2}}\\
&\quad + \beta n_2\|D(u) - D(v)\|\big]^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \big[\big(\|u - v\|^2 - 2\beta(h\|u - v\|^2 - r\|C(u) - C(v)\|^2)\\
&\quad + \beta^2 c^2 n_1^2\|u - v\|^2\big)^{\frac{1}{2}} + \beta d n_2\|u - v\|\big]^2\\
&\leq \big(1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2\big)\\
&\quad \times \big(\sqrt{1 - 2\beta(h - c^2 r) + \beta^2 c^2 n_1^2} + \beta d n_2\big)^2\|u - v\|^2\\
&= \theta^2\|u - v\|^2, \quad \forall u, v \in H,
\end{aligned}
$$

that is,

$$
\|F(u) - F(v)\| \leq \theta\|u - v\|, \quad \forall u, v \in H. \tag{2.12}
$$

Therefore, F is a contraction mapping, moreover, it has a unique fixed point $x \in X$. Similarly, we derive that G has also a unique fixed point $y \in X$.

Now we prove that x and y satisfy (2.1).

Put $z = P_K(y - \alpha(M(A(y), B(y)) - f))$. As y is a fixed point of G, we have

$$
\begin{aligned}
y &= G(y)\\
&= P_K\big\{P_K\big[y - \alpha(M(A(y), B(y)) - f)\big]
\end{aligned}
$$

$$
\begin{aligned}
&- \beta\big[N\big(C(P_K(y-\alpha(M(A(y),B(y))-f))),\\
&D(P_K(y-\alpha(M(A(y),B(y))-f)))\big)-g\big]\big\}\\
&= P_K(z-\beta(N(C(z),D(z))-g)).
\end{aligned} \tag{2.13}
$$

From (2.13), it is easy to see that

$$
\begin{aligned}
F(z) &= P_K\big\{P_K\big[z-\beta(N(C(z),D(z))-g)\big]\\
&\quad - \alpha\big[M\big(A(P_K(z-\beta(N(C(z),D(z))-g))),\\
&\quad B(P_K(z-\beta(N(C(z),D(z))-g)))\big)-f\big]\big\}\\
&= P_K(y-\alpha(M(A(y),B(y))-f)) = z.
\end{aligned} \tag{2.14}
$$

(2.14) ensures that z is a fixed point of F. While x is the unique fixed point of F in X, hence $x = z = P_K(y-\alpha(M(A(y),B(y))-f))$. That is, (2.1) holds. It follows from Lemma 2.1 that the system of generalized nonlinear variational inequalities (1.1) has a solution $(x,y) \in H \times H$.

Next we claim that (x,y) is the unique solution of the system of generalized nonlinear variational inequalities (1.1). In fact, if $(u,v) \in H \times H$ is also a solution of the system of generalized nonlinear variational inequalities (1.1), by Lemma 2.1, we infer that $u = F(u)$ and $v = G(v)$. By the uniqueness of fixed points of F and G, respectively, we know that $u = x$ and $v = y$. This completes the proof. $\square$

Theorem 2.2. *Let the assumptions in Theorem* 2.1 *hold. If there exist positive constants α and β satisfying* (2.11), *then for any given $f, g \in H$, the system of generalized nonlinear variational inequalities* (1.1) *has a unique solution $(x,y) \in H \times H$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, where $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ are defined by Algorithm* 2.1.

Proof. It follows from Theorem 2.1 that the system of generalized nonlinear variational inequalities (1.1) has a unique solution $(x,y) \in H \times H$. Now we prove that the sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ generated by Algorithm 2.1 converge strongly to x and y, respectively. It follows from the proof of Theorem 2.1 that (2.12) holds. In light of (2.5) and (2.12), we infer that

$$
\begin{aligned}
\|s_n - x\| &= \|(1-c_n)(x_n - x) + c_n(F(x_n) - F(x)) + p_n\|\\
&\le (1-c_n)\|x_n - x\| + c_n\|F(x_n) - F(x)\| + \|p_n\|\\
&\le (1-c_n)\|x_n - x\| + c_n\theta\|x_n - x\| + \|p_n\|\\
&\le \|x_n - x\| + \|p_n\|, \quad \forall n \ge 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_{n+1} - x\|\\
&= \|(1-a_n)(x_n - x) + a_n(F(s_n) - F(x)) + u_n\|\\
&\le (1-a_n)\|x_n - x\| + a_n\|F(s_n) - F(x)\| + \|u_n\|
\end{aligned}
$$

$$\begin{aligned} &\leq (1-a_n)\|x_n - x\| + a_n\theta\|s_n - x\| + \|u_n\| \\ &\leq (1-(1-\theta)a_n)\|x_n - x\| + a_n\|p_n\| + \|u_n\|, \quad \forall n \geq 0, \end{aligned} \tag{2.15}$$

where F is defined by (2.3). Similarly, we can get that

$$\begin{aligned} \|y_{n+1} - y\| \leq (1-(1-\theta)b_n)\|y_n - y\| \\ + b_n\|q_n\| + \|v_n\|, \quad \forall n \geq 0. \end{aligned} \tag{2.16}$$

From Lemma 1.2, (2.6), (2.15) and (2.16), it follows that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. This completes the proof. □

Theorem 2.3. *Let the assumptions in Theorem 2.1 hold. If there exist positive constants α and β satisfying (2.11), then for any given $f, g \in H$, the system of generalized nonlinear variational inequalities (1.1) has a unique solution $(x, y) \in H \times H$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, where $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are defined by Algorithm 2.2.*

Proof. Theorem 2.1 ensures that the system of generalized nonlinear variational inequalities (1.1) has a unique solution $(x, y) \in H \times H$. Now we claim that the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ generated by Algorithm 2.2 converge strongly to x and y, respectively. As in the proof of Theorem 2.1, we conclude that

$$\begin{aligned} &\|y_n - y\| \\ &= \|P_K[x_n - \beta(N(C(x_n), D(x_n)) - g)] + u_n \\ &\quad - P_K[x - \beta(N(C(x), D(x)) - g)]\| \\ &\leq \|x_n - x - \beta[N(C(x_n), D(x_n)) - N(C(x), D(x))]\| + \|u_n\| \\ &\leq \big(\sqrt{1 - 2\beta(h - c^2 r) + \beta^2 c^2 n_1^2} + \beta d n_2\big)\|x_n - x\| \\ &\quad + \|u_n\|, \quad \forall n \geq 0, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} &\|x_{n+1} - x\| \\ &= \|(1-a_n)x_n + a_n P_K(y_n - \alpha(M(A(y_n), B(y_n)) - f)) + v_n \\ &\quad - [(1-a_n)x + a_n P_K(y - \alpha(M(A(y), B(y)) - f))]\| \\ &\leq (1-a_n)\|x_n - x\| + a_n\|y_n - \alpha(M(A(y_n), B(y_n)) - f) \\ &\quad - [y - \alpha(M(A(y), B(y)) - f)]\| + \|v_n\| \\ &\leq (1-a_n)\|x_n - x\| + a_n\|y_n - y \\ &\quad - \alpha[M(A(y_n), B(y_n)) - M(A(y), B(y))]\| + \|v_n\| \\ &\leq (1-a_n)\|x_n - x\| \\ &\quad + a_n\sqrt{1 - 2\alpha(p - bm_2 q) + \alpha^2(am_1 + bm_2)^2}\|y_n - y\| + \|v_n\| \\ &\leq (1-(1-\theta)a_n)\|x_n - x\| + a_n\|u_n\| + \|v_n\|, \quad \forall n \geq 0. \end{aligned} \tag{2.18}$$

Lemma 1.2, (2.8) and (2.18) ensure that $\lim_{n\to\infty} x_n = x$. Moreover, it follows from (2.8) and (2.17) that $\lim_{n\to\infty} y_n = y$. This completes the proof. □

As in the proof of the Theorems 2.1, 2.2 and 2.3, we have the following results.

Theorem 2.4. *Let the assumptions in Theorem* 2.1 *hold. If there exist positive constants* α *and* β *satisfying* (2.11), *then for any given* $f, g \in H$, *the system of generalized nonlinear variational inequalities* (1.1) *has a unique solution* $(x, y) \in H \times H$ *and* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *where* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *are defined by Algorithm* 2.3.

Theorem 2.5. *Let the assumptions in Theorem* 2.1 *hold. If there exist positive constants* α *and* β *satisfying* (2.11), *then for any given* $f, g \in H$, *the system of generalized nonlinear variational inequalities* (1.1) *has a unique solution* $(x, y) \in H \times H$ *and* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *where* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *are defined by Algorithm* 2.4.

Remark 2.2. Theorems 2.1$\sim$2.5 extend, improve and unify Theorems 2.1$\sim$2.3 in Verma [8] and Theorems 2.1$\sim$2.5 in Wu, Liu, Shim and Kang [9].

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