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RELATION BETWEEN FRACTAL MEASURES AND CANTOR MEASURES

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ABSTRACT. We investigate the relation between Hausdorff(packing) measure and lower(packing) Cantor measure on a deranged Cantor set. If the infimum of some distortion of contraction ratios is positive, then Hausdorff(packing) measure and lower(packing) Cantor measure of a deranged Cantor set are equivalent except for some singular behavior for packing measure case. It is a generalization of already known result on the perturbed Cantor set.

1. Introduction

We define a deranged Cantor set ([3]). Let I_ϕ =[0,1]. We can obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ deleting middle open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid = c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$,where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^\infty E_n$ a deranged Cantor set. We note that if $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$ for each n then $F = \bigcap_{n=0}^\infty E_n$ is called a perturbed Cantor set ([1]). We recall the s-dimensional Hausdorff measure of F :

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^\infty \mid U_n \mid^s : \{U_n\}_{n=1}^\infty$ is a δ-cover of $F\}$, and the Hausdorff dimension of F :

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}(= \inf\{s > 0 : H^s(F) = 0\})(\text{see}[5]).$$

Also we recall the s-dimensional packing measure of F:

$$p^s(F) = \inf\{\sum_{n=1}^\infty P^s(F_n) : \bigcup_{n=1}^\infty F_n = F\},$$

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where $P^s(E) = \lim_{\delta\to 0} P^s_\delta(E)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^{\infty} \mid U_n \mid^s$: $\{U_n\}$ is a δ-packing of E $\}$, and the packing dimension of F:

$$\dim_p(F) = \sup\{s > 0 : p^s(F) = \infty\}(= \inf\{s > 0 : p^s(F) = 0\})([5]).$$

We introduce functions $h^s(F) = \liminf_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s$ and $q^s(F) = \limsup_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s$ for $s \in$ (0,1) and a deranged Cantor set F. Clearly $h^s(F)$ and $q^s(F)$ are decreasing functions for s.

For some calculation, we also define

$$h^s(F_\tau) = \liminf_{n\to\infty} \sum_{\sigma\in\tau\times\{1,2\}^n} \mid I_\sigma \mid^s$$

and

$$q^s(F_\tau) = \limsup_{n\to\infty} \sum_{\sigma\in\tau\times\{1,2\}^n} \mid I_\sigma \mid^s$$

for $s \in$ (0,1) and each $\tau \in \{1,2\}^m$, where $m = 0, 1, 2, \ldots$. From now on we define $F_\tau = F \cap I_\tau$. So $F = F \cap I_\phi$. Further h^s and q^s are called *lower and upper Cantor measure* on F respectively.

Using h^s and q^s, we ([4]) defined the lower Cantor dimension and the upper Cantor dimension of a deranged Cantor set F by $\dim_{\underline{C}}(F) = \sup\{s > 0 : h^s(F) = \infty\}$ and $\dim_{\overline{C}}(F) = \sup\{s > 0 : q^s(F) = \infty\}$. Then $\dim_{\underline{C}}(F) = \inf\{s > 0 : h^s(F) = 0\}$ and $\dim_{\overline{C}}(F) = \inf\{s > 0 : q^s(F) = 0\}$ since $h^s(F)$ and $q^s(F)$ are decreasing functions for s. We note $\dim_{\underline{C}}$ and $\dim_{\overline{C}}$ are just functions whose domains are the class of the deranged Cantor sets. We note that if c_τ are given, then a deranged Cantor set is determined.

Lemma. *The condition*

$$\inf_{k\in N\cup\{0\}} \inf_{\tau,\sigma\in\{1,2\}^k} \inf_{v\in\{1,2\}^n, n\in N} \frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} > 0$$

is equivalent to the condition that for all τ,$\sigma \in \{1,2\}^k$*where* k *is any non-negative integer,*

$$\frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\ldots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\ldots,l_m}} \geq B$$

for all m *where* $B > 0$.

Proof. Consider

$$c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\ldots,l_m} = \frac{|I_{\tau v}|}{|I_\tau|}$$

and

$$c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\ldots,l_m} = \frac{|I_{\sigma v}|}{|I_\sigma|}.$$

It follows from the definition of infimum and the above consideration. □

Proposition 1. *The contraction distortion condition*

$$\inf_{k\in N}\inf_{\tau,\sigma\in\{1,2\}^k}\frac{|I_\tau|}{|I_\sigma|}>0$$

implies the condition that for all τ,$\sigma \in \{1,2\}^k$*where* k *is any non-negative integer,*

$$\frac{c_{\tau,l_1}c_{\tau,l_1,l_2}\cdots c_{\tau,l_1,l_2,\ldots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_1,l_2}\cdots c_{\sigma,l_1,l_2,\ldots,l_m}}\geq B$$

for all m *where* $B>0$.

Proof. From the above Lemma we only need to show that

$$\inf_{k\in N}\inf_{\tau,\sigma\in\{1,2\}^k}\frac{|I_\tau|}{|I_\sigma|}>0$$

implies the condition that

$$\inf_{k\in N\cup\{0\}}\inf_{\tau,\sigma\in\{1,2\}^k}\inf_{v\in\{1,2\}^n,n\in N}\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|}>0.$$

Assume

$$\inf_{k\in N}\inf_{\tau,\sigma\in\{1,2\}^k}\frac{|I_\tau|}{|I_\sigma|}>0.$$

By the definition of the infimum, there is a real positive number A such that $\inf_{\tau,\sigma\in\{1,2\}^k}\frac{|I_\tau|}{|I_\sigma|}>A$ for all $k\in N$. Clearly

$$\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|}\geq A^2$$

for all $\tau,\sigma\in\{1,2\}^k$,$k\in N\cup\{0\}$ and $v\in\{1,2\}^n, n\in N$. □

In this paper, we call F *a quasi-perturbed Cantor set* if F is a deranged Cantor set satisfying the contraction distortion condition

$$\inf_{k\in N\cup\{0\}}\inf_{\tau,\sigma\in\{1,2\}^k}\inf_{v\in\{1,2\}^n,n\in N}\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|}>0.$$

We note that a perturbed Cantor set and a cookie-cutter repeller ([4], [6]) are special examples of quasi-perturbed Cantor sets. This is why we call the above set as a quasi-perturbed Cantor set. Precisely we state it as follows.

Remark. In the perturbed Cantor set, we ([4]) note that

$$\frac{c_{\tau,l_1}c_{\tau,l_1,l_2}\cdots c_{\tau,l_1,l_2,\ldots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_1,l_2}\cdots c_{\sigma,l_1,l_2,\ldots,l_m}}=1$$

for all τ,$\sigma\in\{1,2\}^k$where k is any non-negative integer. But we note that there are many perturbed Cantor sets satisfying

$$\inf_{k\in N}\inf_{\tau,\sigma\in\{1,2\}^k}\frac{|I_\tau|}{|I_\sigma|}=0.$$

Remark. ([4]). If f is a positively oriented cookie-cutter map in the sense that $f' > 1$, then the repeller F of f is a quasi-perturbed Cantor set (in this paper, we assume that the cookie-cutter map f is of differentiability of class C^2).

We are now ready to study the ratio geometry of a quasi-perturbed Cantor set.

2. Main results

In this section, F means a deranged Cantor set determined by $\{c_\tau\}$ with $\tau \in \{1,2\}^n$ where $n = 1, 2, \ldots$. Hereafter we only consider a deranged Cantor set whose contraction ratios $c_{\tau,1}$, $c_{\tau,2}$ and gap ratios $d_\tau (= 1 - (c_{\tau,1} + c_{\tau,2}))$ are uniformly bounded away from 0(cf. [2]).

Proposition 2. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $q^t(F_\tau) < \infty$ for some $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$, then $q^t(F_\sigma) < \infty$ for all $\sigma \in \{1,2\}^n$. Similarly if $q^t(F_\tau) = \infty$ for some $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$, then $q^t(F_\sigma) = \infty$ for all $\sigma \in \{1,2\}^n$.*

Proof. Fix $n \in N$ and $\tau \in \{1,2\}^n$ and suppose that $q^t(F_\tau) < \infty$. Since

$$\inf_{k \in N \cup \{0\}} \inf_{\tau, \sigma \in \{1,2\}^k} \inf_{v \in \{1,2\}^n, n \in N} \frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} > 0,$$

we have a positive number B such that $\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} \leq B$ for all $\sigma \in \{1,2\}^n$.

Then by definition

$$q^t(F_\sigma) = \limsup_{n \to \infty} \sum_{v \in \sigma \times \{1,2\}^n} | I_{\sigma v} |^t .$$

So we have

$$q^t(F_\sigma) \leq \limsup_{n \to \infty} \sum_{v \in \sigma \times \{1,2\}^n} B^t \frac{|I_{\tau v}|^t |I_\sigma|^t}{|I_\tau|^t} = B^t \frac{|I_\sigma|^t}{|I_\tau|^t} q^t(F_\tau) < \infty.$$

Similarly it holds for the second argument since we have a positive number B' such that $\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} \geq B'$ for all $\sigma \in \{1,2\}^n$. $\square$

Proposition 3. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $h^t(F_\tau) < \infty$ for some $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$, then $h^t(F_\sigma) < \infty$ for all $\sigma \in \{1,2\}^n$. Similarly if $h^t(F_\tau) = \infty$ for some $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$, then $h^t(F_\sigma) = \infty$ for all $\sigma \in \{1,2\}^n$.*

Proof. Fix $n \in N$ and $\tau \in \{1,2\}^n$ and suppose that $h^t(F_\tau) < \infty$. Since

$$\inf_{k \in N \cup \{0\}} \inf_{\tau, \sigma \in \{1,2\}^k} \inf_{v \in \{1,2\}^n, n \in N} \frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} > 0,$$

we have a positive number B such that $\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} \leq B$ for all $\sigma \in \{1,2\}^n$.

Then by definition

$$h^t(F_\sigma) = \liminf_{n\to\infty} \sum_{v\in\sigma\times\{1,2\}^n} | I_{\sigma v} |^t .$$

So we have

$$h^t(F_\sigma) \le \liminf_{n\to\infty} \sum_{v\in\sigma\times\{1,2\}^n} B^t \frac{|I_{\tau v}|^t |I_\sigma|^t}{|I_\tau|^t} = B^t \frac{|I_\sigma|^t}{|I_\tau|^t} h^t(F_\tau) < \infty.$$

Similarly it holds for the second argument since we have a positive number B' such that $\frac{|I_{\sigma v}||I_\tau|}{|I_{\tau v}||I_\sigma|} \ge B'$ for all $\sigma \in \{1,2\}^n$. □

Proposition 4. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If the t-dimensional packing pre-measure $P^t(F_\tau) < \infty$ for some $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$, then $q^t(F_\sigma) < \infty$ for all $\sigma \in \{1,2\}^n$.*

Proof. Fix $n \in N$ and $\tau \in \{1,2\}^n$ and suppose that $P^t(F_\tau) < \infty$. By the definition of t-dimensional pre-packing measure, we have $q^t(F_\tau) < \infty$. By the above Proposition we have $q^t(F_\sigma) < \infty$ for all $\sigma \in \{1,2\}^n$. □

Proposition 5. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $q^t(F) = \infty$, then $q^t(F_\sigma) = \infty$ for all $\sigma \in \{1,2\}^n$.*

Proof. Assume that $q^t(F_\tau) < \infty$ for some $\tau \in \{1,2\}^n$. By the above Proposition we have $q^t(F_\sigma) < \infty$ for all $\sigma \in \{1,2\}^n$, which gives

$$q^t(F) \le \sum_{\sigma\in\{1,2\}^n} q^t(F_\sigma) < \infty.$$

□

Proposition 6. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $q^t(F) = \infty$, then $p^t(F) = \infty$.*

Proof. Assume that $q^t(F) = \infty$ for some $t \in (0,1)$. By the above Proposition $q^t(F_\sigma) = \infty$ for all $\sigma \in \{1,2\}^n$ for each $n \in N$. Hence $P^t(F_\sigma) \ge q^t(F_\sigma) = \infty$ for all $\sigma \in \{1,2\}^n$.

Suppose that $\bigcup_{i=1}^\infty F_i = F$. Then by the Baire category theorem we have F_i whose closure has a non-empty interior in F for some $i \in N$. Hence there is $\tau \in \{1,2\}^n$ such that $F_\tau \subset \overline{F_i}$. Since $P^t(F_i) = P^t(\overline{F_i}) \ge P^t(F_\tau) = \infty$, $p^t(F) = \infty$. □

Proposition 7. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $q^t(F) < \infty$, then $p^t(F) < \infty$. Further if $q^t(F) = 0$, then $p^t(F) = 0$.*

Proof. It follows from the same arguments of the proofs in the above Proposition and the theorem 2 in [4]. □

Proposition 8. *Let F be a quasi-perturbed Cantor set and $t \in (0,1)$. If $h^t(F) = \infty$, then $H^t(F) = \infty$.*

Proof. If $h^t(F) = \liminf_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^t = \infty$, for any large $a > 0$ we see that $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^t > a$ for all but finitely many n. It follows from the end part of the proof of the theorem 1 in [4]. $\square$

Remark. In a deranged Cantor set, by the definition, if $h^t(F) < \infty$, then $H^t(F) < \infty$ since $H^t(F) \le h^t(F)$. Further if $h^t(F) > 0$, then $H^t(F) > 0$ from the theorem 1 in [4].

Corollary 9. *Let F be a quasi-perturbed Cantor set. Then Hausdorff measure H^t and lower Cantor measure h^t are equivalent and the packing measure p^t and the upper Cantor measure q^t are equivalent except that if $p^t(F) = 0$ then $q^t(F) = 0$.*

Proof. It follows from the above Propositions 6,7,8 and Remark. $\square$

Remark. We remark that Meinershagen ([7]) showed the relation between Hausdorff and packing measures and Cantor measures on the perturbed Cantor set with similar results. We extend her results on the perturbed Cantor set to those on the quasi-perturbed Cantor set which is a generalized form of a perturbed Cantor set. Similarly with her results we also cannot guarantee that if $p^t(F) = 0$ then $q^t(F) = 0$.

References

[1] I. S. Baek, *Dimensions of the perturbed Cantor set*, Real Analysis Exchange **19** (1994), no. 1, 269–273.

[2] ———, *Hausdorff dimension of perturbed Cantor sets without some boundedness condition*, Acta Math. Hungar. **99** (2003), no. 4, 279–283.

[3] ———, *Cantor dimension and its application*, Bull. Korean Math. Soc. **41** (2004), no. 1, 13–18.

[4] ———, *Dimensions of a deranged Cantor set with specific contraction ratios*, Bull. Korean Math. Soc. **41** (2004), no. 2, 269–274.

[5] K. J. Falconer, *Fractal geometry*, John Wiley and Sons, 1990.

[6] ———, *Techniques in fractal geometry*, John Wiley and Sons, 1997.

[7] S. Meinershagen, *Hausdorff and packing measure on a perturbed Cantor set*, Real Analysis Exchange **27** (2002), no. 1, 177–190.

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