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A NOTE ON THE MONOTONE INTERVAL-VALUED SET FUNCTION DEFINED BY THE INTERVAL-VALUED CHOQUET INTEGRAL

LEE-CHAE JANG

ABSTRACT. At first, we consider nonnegative monotone interval-valued set functions and nonnegative measurable interval-valued functions. In this paper we investigate some properties and structural characteristics of the monotone interval-valued set function defined by an interval-valued Choquet integral.

1. Introduction

In a previous work [19] the authors investigated monotone set function defined by Choquet integral ([3, 4, 12, 13, 14]) instead of fuzzy integral ([15, 16, 17, 18]). This construction is a useful method to form sound monotone set functions, including fuzzy measures, in various application areas, such as decision making, information theory, expected utility theory, and risk analysis.

Let X be a set and (X, Ω) a measurable space. A nonnegative set function μ is called *a fuzzy measure* if it is monotone and $\mu(\emptyset) = 0$. Then we consider the interval-valued set function ν defined by an interval-valued Choquet integral

$$\bar{\nu}(A) = (C)\int_A \bar{f}d\mu, \quad \forall A \in \Omega$$

is monotone on Ω with $\bar{\nu}(\emptyset) = [0, 0]$ ([9]), where $\bar{f}$ is an interval-valued function.

We note that set-valued Choquet integrals was first introduced by Jang and Kwon ([6]) and restudied by Zhang, Guo and Lia ([21]) and that the theory about set-valued integrals has drawn much attention due to numerous applications in mathematics, economics, theory of control and many other fields. In the papers ([6, 7, 8, 9, 10, 20, 21]), they have been studied some properties of set-valued Choquets and interval-valued Choquet integrals. In this paper, we investigate some basic properties and structural characteristics of monotone interval-valued Choquet integrals.

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2. Preliminaries and definitions

A fuzzy measure μ is said to be *lower semi-continuous* if for every increasing sequence $\{A_n\}$ in Ω, we have $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$. A fuzzy measure μ is said to be *upper semi-continuous* if for every decreasing sequence $\{A_n\}$ in Ω and $\mu(A_1) < \infty$, we have $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$. If μ is both lower semi-continuous and upper semi-continuous, it is said to be *continuous*. A fuzzy measure μ is said to be *finite* if $\mu(X)$ is finite.

Definition 2.1. ([3, 4, 12, 13, 14]) (1) *The Choquet integral* of a measurable function f with respect to a fuzzy measure μ on $A \in \Omega$ is defined by

$$(C)\int_A f d\mu = \int_0^{\infty} \mu(\{x|f(x) > \alpha\} \cap A)d\alpha$$

where the integrand on the right-hand side is an ordinary one.

(2) A measurable function f is called *c-integrable* if the Choquet integral of f can be defined and its value is finite.

Instead of $(C)\int_X f d\mu$, we will write $(C)\int f d\mu$. Throughout this paper, R^+ will denote the interval $[0,\infty)$. The Choquet integral is a generalization of the Lebesgue integral, since they coincide when μ is a classical (σ-additive) measure.

Definition 2.2. ([4]) A set $N \in \Omega$ is called a null set (with respect to μ) if $\mu(A \cap N) = \mu(A)$ for all $A \in \Omega$.

We note that $[P(x)\mu - a.e.$ on $A]$ means there exists a null set N such $P(x)$ is true for all $x \in A - N$ where $P(x)$ is a proposition concerning the point of A.

Definition 2.3. ([3, 4, 12, 13, 14]) Let f, g be nonnegative measurable functions. We say that f and g are comonotonic, in symbol $f \sim g$ if

$$f(x) < f(x') \Rightarrow g(x) \le g(x') \text{ for all } x, x' \in X.$$

Theorem 2.4. ([3,4,12,13,14]) *Let f, g, h be nonnegative measurable functions. Then we have*

(1) $f \sim f$,
(2) $f \sim g \Rightarrow g \sim f$,
(3) $f \sim a$ *for all* $a \in R^+$,
(4) $f \sim g$ *and* $g \sim h \Rightarrow f \sim (g+h)$.

Theorem 2.5. ([3,4,12,13,14]) *Let f, g be nonnegative measurable functions. Then we have the followings.*

(1) *If* $f \le g$, *then* $(C)\int f d\mu \le (C)\int g d\mu$.
(2) *If* $A \subset B$ *and* $A, B \in \Omega$, *then* $(C)\int_A f d\mu \le (C)\int_B f d\mu$.
(3) *If* $f \sim g$ *and* $a, b \in R^+$, *then*

$$(C)\int (af + bg)d\mu = a(C)\int f d\mu + b(C)\int g d\mu.$$

(4) *If* $(f \vee g)(x) = f(x) \vee g(x)$ *and* $(f \wedge g)(x) = f(x) \wedge g(x)$ *for all* $x \in X$, *then*

$$(C)\int f \vee g d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu$$

and

$$(C)\int f \wedge g d\mu \leq (C)\int f d\mu \wedge (C)\int g d\mu.$$

We denote $I(R^+)$ by

$$I(R^+) = \{\bar{a} = [a^-, a^+] | a^- \leq a^+, a^-, a^+ \in R^+\}.$$

For any $a \in R^+$, we define $a = [a, a]$. Obviously, $a \in I(R^+)$.

Definition 2.6. If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(R^+)$, then we define:

(1) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(2) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(3) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(4) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(5) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^- \leq a^+ \leq b^+$.

Theorem 2.7. *Let* $\bar{a}, \bar{b}, \bar{c} \in I(R^+)$. *Then we have*

(1) *idempotent law*: $\bar{a} \wedge \bar{a} = \bar{a}$, $\bar{a} \vee \bar{a} = \bar{a}$,
(2) *commutative law*: $\bar{a} \wedge \bar{b} = \bar{b} \wedge \bar{a}, \bar{a} \vee \bar{b} = \bar{b} \vee \bar{a}$,
(3) *associative law*:

$$\bar{a} \wedge (\bar{b} \wedge \bar{c}) = (\bar{a} \wedge \bar{b}) \wedge \bar{c}$$

$$\bar{a} \vee (\bar{b} \vee \bar{c}) = (\bar{a} \vee \bar{b}) \vee \bar{c}$$

(4) *absorption law*: $\bar{a} \wedge (\bar{b} \vee \bar{b}) = \bar{a} \vee (\bar{a} \wedge \bar{b}) = \bar{a}$
(5) *distributive law*:

$$\bar{a} \wedge (\bar{b} \vee \bar{c}) = (\bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge \bar{c})$$

$$\bar{a} \vee (\bar{b} \wedge \bar{c}) = (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}).$$

Clearly, we have the following theorem for multiplication and Hausdorff metric on $I(R^+)$.

Theorem 2.8. (1) *If we define* $\bar{a} \cdot \bar{b} = \{x \cdot b | x \in \bar{a}, y \in \bar{b}\}$ *for* $\bar{a} \cdot \bar{b} \in I(R^+)$, *then we have*

$$\bar{a} \cdot \bar{b} = [a^- \cdot b^-, a^+ \cdot b^+].$$

(2) *If* $d_H : I(R^+) \times I(R^+) \to [0, \infty)$ *is a Hausdorff metric, then we have*

$$d_H(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

3. Interval-valued Choquet integrals

Let $C(R^+)$ be the class of closed subsets of R^+. We denote a real-valued function $f : X \to R^+$, a closed set-valued function $\bar{f} : X \to C(R^+) \setminus \{\emptyset\}$.

Definition 3.1. ([1, 2]) A closed set-valued function $\bar{f}$ is said to be *measurable* if for each open set $O \subset R^+$,

$$\bar{f}^{-1}(O) = \{x \in X | \bar{f}(x) \cap O \neq \emptyset\} \in \Omega.$$

Definition 3.2. ([8, 19, 21]) Let $\{A_n\} \subset C(R^+)$ be a sequence and $A \in C(R^+)$. We define

(1) $A_n \uparrow (\downarrow) A$(order) if and only if $d_H(A_n, A) \to 0$ as $n \to \infty$ and $A_n \leq A_{n+1} (A_n \geq A_{n+1})$ for all $n = 1, 2, \ldots,$

(2) $A_n \uparrow (\downarrow) A$(inclusion) if and only if $d_H(A_n, A) \to 0$ and $A_n \subset A_{n+1} (A_n \supset A_{n+1})$ for all $n = 1, 2, \ldots$.

Definition 3.3. ([6, 7, 8, 9, 10, 11, 21]) (1) Let $A \in \Omega$. *The Choquet integral* of $\bar{f}$ on A is defined by

$$(C) \int_A \bar{f} d\mu = \{(C) \int_A f d\mu | f \in S(\bar{f})\}$$

where $S(\bar{f})$ is the family of measurable selections of $\bar{f}$.

(2) $\bar{f}$ is said to be *c-integrable* if $(C) \int \bar{f} d\mu \neq \emptyset$.

(3) $\bar{f}$ is said to be *Choquet integrably bounded* if there is a c-integrable function g such that

$$\| \bar{f} \| = \sup_{r \in \bar{f}(x)} |r| \leq g(x), \quad \text{for all } x \in X.$$

Instead of $(C) \int_X \bar{f} d\mu$, we write $(C) \int \bar{f} d\mu$. Obviously, $(C) \int \bar{f} d\mu$ may be empty. We remark that if $A, B \in C(X)$ (the class of closed subsets of X), then $A \leq B$ means $\inf A \leq \inf B$ and $\sup A \leq \sup B$.

Theorem 3.4. *If a closed set-valued function $\bar{f}$ is c-integrable, then*

$$A \leq B \text{ and } A, B \in C(X) \Rightarrow (C) \int_A \bar{f} d\mu \leq (C) \int_B \bar{f} d\mu$$

and

$$A \subset B \text{ and } A, B \in C(X) \Rightarrow (C) \int_A \bar{f} d\mu \subset (C) \int_B \bar{f} d\mu.$$

4. Main results

In this section, we investigate structural characteristics of monotone interval-valued set functions defined by the interval-valued Choquet integral. The concept of absolute continuity in classical measure theory has been generalized in as many so 21 different types for fuzzy measures (or continuous fuzzy measures) in [18]. In Z. Wang et al. [11] they obtained that these generalizations were applicable to nonnegative monotone set functions. We consider two of the 21 generalized types of absolutely continuity, which are labeled in [18] as types I and IV; they are defined as follows.

Definition 4.1. Let μ be a fuzzy measure on Ω. We say that a nonnegative monotone interval-valued set function $\bar{\nu}$ on Ω is absolutely continuous of type I with respect to μ, denoted by $\bar{\nu} \ll_I \mu$ if and only if $\bar{\nu}(A) = [0,0]$ whenever $A \in \Omega$ and $\mu(A) = 0$; we say that $\bar{\nu}$ on Ω is absolutely continuous of type VI with respect to μ, denoted by $\bar{\nu} \ll_{VI} \mu$ if and only if $d_H(\bar{\nu}(A_n), [0,0]) \to 0$ as $n \to \infty$ whenever $\{A_n\} \subset \Omega$ and $\mu(A_n) \to 0$.

Using Theorem 3.4, it is easy to show that the interval-valued set function $\bar{\nu}_{\bar{f}}$ on Ω defined by

$$\bar{\nu}_{\bar{f}}(A) = (C)\int_A \bar{f} d\mu, \quad \forall A \in \Omega \tag{4.1}$$

is also nonnegative monotone and vanishing at $\emptyset$. Then by Theorem 3.4, we also obtain that if $\bar{f}$ is Choquet integrably bounded and a fuzzy measure μ is continuous, there exist two nonnegative monotone interval-valued set functions ν_{f^-}, ν_{f^+} such that

$$\bar{\nu}(A) = [\nu_{f^-}(A), \nu_{f^+}(A)], \quad \forall A \in \Omega \tag{4.2}$$

where $\nu_{f^-}(A) = (C)\int_A f^- d\mu$ and $\nu_{f^+}(A) = (C)\int_A f^+ d\mu$.

Theorem 4.2. ([19]) *Let $A \in \Omega$ and ν_g be defined in terms of a fuzzy measure μ and a real -valued function g by $\nu_g(A) = \int_A g d\mu$.*

(1) *If g is measurable, then $\nu_g \ll_I \mu$.*

(2) *If g is Choquet integrable, then $\nu_g \ll_{IV} \mu$.*

Theorem 4.3. *Let an interval-valued set function $\bar{\nu}_{\bar{f}}$ be defined in terms of μ and $\bar{f}$ by (4.1). If $\bar{f}$ is Choquet integrably bounded and a fuzzy measure μ is continuous, then $\bar{\nu}_{\bar{f}} \ll_I \mu$.*

Proof. We note that $\bar{f} = [f^-, f^+]$ and that if $\bar{f}$ is Choquet integrably bounded, then clearly f^+, f^- are c-integrable. By Theorem 4.2(1)),

$$\nu_{f^-} \ll_I \mu \quad \text{and} \quad \nu_{f^+} \ll_I \mu.$$

Thus, if $A \in \Omega$ and $\mu(A) = 0$, then we have

$$\nu_{f^-}(A) = 0 \quad \text{and} \quad \nu_{f^+}(A) = 0.$$

Using the equation (4.2), we have $\bar{\nu}_{\bar{f}}(A) = [\nu_{f^-}(A), \nu_{f^+}(A)] = [0,0]$. Therefore we have $\bar{\nu}_{\bar{f}} \ll_I \mu$. □

Theorem 4.4. *Let an interval-valued set function $\bar{\nu}_{\bar{f}}$ be defined in terms of μ and $\bar{f}$ by (4.1). If $\bar{f}$ is Choquet integrably bounded and a fuzzy measure μ is continuous, then $\bar{\nu}_{\bar{f}} \ll_{VI} \nu$.*

Proof. By the equation (4.2), we have $\bar{\nu}_{\bar{f}} = [\nu_{f^-}, \nu_{f^+}]$. Then by Theorem 4.2(2), we obtain

$$\nu_{f^-} \ll_{VI} \mu \quad \text{and} \quad \nu_{f^+} \ll_{VI} \mu.$$

Thus, if $\{A_n\} \subset \Omega$ and $\mu(A_n) \to 0$ as $n \to \infty$, then

$$\nu_{f^-}(A_n) \to 0 \;\text{ and }\; \nu_{f^-}(A_n) \to 0 \;\text{ as }\; n \to \infty.$$

Thus, by Theorem 2.8(2),

$$d_H(\bar{\nu}_{\bar{f}}(A_n), [0,0]) = \max\{|\nu_{f^-}(A_n)|, |\nu_{f^+}(A_n)|\} \to 0$$

as $n \to \infty$. Therefore we have $\bar{\nu}_{\bar{f}} \ll_I \mu$. □

Now, we show that the lower semi-continuity and the upper semi-continuity of μ are preserved in $\bar{\nu}_{\bar{f}}$ when $\bar{\nu}_{\bar{f}}$ is constructed from a fuzzy measure μ and an interval-valued function $\bar{f}$ by the interval-valued Choquet integral according to (4.1).

Definition 4.5. (1) A nonnegative monotonic interval-valued set function $\bar{\nu}$ is said to be *lower semi-continuous in the meaning of order* if $\bar{\nu}(A_n) \uparrow [0,0]$ (order), whenever $\{A_n\}$ is an increasing sequence in Ω and $\mu(A_n) \to 0$ as $n \to \infty$.

(2)A nonnegative monotonic interval-valued set function $\bar{\nu}$ is said to be *upper semi-continuous in the meaning of inclusion* if $\bar{\nu}(A_n) \downarrow [0,0]$ (inclusion), whenever $\{A_n\}$ is an decreasing sequence in Ω, $\mu(A_1) < \infty$ and $\mu(A_n) \to 0$ as $n \to \infty$.

(3) A nonnegative monotonic interval-valued set function $\bar{\nu}$ is said to be *continuous in the meaning of order* (*in the meaning of inclusion*) if it is both a lower semi-continuous in meaning of order (in meaning of inclusion) and an upper semi-continuous in meaning of order (in the meaning of inclusion).

Theorem 4.6. ([19]) *Let $A \in \Omega$ and ν_g be defined in terms of a fuzzy measure μ and a real -valued function g by $\nu_g(A) = \int_A g d\mu$.*

(1) *If μ is lower semi-continuous and g is Choquet integrable, then ν_g is lower semi-continuous in the meaning of inclusion.*

(2) *If μ is upper semi-continuous and g is Choquet integrable, then ν_g is upper semi-continuous in the meaning of inclusion.*

Theorem 4.7. (1) *Let an interval-valued set function $\bar{\nu}_{\bar{f}}$ be defined in terms of μ and $\bar{f}$ by (4.1). If a fuzzy measure μ is a lower semicontinuous and $\bar{f}$ is Choquet integrably bounded, then $\bar{\nu}_{\bar{f}}$ is a lower semi-continuous in the meaning of inclusion.*

(2) *Let an interval-valued set function $\bar{\nu}_{\bar{f}}$ be defined in terms of μ and $\bar{f}$ by (4.1). If a fuzzy measure μ is an upper semicontinuous and $\bar{f}$ is Choquet integrably bounded, then $\bar{\nu}_{\bar{f}}$ is an upper semi-continuous in the meaning of inclusion.*

Proof. (1) Suppose that a fuzzy measure μ is a lower semi-continuous. Let $\{A_n\}$ be a sequence in Ω with $A_n \subset A_{n+1}$ for $n = 1, 2, \dots$ and $\mu(A_n) \to 0$ as $n \to \infty$. Since $\bar{f}$ is Choquet integrably bounded, there exists a c-integrable function g such that

$$f \leq g \text{ for all selections } f \in S(\bar{f}).$$

Thus, by Theorem 4.6(1), ν_g is a lower semi-continuous, that is, $\nu_g(A_n) \to 0$ as $n \to \infty$. Thus

$$\begin{aligned} & d_H(\bar{\nu}_{\bar{f}}(A_n), [0,0]) \\ = \; & \max\{\sup_{u \in \bar{\nu}_{\bar{f}}(A_n)} \inf_{v \in [0,0]} |u - v|, \; \sup_{v \in [0,0]} \inf_{u \in \bar{\nu}_{\bar{f}}(A_n)} |u - v|\} \\ = \; & \max\{\sup_{u \in \bar{\nu}_{\bar{f}}(A_n)} |u|, \; \inf_{u \in \bar{\nu}_{\bar{f}}(A_n)} |u|\} \\ = \; & \sup_{u \in \bar{\nu}_{\bar{f}}(A_n)} |u| \\ \leq \; & |\nu_g(A_n)|. \end{aligned}$$

Thus, we have

$$0 \leq \lim_{n \to \infty} d_H(\bar{\nu}_{\bar{f}}(A_n), [0,0]) \leq \lim_{n \to \infty} |\nu_g(A_n)| = 0,$$

that is, $\bar{\nu}_{\bar{f}}$ is lower semi-continuous in the meaning of inclusion.

(2) The proof is similar to the proof of (1). □

By the same method of the proof of Theorem 4.7, it is to show that under the same hypothesis of Theorem 4.7 we have $\bar{\nu}_{\bar{f}}$ is a lower (an upper) semi-continuous in the meaning of order.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
COLLEGE OF SCIENCE KONKUK UNIVERSITY
CHUNGJU 380-701, KOREA
E-mail address: leechae.jang@kku.ac.kr