

COMPLEMENTED SUBLATTICE OF THE BANACH ENVELOPE OF Weak L_1 ISOMORPHIC TO ℓ^p

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ABSTRACT. In this paper we investigate the ℓ^p space structure of the Banach envelope of Weak L_1 . In particular, the Banach envelope of Weak L_1 contains a complemented Banach sublattice that is isometrically isomorphic to the nonseparable Banach lattice ℓ^p , ($1 \leq p < \infty$) as well as the separable case.

1. Introduction

The space Weak L_1 , as a Lorentz space $L(1, \infty)$, was introduced in analysis because key operators of harmonic analysis do not map L_1 into L_1 . Examples of such operators are the Hardy-Littlewood maximal function and the Hilbert transform. In this viewpoint, it became natural to investigate Weak L_1 consisting of all (equivalence classes of) μ -measurable functions f on X for which $\|f\| = \sup_{a>0} a\mu(\{x \in X : |f(x)| > a\}) < \infty$. This space with the topology defined by the quasinorm $\|\cdot\|$ is called the Weak L_1 . Let U be the convex hull of $V = \{f \in \text{Weak } L_1 : \|f\| \leq 1\}$ and let $\|\cdot\|$ be the Minkowski functional of U . Since V is solid, i.e., $f \in V, |g| \leq |f|$ implies $g \in V$, U is solid. Hence $\|\cdot\|$ is a lattice seminorm on Weak L_1 and $\|\cdot\| \leq \|\cdot\|$. Let I be the ideal $\{f : \|f\| = 0\}$. Then Weak L_1/I with the quotient order and the norm $\|f + I\| = \|f\|$ is a normed vector lattice. Its completion is a Banach lattice, which is called *the Banach envelope of Weak L_1* (which we will be denoted by \mathbf{W}). By identifying f with $f + I$, we can identify Weak L_1 with a dense sublattice of \mathbf{W} . It has been shown in [3] and [4] that in the nonatomic case

$$(1.1) \quad \|f\|_{\mathbf{W}} = \lim_{n \rightarrow \infty} \sup_{\frac{q}{p} \geq n} \frac{1}{\ln \frac{q}{p}} \int_{\{p \leq |f| \leq q\}} |f| d\mu.$$

Later on, in [4] actually the Banach envelope seminorm on \mathbf{W} was calculated to be exactly as above. Note that the seminorm on \mathbf{W} defined in (1.1) is a lattice seminorm. This is not quite obvious, but using integration by parts, one

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can readily show that the seminorm $\|\cdot\|_W$ is exactly same as (see [7]),

$$(1.2) \quad \|f\|_W = \lim_{n \rightarrow \infty} \sup_{\frac{a}{p} \geq n} \frac{1}{\ln \frac{a}{p}} \int_p^a \mu(\{x \in X : |f| > t\}) dt.$$

Even though Weak L_1 is complete with respect to the quasinorm $\|f\|$, it is not complete with respect to the seminorm $\|\cdot\|_W$. This is due to M. Cwikel and C. Fefferman ([3] and [4]). We will only consider Weak L_1 and its Banach envelope in the case where (X, Σ, μ) is a finite nonatomic separable measure space. Thus, we can restrict ourselves to the cases $X = (0, 1]$, and μ notes the Lebesgue measure on $(0, 1]$. Throughout this paper, μ denotes the Lebesgue measure on $(0, 1]$ and L_p , $(1 \leq p \leq \infty)$ denotes the space $L_p((0, 1], \mu)$.

In [3], M. Cwikel and Y. Sager showed that Weak L_1 has a non trivial dual space and it is known that (except for some trivial measure space), Weak L_1 is not normable. In [7], J. Kupka and T. Peck studied the structure of Weak L_1 . They showed that the space L_∞ is dense in the dual of Weak L_1 endowed with *weak**-topology and showed lattice embeddings of L_1 , $\ell^1([0, 1])$, ℓ^∞ and $c_0([0, 1])$ into \mathbf{W} . Later on, T. Peck and M. Talagrand ([11]) proved that every separable order continuous Banach lattice is lattice isometric to a sublattice of \mathbf{W} . Finally, H. Lotz and T. Peck ([10]) removed the hypothesis of order continuity in the separable case.

In [7, theorem 3.5], J. Kupka and N. T. Peck showed that $\ell^1[0, 1]$ is a Banach subspace of the Banach envelope of Weak L_1 . Moreover, they showed that $\ell^1[0, 1]$ is isometrically isomorphic to a complemented sublattice of the Banach envelope of Weak L_1 . In this paper we will show that the Banach envelope of Weak L_1 contains a sublattice that is isometric, order isomorphic to $\ell^p(S)$, $1 \leq p < \infty$ and this space is also a complemented subspace of the Banach envelope of Weak L_1 .

To study this subject, we need some basic facts about the dual of Weak L_1 . For the construction of the dual of Weak L_1 , we will use the idea of J. Kupka and T. Peck in [7]. We would like to convert the nonlinear limit superior expression (1.1) for $\|\cdot\|_W$ into a linear limit expression by directing the numbers $I_a^b(f) = \frac{1}{\ln \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f| d\mu$ in some fashion. For this, we introduce an ultrafilter \mathcal{U} so that the limit of the $I_a^b(f)$ along \mathcal{U} determines a canonical integral-like linear functional $I_{\mathcal{U}} \in \mathbf{W}^*$. We now begin with the discussion of \mathcal{U} . For $n = 1, 2, 3, \dots$, let

$$F_n = \{(a, b) : 1 \leq a < b; \frac{b}{a} \geq n\},$$

and then define

$$\mathcal{F} = \{F_n : n \geq 1\}.$$

Treating \mathcal{F} as a filter of subsets of the set $S = [1, \infty) \times [1, \infty)$, we obtain from Zorn's lemma an ultrafilter \mathcal{U} of subsets of S such that $\mathcal{F} \subset \mathcal{U}$. From now on, we will fix the ultrafilter \mathcal{U} . The significance of the ultrafilter property lies in the fact that, for every function $f \in \text{Weak } L_1$, and for every integer n

sufficiently large, the set $\{I_a^b(f) : (a, b) \in F_n\}$ is bounded, so that the limit $l = \lim_{\mathcal{U}} I_a^b(f)$ always exists. [Recall that l is defined by the requirement : For every $\epsilon > 0$, there is a set $U \in \mathcal{U}$ such that $|I_a^b(f) - l| < \epsilon$ whenever $(a, b) \in U$. We have stipulated that $a \geq 1$ in the definition of the set F_n in order to avoid choices of \mathcal{U} for which the limit uselessly vanishes on Weak L_1 .]

Define the “ersatz integral” for every nonnegative function $f \in \text{Weak } L_1$ by

$$(1.3) \quad I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f) = \lim_{\mathcal{U}} \frac{1}{\ln \frac{b}{a}} \int_{\{a \leq f \leq b\}} f d\mu.$$

Now we can give the main properties of the ersatz integral $I_{\mathcal{U}}$ with no proof.

Lemma 1.1 (J. Kupka and N. T. Peck). *Let $f, g \in \text{Weak } L_1$ be nonnegative, and let $r > 0$. Then we have*

- i) $I_{\mathcal{U}}(rf) = rI_{\mathcal{U}}(f)$.
- ii) $I_{\mathcal{U}}(f + g) = I_{\mathcal{U}}(f) + I_{\mathcal{U}}(g)$.
- iii) *If $f \leq g$, then $I_{\mathcal{U}}(f) \leq I_{\mathcal{U}}(g)$.*
- iv) $I_{\mathcal{U}}(f) \leq \|f\|_W$.

We define $I_{\mathcal{U}}(f)$ for an arbitrary function $f \in \text{Weak } L_1$ by

$$(1.4) \quad I_{\mathcal{U}}(f) = I_{\mathcal{U}}(f^+) - I_{\mathcal{U}}(f^-).$$

Then we have

- v) $I_{\mathcal{U}}$ is linear.
- vi) $|I_{\mathcal{U}}(f)| \leq \|f\|_W$ for all $f \in \text{Weak } L_1$.
- vii) $I_{\mathcal{U}}$ vanishes on the ideal $I = \{f : \|f\|_W = 0\}$

and hence determines a well defined, bounded linear functional on \mathbf{W} .

Also J. Kupka and N. T. Peck showed that L_{∞} space is dense in the Weak L_1^* . We give the theorem with no proof.

Theorem 1.2 ([7]). *Define a linear operator $T_{\mathcal{U}} : L_{\infty}(\mu) \rightarrow \text{Weak } L_1^*$ by $T_{\mathcal{U}}(m) : f \mapsto I_{\mathcal{U}}(mf)$ for all $m \in L_{\infty}(\mu)$, and for all $f \in \text{Weak } L_1$. Then $T_{\mathcal{U}}$ constitutes an isometric, order isomorphism of $L_{\infty}(\mu)$ into $\text{Weak } L_1^*$. Moreover, the linear span of the subspace $T_{\mathcal{U}}(L_{\infty}(\mu))$, as \mathcal{U} ranges over the collection of ultrafilters which contains \mathcal{F} , constitutes a norming and hence a weak* dense subspace of $\text{Weak } L_1^*$.*

The standard integral not only constitutes a distinguished linear functional, but it also serves to define the L_1 norm. Likewise our “ersatz integral” $I_{\mathcal{U}}$ determines an L_1 -like seminorm on Weak L_1 and, by above lemma 1.1, vii) on \mathbf{W} . Thus, for $f \in \text{Weak } L_1$, define

$$(1.5) \quad \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|).$$

We now give a theorem about linear functionals on \mathbf{W} which is actually due to J. Kupka and N. T. Peck (see [7, 2.20]).

Theorem 1.3. *For a ultrafilter \mathcal{U} defined as above, let $f \in \mathbf{W}$ be a nonnegative function with $\|f\|_{\mathcal{U}} = 1$. Then for any $g \in \mathbf{W}$, disjointly supported from f , we can find $\phi \in \mathbf{W}^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$ and $\phi(g) = 0$.*

Now we shall see that the set

$$(1.6) \quad L(\mathcal{U}) = \{f \in \text{Weak } L_1 : \|f\|_W = \|f\|_{\mathcal{U}}\}$$

on which $\|\cdot\|_W$ shares the vital L_1 additivity property (lemma 1.1 ii)), is big enough to contain a copy of the given L_1 space. So for the remainder of the paper, we take a nonnegative function $f \in \mathbf{W}$ with $\|f\|_W = 1$ to be fixed, and we take an ultrafilter with $\|f\|_W = \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$ to be fixed. For the more properties of the set $L(\mathcal{U})$, refer to [7, lemma 3.2, 3.3]. From the above theorem, we can get the following corollary by an inductive argument.

Corollary 1.4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in \mathbf{W} with $\|f_n\|_W = 1$, for all $n = 1, 2, 3, \dots$ and such that the f_n have pairwise disjoint supports. Then for each n , there exists a linear functional ϕ_n on \mathbf{W} such that $\phi_n(f) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$, if $n \neq m$.*

Proof. We can show this by an inductive argument successively applying above theorem 1.3, for each f_n . For given $f_1 \in \mathbf{W}$, by theorem 1.3, we can choose ϕ_1 with $\phi_1(f_1) = \|\phi_1\| = 1$ and $\phi_1(f_j) = 0$, for all $j = 2, 3, \dots$. If we selected $\phi_1, \phi_2, \dots, \phi_n$ satisfying all the conclusions of corollary, then ϕ_{n+1} can be selected by applying theorem 1.3 again. This proves the corollary. \square

Lemma 1.5. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in \mathbf{W} such that the f_n have pairwise disjoint supports with $\|f_n\|_W = 1$, for all $n = 1, 2, 3, \dots$ and let $(\phi_n)_{n=1}^{\infty}$ be a sequence of linear functionals on \mathbf{W} selected as in corollary 1.4. Then for any $f \in \mathbf{W}$, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_W$.*

Proof. For an arbitrary function $f \in \mathbf{W}$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = \text{supp}(f_n)$, and f_n is bounded on $E_{n,k}^c$ for all k (See Theorem 1.3.). Fix $n \neq m$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $(E_{m,k})_{k=1}^{\infty}$ the corresponding sequence for f_m .

Let $r = \text{sgn} I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$, $s = \text{sgn} I_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)$. Put $m = r\chi_{E_{n,k}} + s\chi_{E_{m,k}}$ so that $\|m\|_{\infty} = 1$. We identify $\widehat{m} = T_{\mathcal{U}}(m)$ as a positive linear functional on $\text{Weak } L_1$ (also on \mathbf{W}) where $T_{\mathcal{U}}$ is an isometric isomorphism of L_{∞} into $\text{Weak } L_1^*$ in Theorem 2.2. Then we have

$$(1.7) \quad \begin{aligned} \widehat{m}(f) &= |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)| + |I_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)| \\ &= I_{\mathcal{U}}(m \cdot f) \\ &\leq \|m\|_{\infty} \|f\|_{\mathcal{U}} \quad \text{since } \|m\|_{\infty} = 1 \\ &= \|f\|_{\mathcal{U}} \quad \text{by lemma 1.1 iv)} \\ &\leq \|f\|_W. \end{aligned}$$

By the additive rule for nets, we can say that in the limit

$$(1.8) \quad \begin{aligned} |\phi_n(f)| + |\phi_m(f)| &\leq \|f\|_{\mathcal{U}} && \text{by lemma 1.1 iv)} \\ &\leq \|f\|_W. \end{aligned}$$

To show that $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_W$, it suffices to show that for any $N \in \mathbf{N}$, $\sum_{n=1}^N |\phi_n(f)| \leq \|f\|_W$. For $n = 1, 2, 3, \dots$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $E_n = \text{supp}(f_n)$. Let $r_n = \text{sgn} I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$. Put $m = \sum_{n=1}^N r_n \chi_{E_{n,k}}$. Then we have $\|m\|_{\infty} = 1$. By the same argument as above, one can get

$$(1.9) \quad \begin{aligned} \widehat{m}(f) &= \sum_{m=1}^N |I_{\mathcal{U}}(\chi_{m,k} \cdot f)| \\ &= I_{\mathcal{U}}(mf) \\ &\leq \|m\|_{\infty} \|f\|_{\mathcal{U}} && \text{by lemma 1.1 iv)} \\ &\leq \|f\|_W. \end{aligned}$$

By the additive rule for nets, in the limit

$$(1.10) \quad \begin{aligned} \sum_{n=1}^N |\phi_n(f)| &\leq \|f\|_{\mathcal{U}} && \text{by lemma 1.1 iv)} \\ &\leq \|f\|_W. \end{aligned}$$

We can therefore say that $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_W$. This proves the lemma. \square

2. ℓ^p space structure of the Banach envelope of Weak L_1

First, we will investigate a complemented sublattice of \mathbf{W} that is isometrically isomorphic to ℓ^p , $1 \leq p < \infty$. The space ℓ^p is a classical Banach lattice which is separable and order continuous. To prove this, we need a theorem which was done by H. Lotz and T. Peck in [10].

Theorem 2.1 ([10]). *Let E be a separable Banach lattice. Then E is lattice isometric to a closed sublattice of the Banach envelope of weak L_1 .*

For $1 \leq p < \infty$, ℓ^p is a separable Banach lattice with order continuous norm. Hence by Theorem 2.1, there exists a lattice isometry T from ℓ^p into \mathbf{W} . Then we can embed ℓ^p into \mathbf{W} under a lattice isometry. So one can ask the natural question: What can we say about the range of T for ℓ^p , $1 \leq p < \infty$? The following theorem says that the Banach envelope of Weak L_1 contains a complemented sublattice that is isometrically isomorphic to ℓ^p , $1 \leq p < \infty$.

Theorem 2.2. *For $1 \leq p < \infty$, let $T : \ell^p \rightarrow \mathbf{W}$ be a lattice isometry given by $T(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n=1}^{\infty} a_n f_n$ where $(e_{n=1})_{n=1}^{\infty}$ is the usual basis of ℓ^p . Then the range of T is a complemented sublattice of \mathbf{W} .*

Proof. Let $T : \ell^p \rightarrow \mathbf{W}$ be a lattice isometry, and let $(e_{i=1})_{i=1}^{\infty}$ be the usual basis for ℓ^p . Define $Te_i = f_i$, $i = 1, 2, 3, \dots$. Then since T preserves lattice structure, f_i are nonnegative pairwise disjoint and $\|f_i\|_W = 1$, for all $i = 1, 2, 3, \dots$. Now we can find linear functionals ϕ_n on \mathbf{W} such that $\phi_n(f_n) = 1$, $\phi_n(f_m) = 0$ if $n \neq m$ and $\|\phi_n\| = 1$, for all $n = 1, 2, 3, \dots$

For arbitrary function $f \in \mathbf{W}$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = \text{supp}(f_n)$ and f_n is bounded on $E_{n,k}^C$ for all k and $n = 1, 2, 3, \dots$

Now we define a contractive projection $P : \mathbf{W} \rightarrow T\ell^p$ from \mathbf{W} onto $T\ell^p$, by

$$(2.1) \quad P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n.$$

First, we need to show that P is well defined. Since $(f_n)_{n=1}^{\infty}$ is a copy of the usual ℓ^p basis in \mathbf{W} , we have for $1 \leq p < \infty$,

$$(2.2) \quad \begin{aligned} \left\| \sum_{n=1}^{\infty} \phi_n(f) f_n \right\|_W &= \left(\sum_{n=1}^{\infty} |\phi_n(f)|^p \right)^{\frac{1}{p}} \\ &\leq \sum_{n=1}^{\infty} |\phi_n(f)| \\ &\leq \|f\|_{\mathcal{U}} \quad \text{by Lemma 1.5} \\ &\leq \|f\|_W. \end{aligned}$$

Hence, $P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n \in T\ell^p$, for all $f \in \mathbf{W}$.

Next, we need to show that $\|P\| = 1$. By (2.2), we have $\|P(f)\|_W \leq \|f\|_W$. Hence $\|P\| \leq 1$. On the other hand, since $f_n \in T\ell^p \subset \mathbf{W}$, we have

$$(2.3) \quad \begin{aligned} P(f_m) &= \sum_{n=1}^{\infty} \phi_n(f_m) f_n \\ &= \phi_m(f_m) f_m = f_m. \end{aligned}$$

This shows $\|P\| = 1$.

Finally, we need to show that $P^2 = P$. For $f \in \mathbf{W}$,

$$(2.4) \quad \begin{aligned} P^2(f) &= P\left(\sum_{n=1}^{\infty} \phi_n(f) f_n\right) \\ &= \sum_{j=1}^{\infty} \phi_j\left(\sum_{n=1}^{\infty} \phi_n(f) f_n\right) f_j, \quad \text{by } \phi_n(f_j) = \delta_{n,j} \\ &= \sum_{j=1}^{\infty} \phi_j(f) f_j \\ &= P(f). \end{aligned}$$

Therefore P is the norm one projection from \mathbf{W} onto $T\ell^p$. This proves the theorem. \square

Next, we study the structure of nonseparable complemented sublattices of \mathbf{W} . As a typical example of such a nonseparable Banach lattice, one can give $\ell^p(S)$, for $1 \leq p < \infty$ and $|S| \leq 2^{\aleph_0}$. To study these spaces, we need some preliminaries from [11]. The *dyadic tree* T is the set $\cup_{n=1}^{\infty} \{0, 1\}^n$. A *node* is an element of T : the node φ is at *level* n if $\varphi \in \{0, 1\}^n$, and we denote this by writing $|\varphi| = n$. The nodes can be ordered : we say $\varphi < \psi$ if

$$(2.5) \quad \begin{aligned} &\text{either } |\varphi| = |\psi| = n \text{ say, and } \varphi = (\epsilon_1, \dots, \epsilon_n), \psi = (\delta_1, \dots, \delta_n) \\ &\text{with } \epsilon_i = \delta_i, 1 \leq i \leq j, \text{ and } \epsilon_{j+1} = 0, \delta_{j+1} = 1; \text{ or} \end{aligned}$$

$$(2.6) \quad \begin{aligned} &\psi = (\epsilon_1, \dots, \epsilon_m), \psi = (\delta_1, \dots, \delta_n), \text{ with } m < n \\ &\text{and } \epsilon_i = \delta_i \text{ for } 1 \leq i \leq m. \end{aligned}$$

Note that this is not a partial order. However, the ordering defined by the above (2.6) is used only to define the term “branch”, while in the application we will be using only the lexicographic order (2.5). A *branch* of T is a sequence of nodes $\varphi(0), \varphi(1), \dots$ such that $|\varphi(n)| = n$ and $\varphi(n) < \varphi(n + 1)$, for all n in the order defined as above (2.6). We will let $B(n)$ denote $\varphi(n)$, where $B = (\varphi(0), \varphi(1), \dots)$. One can order the branches by saying that $B_1 \prec B_2$ if $B_1(j) = B_2(j)$ for $0 \leq j < k$ and $B_1(k) < B_2(k)$ for some k in the lexicographic order (2.5). Let \mathcal{B} be the set of all branches. For future reference, we state the theorem in [11] with no proof.

Theorem 2.3. *Let E be a Banach lattice with generalized basis $(x_B : B \in \mathcal{B})$. Assume that (x_B) satisfies the following conditions:*

- i) *if $x = \sum a_B x_B$, then $x \geq 0$ if and only if each $a_B \geq 0$.*
- ii) *(spreading invariance) whenever $n \in \mathbf{N}$, a_1, \dots, a_n are reals and $B_1 < B_2 < \dots < B_n, B'_1 < B'_2 < \dots < B'_n$ are branches, then*

$$\left\| \sum_{i=1}^n a_i x_{B_i} \right\| = \left\| \sum_{i=1}^n a_i x_{B'_i} \right\|.$$

Then there is a lattice isometry of E into \mathbf{W} .

To get the main result, we need to see how to define a lattice isometry T from E into \mathbf{W} . For each $x_B \in E$, we should construct $e_B \in \mathbf{W}$ such that $Tx_B = e_B$. T. Peck and M. Talagrand defined $e_B, \forall B \in \mathcal{B}$ by

$$(2.7) \quad e_B = \sum_{m=1}^{\infty} \sum_{s \in \text{spr}_{g(m)}(G(m))} e_{B_{(g(m)), s, m}}.$$

Here, $e_{B(g(m)),s,m} = e_{i,s,m}$ where $B(g(m))$ is the i^{th} node at level $g(m)$ in the lexicographic ordering and for $i \in S(s,m)$, define

$$(2.8) \quad \begin{aligned} e_{i,s,m}(x) &= \frac{b_{i,s,m}}{x - u_{i,s,m}}, \quad x \in [v_{i,s,m}, w_{i,s,m}] \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

where for the other conditions we refer to [7, theorem 5].

Remark 2.4. In (2.7), $(e_B)_{B \in \mathcal{B}}$ are nonnegative pairwise disjointly supported elements in \mathbf{W} . For all $(a_i)_{i=1}^k$, we have

$$(2.9) \quad \left\| \sum_{i=1}^k a_i e_{B_i} \right\|_W = \left\| \sum_{i=1}^k a_i x_{B_i} \right\|.$$

We now can prove the main theorem for a nonseparable complemented sublattice in \mathbf{W} .

Theorem 2.5. *Let E be a Banach lattice with generalized basis $(x_B : B \in \mathcal{B})$. Assume that $(x_B)_{B \in \mathcal{B}}$ satisfies the following conditions;*

- i) if $x = \sum a_B x_B$, then $x \geq 0$ if and only if each $a_B \geq 0$.
- ii) (*spreading invariance*) whenever $n \in \mathbf{N}$, a_1, \dots, a_n are reals and $B_1 < B_2 < \dots < B_n$, $B'_1 < B'_2 < \dots < B'_n$ are branches, then $\left\| \sum_{i=1}^n a_i x_{B_i} \right\| = \left\| \sum_{i=1}^n a_i x'_{B'_i} \right\|$,

and let $T : E \rightarrow \mathbf{W}$ be a lattice isometry defined by above theorem 2.3. Then the range of T is a complemented sublattice of \mathbf{W} .

Proof. We define $T : E \rightarrow \mathbf{W}$ by

$$(2.10) \quad Tx_B = e_B = \sum_{m=1}^{\infty} \sum_{s \in spr_{g(m)}} e_{B(g(m)),s,m}$$

in the above theorem 2.3. By remark 2.4, $(e_B)_{B \in \mathcal{B}}$ are pairwise disjointly supported, and by (2.9), we have $\left\| \sum_{i=1}^k a_i x_{B_i} \right\| = \left\| \sum_{i=1}^k a_i e_{B_i} \right\|_W$ for all $(a_i)_{i=1}^k$. By normalizing, we can assume $\|e_B\|_W = 1$, for all $B \in \mathcal{B}$. By the corollary 1.4, we can choose a linear functional φ_B on \mathbf{W} such that $\|\varphi\| = 1$, $\varphi_B(e_B) = 1$, and $\varphi_B(e_{B'}) = 0$ if $B \neq B'$. For arbitrary $f \in \mathbf{W}$, $\varphi_B(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{D_{B,k}} \cdot f)\}$ where $(D_{B,k})_{k=1}^{\infty}$ is the decreasing sequence of subsets of $D_B = supp(e_B)$, $\bigcap_{k=1}^{\infty} D_{B,k} = \emptyset$ and e_B is bounded on $D_{B,k}^c$ for all $k = 1, 2, \dots$

Now define

$P : \mathbf{W} \rightarrow TE$ by

$$(2.11) \quad P(f) = \sum_{B \in \mathcal{B}} \varphi_B(f) e_B.$$

$\varphi_B(f)$ is non-zero for only countably many B . Hence we can write

$$(2.12) \quad P(f) = \sum_{i=1}^{\infty} \varphi_{B_i}(f) e_{B_i}.$$

Then

$$(2.13) \quad \begin{aligned} \|P(f)\|_W &= \left\| \sum_{i=1}^{\infty} \varphi_{B_i}(f) e_{B_i} \right\|_W \\ &\leq \sum_{i=1}^{\infty} |\varphi_{B_i}(f)| \|e_{B_i}\|_W \\ &= \sum_{i=1}^{\infty} |\varphi_{B_i}(f)| \\ &\leq \|f\|_W \quad \text{by lemma 1.5.} \end{aligned}$$

Now the rest of the argument that P is a norm one is just a routine proof. Therefore TE is a complemented sublattice in \mathbf{W} . This proves the theorem. \square

Now from this theorem, we immediately can get the following corollary.

Corollary 2.6. *Suppose E is a Banach lattice with generalized basis $\{x_s : s \in S\}$. Assume $\sum a_i x_{s_i} \geq 0$ if and only if each $a_i \geq 0$ and that the norm is symmetric. Then if $|S| \leq 2^{\aleph_0}$, the Banach envelope of Weak L_1 contains a complemented sublattice that is isometrically isomorphic to E .*

Proof. By theorem 2.3, let $T : E \rightarrow \mathbf{W}$ be a lattice isometry. The given condition about the basis implies that the hypotheses in theorem 2.5 are satisfied. Hence TE is a complemented sublattice of \mathbf{W} that is isometrically isomorphic to E . This proves the corollary. \square

Finally we can see that \mathbf{W} has a complemented sublattice that is lattice isomorphic to $\ell^p(S)$ where $|S| \leq 2^{\aleph_0}$, $1 \leq p < \infty$.

Corollary 2.7. *For $1 \leq p < \infty$ and $|S| \leq 2^{\aleph_0}$, the Banach envelope of Weak L_1 contains a complemented sublattice that is isometrically isomorphic to $\ell^p(S)$.*

Proof. The usual generalized basis $(e_s)_{s \in S}$ satisfies all conditions in corollary 2.6 with coordinatewise partial order. Then again by theorem 2.3, there is a lattice isometry T from $\ell^p(S)$ into \mathbf{W} . Again by corollary 2.6, the range of T , $T\ell^p(S)$ is a complemented sublattice of \mathbf{W} . This proves the corollary. \square

References

- [1] M. Cwikel and Y. Sagher, $L(p, \infty)^*$, Indiana Uni. Math. J. **21** (1972), 782–786.
- [2] M. Cwikel, *On the conjugates of some function spaces*, Studia Math. **45** (1973), 49–55.
- [3] M. Cwikel and C. Fefferman, *Maximal seminorm on wL_1* , **69** (1980), 149–154.
- [4] ———, *The canonical seminorm on wL_1* , Studia Math. **78** (1984), 275–278.
- [5] N. Dunford and J. T. Schwartz, *Linear Operator I: General Theory. Pure and Applied Mathematics*, New York: Interscience VII, 1967.

- [6] JeongHeung Kang, *Banach subspaces and envelope norm of wL_1* , Bull. Korean. Math. Soc. **35** (1998), no. 3, 409–420.
- [7] J. Kupka and T. Peck, *The L_1 -structure of wL_1* , Math. Ann. **269** (1984), 235–262.
- [8] Denny H. Leung, *The normed and Banach envelopes of $weakL^1$* , Isreal J. Math. **121** (2001), 247–264.
- [9] Lindenstrauss and Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
- [10] H. P. Lotz and T. Peck, *Sublattices of the Banach envelope of $Weak L_1$* , Proc. Amer. Math. Soc. **126** (1998), 75–84.
- [11] T. Peck and M. Talagrand, *Banach sublattices of $Weak L_1$* , Israel. J. Math. **59** (1987), 257–271.

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