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L^p BOUNDS FOR THE PARABOLIC MARCINKIEWICZ INTEGRAL WITH ROUGH KERNELS

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ABSTRACT. We prove the L^p $(1 < p < \infty)$ boundedness of the parabolic Marcinkiewicz integral with the kernel function $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$. The result is an improvement and extension of some known results.

1. Introduction

Let $\alpha_1, \ldots, \alpha_n$ be fixed real numbers, $\alpha_i \geq 1$. For fixed $x \in \mathbb{R}^n$, the function $F(x, \rho) = \sum_{i=1}^n \frac{{x_i}^2}{\rho^{2\alpha_i}}$ is a decreasing function in $\rho > 0$. We denote the unique solution of the equation $F(x, \rho) = 1$ by $\rho(x)$. In [8], Fabes and Rivière showed that $\rho(x)$ is a metric on $\mathbb{R}^n$, and $(\mathbb{R}^n, \rho)$ is called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$.

For $\lambda > 0$, let $A_\lambda = \begin{pmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{pmatrix}$. Suppose that $\Omega(x)$ is a real valued and measurable function defined on $\mathbb{R}^n$. We say $\Omega(x)$ is homogeneous of degree zero with respect to A_λ, if for any $\lambda > 0$ and $x \in \mathbb{R}^n$

$$\Omega(A_\lambda x) = \Omega(x). \tag{1.1}$$

Moreover, $\Omega(x)$ satisfies the following condition

$$\int_{S^{n-1}} \Omega(x')J(x')d\sigma(x') = 0, \tag{1.2}$$

where $J(x')$ is a function defined on the unit sphere S^{n-1} in $\mathbb{R}^n$, which will be defined in Section 2.

In 1966, Fabes and Rivière [8] proved that if $\Omega \in C^1(S^{n-1})$ satisfying (1.1) and (1.2), then the parabolic singular integral operator T_Ω is bounded

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on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^\alpha} f(x-y)\, dy \quad \text{and} \quad \alpha = \sum_{i=1}^n \alpha_i.$$

In 1976, Nagel, Rivière and Wainger [10] improved the above result. They showed T_Ω is still bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if replacing $\Omega \in C^1(S^{n-1})$ by a weaker condition $\Omega \in L\log^+ L(S^{n-1})$.

Inspired by the works in [8] and [10], recently, Ding, Xue and Yabuta [6] defined the parabolic Marcinkiewicz integral by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{\rho(y)\le t} \frac{\Omega(y)}{\rho(y)^{\alpha-1}} f(x-y)dy.$$

The authors of [6] gave the $L^p\,(1 < p < \infty)$ boundedness of the parabolic Marcinkiewicz integral:

Theorem A. *If $\Omega \in L^q(S^{n-1})(q > 1)$ satisfies* (1.1) *and* (1.2). *Then*

$$\|\mu_\Omega(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

On the other hand, we note that if $\alpha_1 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$, $\alpha = n$ and $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$. In this case, μ_Ω is just the classical Marcinkiewicz integral (we still denote it by μ_Ω here), which was studied by many authors. (See [12], [2], [4], [5], for example.) In particular, we mention that in 1972, Walsh [14] stated that if $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ satisfies (1.1) and (1.2) with $J(x') \equiv 1$, then μ_Ω is a bounded operator on $L^2(\mathbb{R}^n)$. In 2002, AL-Salman, AL-Qassem, Cheng and Pan [1] showed that the condition $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ is still sufficient for the $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness of μ_Ω.

In this paper, we will prove that the condition $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ is also sufficient for the $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness of the parabolic Marcinkiewicz integral μ_Ω.

Theorem 1. *Let $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ satisfies* (1.1) *and* (1.2). *Then*

$$\|\mu_\Omega(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Note that on S^{n-1},

$$L^q(S^{n-1})\,(q > 1) \subsetneq L\log^+ L(S^{n-1}) \subsetneq L(\log^+ L)^{1/2}(S^{n-1}).$$

Hence our result improves Theorem A and is an extension of the results in [1].

2. Some lemmas

In this section, we give some lemmas which will be used in the proof of Theorem 1. For any $x \in \mathbb{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2} \cos\varphi_{n-1} \\ x_2 &= \rho^{\alpha_2} \cos\varphi_1 \cdots \cos\varphi_{n-2} \sin\varphi_{n-1} \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2 \\ x_n &= \rho^{\alpha_n} \sin\varphi_1. \end{aligned}$$

Then $dx = \rho^{\alpha-1} J(\varphi_1, \ldots, \varphi_{n-1}) d\rho d\sigma$, where $\alpha = \sum_{i=1}^{n} \alpha_i$, $d\sigma$ is the element of area of S^{n-1} and $\rho^{\alpha-1} J(\varphi_1, \ldots, \varphi_{n-1})$ is the Jacobian of the above transform. In [8], it was shown there exists a constant $M \geq 1$ such that $1 \leq J(\varphi_1, \ldots, \varphi_{n-1}) \leq M$ and $J(\varphi_1, \ldots, \varphi_{n-1}) \in C^\infty((0, 2\pi)^{n-2} \times (0, \pi))$. So, it is easy to see that J is also a C^∞ function in the variable $y' \in S^{n-1}$. For simplicity, we denote still it by $J(y')$.

Lemma 2.1 ([9]). *Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\widehat{\Psi}(0) = 0$. Denote $\Psi_t(x) = t^{-\alpha}\Psi(A_{t^{-1}} x)$ for $t > 0$, the Littlewood-Paley g-function related to the transform A is defined by*

$$g_\Psi(f)(x) = \left(\int_0^\infty |\Psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then there exists $C > 0$ such that $\|g_\Psi(f)\|_p \leq C\|f\|_p$ for any $f \in L^p(\mathbb{R}^n), 1 < p < \infty$.

Lemma 2.2. *Suppose that $0 \leq \beta \leq 1$ and m denotes the distinct numbers of $\{\alpha_j\}$. Then for any $x, y \in \mathbb{R}^n$*

$$\left| \int_1^2 e^{-i\langle A_\lambda x, y\rangle} \frac{d\lambda}{\lambda} \right| \leq C|\langle x, y\rangle|^{-\frac{\beta}{m}},$$

where $C > 0$ is independent of x and y.

Proof. By Theorem 1 in [13] and the same argument of Lemma 2 in [6], it is easy to get

$$\left| \int_v^u e^{-i\langle A_\lambda x, y\rangle} d\lambda \right| \leq C|\langle x, y\rangle|^{-\frac{\beta}{m}}, \tag{2.1}$$

where C remains bounded as long as u and v lie in a compact subinterval of $(0, \infty)$.

Since $1/\lambda$ is a decreasing function in λ, then there exists $\xi \in [1, 2]$ such that

$$\int_1^2 e^{-i<A_\lambda x, y>} \frac{d\lambda}{\lambda} = \int_1^\xi e^{-i\langle A_\lambda x, y\rangle} d\lambda + 1/2 \int_\xi^2 e^{-i\langle A_\lambda x, y\rangle} d\lambda.$$

By (2.1), we get the conclusion of Lemma 2.2. Obviously, the constant C is independent of x and y. □

Lemma 2.3 ([11]). *Support that λ_j's and α_j's are fixed real numbers and $\Gamma(t) = (\lambda_1 t^{\alpha_1}, \dots, \lambda_n t^{\alpha_n})$ is a function from $\mathbb{R}_+$ to $\mathbb{R}^n$. For suitable f, the maximal function associated to the homogeneous curve Γ is defined by*

$$\mathcal{M}_\Gamma(f)(x) = \sup_h \frac{1}{h}\left| \int_0^h f(x - \Gamma(t))\, dt \right|, \quad h > 0. \tag{2.2}$$

Then for $1 < p \leq \infty$, there is a constant $C > 0$, independent of λ_j's and f, such that

$$\|\mathcal{M}_\Gamma(f)\|_p \leq C\|f\|_p.$$

To show the following lemma, we need to give some definitions. For a suitable family of measures $\tau = \{\tau_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$, we define the operator Δ_τ and τ^* by

$$\Delta_\tau(f)(x) = \left(\int_{\mathbb{R}} |\tau_t * f(x)|^2\, dt \right)^{1/2}$$

and

$$\tau^*(f)(x) = \sup_{t\in\mathbb{R}} (|\tau_t| * |f|)(x),$$

respectively.

The following lemma is a variation of Lemma 3 in [1].

Lemma 2.4. *Let $b \geq 2$, $B > 0$, $\gamma > 0$, and $q > 1$. Suppose that the family of measures $\{\tau_t : t \in \mathbb{R}\}$ satisfies the following conditions:*

(i) $\|\tau_t\| \leq CB$ *for* $t \in \mathbb{R}$;

(ii) $|\widehat{\tau_t}(\xi)| \leq CB(\min\{|A_{b^t}\xi|, |A_{b^t}\xi|^{-1}\})^{\gamma/\ln b}$ *for* $\xi \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$;

(iii) $\|\tau^*(f)\|_q \leq CB\|f\|_q$ *for* $f \in L^q(\mathbb{R}^n)$,

where $C > 0$ is independent of b, f and B. Then, for every p satisfying $|1/p - 1/2| < 1/(2q)$, there exists a constant $C > 0$, independent of b, f and B, such that for $f \in L^p(\mathbb{R}^n)$

$$\|\Delta_\tau(f)\|_p \leq CB\|f\|_p.$$

Proof. Choose $\psi \in C^\infty(\mathbb{R}^n)$ such that

(i) $\psi(\xi) = \psi(\rho(\xi))$;

(ii) $0 < \psi(\xi) \leq 1$;

(iii) $\mathrm{supp}(\psi) \subset \{y : 4/(5b) \leq \rho(y) \leq 5b/4\}$;

(iv) $\int_0^\infty \psi(\zeta)/\zeta\, d\zeta = 2\ln b$.

Define $\Psi \in C^\infty(\mathbb{R}^n)$ by $\widehat{\Psi}(\xi) = \psi(\rho(\xi)^2)$. For $t > 0$, denote $\Psi_t(x) = t^{-\alpha}\Psi(A_{t^{-1}}x)$. It is easy to check $\widehat{\Psi_t}(\xi) = \psi(t^2\rho(\xi)^2)$. Thus, for $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$f(x) = \frac{1}{\ln b}\int_0^\infty \Psi_t * f(x)\frac{dt}{t} = \int_{-\infty}^\infty \Psi_{b^t} * f(x)\, dt. \tag{2.3}$$

By (2.3) and the Minkowski inequality, we obtain

$$\begin{aligned}\Delta_\tau(f)(x) &= \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \Psi_{b^{s+t}} * \tau_t * f(x)\,ds\right|^2 dt\right)^{1/2} \\ &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\Psi_{b^{s+t}} * \tau_t * f(x)|^2\,dt\right)^{1/2} ds \\ &= \int_{\mathbb{R}} G_s(f)(x)\,ds.\end{aligned}$$

Using the Minkowski inequality again yields

$$\|\Delta_\tau(f)\|_p \le \int_{-\infty}^0 \|G_s(f)\|_p\,ds + \int_0^\infty \|G_s(f)\|_p\,ds. \tag{2.4}$$

Below we show that if $1 < p_0 < \infty$, $p_0 \ne 2$, and $|\frac{1}{2} - \frac{1}{p_0}| = \frac{1}{2q}$, then

$$\|G_s f\|_{p_0} \le CB\|f\|_{p_0}, \tag{2.5}$$

where C is independent of s, f and b.

We first consider the case $1 < p_0 < 2$. By Lemma 2.4 (i), we have

$$\left\|\int_{-\infty}^{\infty} \tau_t * H_{s+t}(\cdot)dt\right\|_1 \le CB\left\|\int_{-\infty}^{\infty} H_t(\cdot)dt\right\|_1, \tag{2.6}$$

where $H_{s+t}(x) = \Psi_{b^{s+t}} * f(x)$. By Lemma 2.4 (iii), we get

$$\|\sup_{t\in\mathbb{R}} |\tau_t * H_{s+t}|\|_q \le \|\tau^*(\sup_{t\in\mathbb{R}} |H_t|)\|_q \le CB\|\sup_{t\in\mathbb{R}} |H_t|\|_q. \tag{2.7}$$

Now if we define a linear operator T by $TH_{s+t}(x) = \tau_t * H_{s+t}(x)$, then (2.6) and (2.7) show that T is a bounded operator on $L^1(L^1(\mathbb{R}), \mathbb{R}^n)$ and $L^q(L^\infty(\mathbb{R}), \mathbb{R}^n)$, respectively. Since $1 < p_0 < 2$ and $\frac{1}{q} = \frac{2}{p_0} - 1$, using the operator interpolation theorem between (2.6) and (2.7), we know that T is also bounded on $L^{p_0}(L^2(\mathbb{R}), \mathbb{R}^n)$. Hence, by Lemma 2.1, we get

$$\begin{aligned}\|G_s(f)\|_{p_0} &= \left\|\left(\int_{-\infty}^{\infty} |\tau_t * H_{s+t}(\cdot)|^2 dt\right)^{1/2}\right\|_{p_0} \\ &\le CB\left\|\left(\int_{-\infty}^{\infty} |H_t(\cdot)|^2 dt\right)^{1/2}\right\|_{p_0} \\ &\le CB\|f\|_{p_0}.\end{aligned}$$

Then we obtain (2.5) for $1 < p_0 < 2$.

Next we consider the case $2 < p_0 < \infty$. Since $1/(2q) = 1/2 - 1/(p_0)$, we get $q = (\frac{p_0}{2})'$. Hence

$$\begin{aligned}\|G_s(f)\|_{p_0} &= \left\| \left(\int_{\mathbb{R}} |\Psi_{b^{s+t}} * \tau_t * f(\cdot)|^2 \, dt \right)^{1/2} \right\|_{p_0} \\ &= \left\{ \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} |\Psi_{b^{s+t}} * \tau_t * f(x)|^2 \, dt \right)^{p_0/2} dx \right)^{2/p_0} \right\}^{1/2} \\ &= \left(\sup_{\substack{\omega \in L^q(\mathbb{R}^n) \\ \|\omega\|_q = 1}} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\Psi_{b^{s+t}} * \tau_t * f(x)|^2 \, dt \ \ \omega(x) \, dx \right| \right)^{1/2}.\end{aligned}$$

Applying Hölder's inequality and noting the fact $\|\tau_t\|_1 \le CB$,

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\Psi_{b^{s+t}} * \tau_t * f(x)|^2 \, dt \ \ \omega(x) \, dx \right| \\ &\le \|\tau_t\|_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\tau_t| * |\Psi_{b^{s+t}} * f(x)|^2 \, dt |\omega(x)| \, dx \\ &\le CB \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} |\Psi_{b^{s+t}} * f(x)|^2 \, dt \right) \tau^*(\widetilde{\omega})(-x) \, dx \\ &\le CB \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} |\Psi_{b^{s+t}} * f(x)|^2 \, dt \right)^{p_0/2} dx \right)^{2/p_0} \|\tau^*(\widetilde{\omega})\|_q,\end{aligned}$$

where $\widetilde{\omega}(x) = \omega(-x)$. Using Lemma 2.1 and Lemma 2.4 (iii), we obtain

$$\|G_s f\|_{p_0}^2 \le \sup_{\substack{\omega \in L^q \\ \|\omega\|_q = 1}} CB \|f\|_{p_0}^2 \|\tau^*(\widetilde{\omega})\|_q \le CB^2 \|f\|_{p_0}^2.$$

Thus we have (2.5) for $2 < p_0 < \infty$. From the proof of (2.5) above, it is easy to check that the constant C is independent of s, b, B and f.

Now we give a delicate estimate for $\|G_s(f)\|_2$. By the Plancherel theorem, we get

$$\|G_s(f)\|_2^2 \le \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\psi(\rho(A_{b^{s+t}}\xi)^2)|^2 |\widehat{\tau_t}(\xi)|^2 \, d\xi \, dt.$$

For $s > 0$, using the estimate $|\widehat{\tau_t}(\xi)| \le CB|A_{b^t}\xi|^{\gamma/\ln b}$, and the properties of ψ, we have

$$\begin{aligned}&\|G_s(f)\|_2^2 \\ &\le CB \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\psi(\rho(A_{b^{s+t}}\xi)^2)|^2 |A_{b^t}\xi|^{2\gamma/\ln b} \, d\xi \, dt \\ &\le CB \int_{\mathbb{R}} \int_{\frac{2}{\sqrt{5}} b^{-s-1/2} \le b^t \rho(\xi) \le \frac{\sqrt{5}}{2} b^{-s+1/2}} |\widehat{f}(\xi)|^2 |A_{b^t}\xi|^{2\gamma/\ln b} \, d\xi \, dt \\ &\le CB \int_{\mathbb{R}} \int_{\frac{2}{\sqrt{5}} b^{-s-1/2} \le b^t \rho \le \frac{\sqrt{5}}{2} b^{-s+1/2}} \int_{S^{n-1}} J(\xi') |\widehat{f}(A_\rho \xi')|^2 \\ &\quad \times |A_{b^t \rho}\xi'|^{2\gamma/\ln b} \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho \, dt\end{aligned}$$

$$\le CB \int_{\mathbb{R}} \int_{\frac{2}{\sqrt{5}} b^{-s-1/2} \le b^t \rho \le \frac{\sqrt{5}}{2} b^{-s+1/2}} \int_{S^{n-1}} J(\xi') |\widehat{f}(A_\rho \xi')|^2$$
$$\times \left((b^{-s+1/2})^{2\alpha_1} + \cdots + (b^{-s+1/2})^{2\alpha_n} \right)^{\gamma/\ln b} \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho \, dt$$
$$\le CB \int_0^\infty \int_{S^{n-1}} J(\xi') |\widehat{f}(A_\rho \xi')|^2 \left((b^{-s+1/2})^{2\alpha_1} + \cdots + (b^{-s+1/2})^{2\alpha_n} \right)^{\gamma/\ln b}$$
$$\times \left(\int_{-s-1/2-\frac{\log \frac{\sqrt{5}}{2}}{\log b} - \frac{\log \rho}{\log b}}^{-s+1/2+\frac{\log \frac{\sqrt{5}}{2}}{\log b} - \frac{\log \rho}{\log b}} dt \right) \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho$$
$$\le CB b^{\max\{\alpha_j\}\gamma/\ln b} (b^{-s})^{2\min\{\alpha_j\}\gamma/\ln b} \int_0^\infty \int_{S^{n-1}} J(\xi') |\widehat{f}(A_\rho \xi')|^2 \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho$$
$$\le CB e^{\max\{\alpha_j\}\gamma} e^{-2s\gamma} \|\widehat{f}\|_2^2$$
$$\le CB e^{-2s\gamma} \|f\|_2^2.$$

Hence we have

$$\|G_s(f)\|_2 \le CBe^{-s\gamma} \|f\|_2 \quad \text{for} \quad s > 0, \tag{2.8}$$

where C is independent of s, b, B and f.

For $s < 0$, using the estimate $|\widehat{\tau_t}(\xi)| \le CB|A_{b^t}\xi|^{-\gamma/\ln b}$ and the same idea of proving (2.8), we have

$$\|G_s(f)\|_2^2 \le CBe^{2s\gamma} \|f\|_2^2.$$

Hence we have

$$\|G_s(f)\|_2 \le CBe^{s\gamma} \|f\|_2 \quad \text{for} \quad s < 0, \tag{2.9}$$

where C is independent of s, b, B and f.

Now, if p satisfies $|1/p - 1/2| < 1/(2q)$, then there exists p_0, such that $|1/(p_0) - 1/2| = 1/(2q)$ and $1/p = \theta/2 + (1-\theta)/p_0$ for some $0 < \theta \le 1$. Applying the Riesz-Thorin interpolation theorem of sublinear operators [3] between (2.5) and (2.8), between (2.5) and (2.9), respectively, we have

$$\|G_s(f)\|_p \le CBe^{-\theta\gamma s} \|f\|_p \quad \text{for} \quad s > 0, \tag{2.10}$$

and

$$\|G_s(f)\|_p \le CBe^{\theta\gamma s} \|f\|_p \quad \text{for} \quad s < 0, \tag{2.11}$$

where the constant C appearing in (2.10) and (2.11) is independent of s, b, B and f. Thus, from (2.4), (2.10) and (2.11), we obtain

$$\|\Delta_\tau(f)\|_p \le \int_{-\infty}^0 \|G_s(f)\|_p \, ds + \int_0^\infty \|G_s(f)\|_p \, ds \le CB\|f\|_p,$$

where C is independent of b, s, B and f. Therefore, we get the conclusion of Lemma 2.4. □

Remark 2.1. Let q vary from 1 to ∞ in Lemma 2.4, then p varies from 1 to ∞. Thus, indeed we get

Lemma 2.5. *Let $b \geq 2$, $B > 0$ and $\gamma > 0$. Suppose that the family of measures $\{\tau_t : t \in \mathbb{R}\}$ satisfies the following:*

(i) $\|\tau_t\| \leq CB$ *for* $t \in \mathbb{R}$;
(ii) $|\widehat{\tau}_t(\xi)| \leq CB(\min\{|A_{b^t}\xi|, |A_{b^t}\xi|^{-1})^{\gamma/\ln b}$ *for* $\xi \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$;
(iii) $\|\tau^*(f)\|_p \leq CB\|f\|_p$ *for* $f \in L^p(\mathbb{R}^n), 1 < p < \infty$,

where $C > 0$ is independent of b, f and B. Then, for every $1 < p < \infty$, there exists a constant $C > 0$, independent of b, f and B, such that $f \in L^p(\mathbb{R}^n)$

$$\|\Delta_\tau(f)\|_p \leq CB\|f\|_p.$$

3. Proof of Theorem 1

In the proof of Theorem 1 we will use some idea taken from [7] and [1]. Denote $\|\Omega\|_p = \|\Omega\|_{L^p(S^{n-1})}$ for $1 \leq p \leq \infty$. For $k \in \mathbb{N}$, let $E_k = \{y' \in S^{n-1} : 2^{k-1} \leq |\Omega(y')| < 2^k\}$, $E_0 = \{y' \in S^{n-1} : |\Omega(y')| < 1\}$. Denote $\delta = \int_{S^{n-1}} J(y')\, d\sigma(y')$, then $\delta > 0$. For $k \in \mathbb{N}$, let

$$\Omega_k(y') = \Omega(y')\chi_{E_k}(y') - \frac{1}{\delta}\int_{E_k} \Omega(x')J(x')\, d\sigma(x').$$

Since Ω satisfies (1.2), we have

$$\int_{S^{n-1}} \Omega_k(y')J(y')\, d\sigma(y') = 0 \quad \text{for} \quad k \in \mathbb{N}. \tag{3.1}$$

On the other hand, by $1 \leq J(x') \leq M$, we get

$$\begin{aligned} \|\Omega_k\|_1 &= \int_{S^{n-1}} \left|\Omega(y')\chi_{E_k}(y') - \frac{1}{\delta}\int_{E_k} \Omega(x')J(x')\, d\sigma(x')\right| d\sigma(y') \\ &\leq \int_{E_k} |\Omega(y')|\, d\sigma(y') + M\frac{\sigma(S^{n-1})}{\delta}\int_{E_k} |\Omega(x')|\, d\sigma(x') \\ &\leq C2^k\sigma(E_k). \end{aligned} \tag{3.2}$$

By the Minkowski inequality, we get

$$\begin{aligned} \|\Omega_k\|_2 &= \left(\int_{S^{n-1}} \left|\Omega(y')\chi_{E_k}(y') - \frac{1}{\delta}\int_{E_k} \Omega(x')J(x')\, d\sigma(x')\right|^2 d\sigma(y')\right)^{1/2} \\ &\leq \left(\int_{E_k} |\Omega(y')|^2 d\sigma(y')\right)^{1/2} + \delta^{-1}\sqrt{\sigma(S^{n-1})}\int_{E_k} |\Omega(x')J(x')|\, d\sigma(x') \\ &\leq 2^k\sqrt{\sigma(E_k)} + \frac{M}{\delta}\sqrt{\sigma(S^{n-1})}2^k\sigma(E_k). \end{aligned}$$

Let $\Lambda = \{k \in \mathbb{N} : \sigma(E_k) > 2^{-4k}\}$. We now give the estimates of $\|\Omega_k\|_2$ for $k \in \Lambda$ and $k \overline{\in} \Lambda$, respectively. If $k \overline{\in} \Lambda$, then

$$\|\Omega_k\|_2 \leq C2^k 2^{-2k} + C2^k 2^{-4k} \leq C2^{-k}. \tag{3.3}$$

If $k \in \Lambda$, then

$$\begin{aligned} \|\Omega_k\|_2 &\le 2^k \sigma(E_k) \frac{1}{\sqrt{\sigma(E_k)}} + \frac{M}{\delta} \sqrt{\sigma(S^{n-1})} 2^k \sigma(E_k) \\ &\le 2^{2k} 2^k \sigma(E_k) + C 2^k \sigma(E_k) \le C 2^{2k} 2^k \sigma(E_k). \end{aligned} \tag{3.4}$$

Let $\Omega_0 = \Omega - \sum_{k\in\Lambda} \Omega_k$. Since

$$\begin{aligned} \Omega(y') &= \sum_{k=0}^{\infty} \left(\Omega(y')\chi_{E_k}(y') - \frac{1}{\delta} \int_{E_k} \Omega(x')J(x')\,d\sigma(x') \right) \\ &= \sum_{k=1}^{\infty} \Omega_k(y') + \Omega(y')\chi_{E_0}(y') - \frac{1}{\delta} \int_{E_0} \Omega(x')J(x')\,d\sigma(x'), \end{aligned}$$

we have

$$\Omega_0(y') = \sum_{k \overline{\in} \Lambda} \Omega_k(y') + \Omega(y')\chi_{E_0}(y') - \frac{1}{\delta} \int_{E_0} \Omega(x')J(x')\,d\sigma(x').$$

Using (3.3) we get

$$\begin{aligned} \|\Omega_0\|_2 \le & \sum_{k \overline{\in} \Lambda} \|\Omega_k\|_2 + \left(\int_{E_0} |\Omega(x')|^2\,d\sigma(x') \right)^{1/2} \\ & + \frac{1}{\delta} \int_{E_0} |\Omega(x')J(x')|\,d\sigma(x') \sqrt{\sigma(S^{n-1})} \\ \le & \, C \sum_{k \overline{\in} \Lambda} 2^{-k} + \sqrt{\sigma(E_0)} + \frac{M}{\delta} \sigma(E_0) \sqrt{\sigma(S^{n-1})} \le C. \end{aligned}$$

It then follows that $\Omega_0 \in L^2(S^{n-1})$. By (1.2) and (3.1), we have

$$\int_{S^{n-1}} \Omega_0(y')J(y')\,d\sigma(y') = 0.$$

For every $k \in \Lambda$, we define the family of measures $\tau^{(k)} = \{\tau_{k,t} : t \in \mathbb{R}\}$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f\,d\tau_{k,t} = 2^{-kt} \int_{\rho(y)\le 2^{kt}} \frac{\Omega_k(y)}{\rho(y)^{\alpha-1}} f(y)\,dy.$$

Then we know

$$\tau_{k,t} = 2^{-kt} \frac{\Omega_k(y)}{\rho(y)^{\alpha-1}} \chi_{\{\rho(y)\le 2^{kt}\}}(y).$$

Now we set $b_k = 2^k$, $B_k = 2^k\sigma(E_k)$ and take β satisfying $0 < \beta < \min\{1/2, m/4, m/2\alpha\}$, $\gamma = \frac{\beta \ln 2}{m}$. Then the family of measures $\tau^{(k)}$ satisfies the following properties: For $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $1 < p < \infty$,

$$\begin{aligned} &\text{(i)} && \|\tau_{k,t}\|_1 \le CB_k, \\ &\text{(ii)} && |\widehat{\tau}_{k,t}(\xi)| \le CB_k(|A_{b_k^t}\xi|)^{-\gamma/\ln b_k}, \\ &\text{(iii)} && |\widehat{\tau}_{k,t}(\xi)| \le CB_k(|A_{b_k^t}\xi|)^{\gamma/\ln b_k}, \\ &\text{(iv)} && \|(\tau^{(k)})^*\|_{p,p} \le CB_k, \end{aligned} \tag{3.5}$$

where C is independent of k and ξ.

In fact, by $1 \le J(x') \le M$ and (3.2),

$$\begin{aligned}\|\tau_{k,t}\|_1 &\le 2^{-kt}\int_{\rho(y)\le 2^{kt}} \frac{|\Omega_k(y)|}{\rho(y)^{\alpha-1}}\,dy\\ &= 2^{-kt}\int_0^{2^{kt}}\int_{S^{n-1}} |\Omega_k(y')|J(y')\,d\sigma(y')\,d\rho\\ &\le C\int_{S^{n-1}}|\Omega_k(y')|\,d\sigma(y') \le C2^k\sigma(E_k) = CB_k.\end{aligned}$$

Thus we get (3.5.i). For (3.5.ii), by Hölder's inequality, we have

$$\begin{aligned}|\widehat{\tau}_{k,t}(\xi)|^2 &\le \sum_{j=-\infty}^{0} 2^j \int_{2^{kt+j-1}}^{2^{kt+j}} \left|\int_{S^{n-1}} J(y')\Omega_k(y')e^{-2\pi i\langle A_\rho y',\xi\rangle}\,d\sigma(y')\right|^2 \frac{d\rho}{\rho}\\ &\le \sum_{j=-\infty}^{0} 2^j \int_{2^{kt+j-1}}^{2^{kt+j}} \int_{S^{n-1}\times S^{n-1}} J(y')J(x')\\ &\quad\times\Omega_k(y')\overline{\Omega_k(x')}e^{-2\pi i\langle A_\rho(y'-x'),\xi\rangle}\,d\sigma(y')d\sigma(x')\frac{d\rho}{\rho}\\ &\le C\int_{S^{n-1}\times S^{n-1}} \Omega_k(y')\overline{\Omega_k(x')}\\ &\quad\times\left(\sum_{j=-\infty}^{0} 2^j\int_{2^{kt+j-1}}^{2^{kt+j}} e^{-2\pi i\langle A_\rho(y'-x'),\xi\rangle}\,\frac{d\rho}{\rho}\right) d\sigma(y')d\sigma(x').\end{aligned}$$

Let $\eta' = \frac{A_{2^{(kt+j-1)}}\xi}{|A_{2^{(kt+j-1)}}\xi|}$. By Lemma 2.2, we know

$$\begin{aligned}\left|\int_{2^{kt+j-1}}^{2^{kt+j}} e^{-2\pi i\langle A_\rho(x'-y'),\xi\rangle}\,\frac{d\rho}{\rho}\right| &= \left|\int_1^2 e^{-2\pi i\langle A_{2^{kt+j-1}\rho}(x'-y'),\xi\rangle}\,\frac{d\rho}{\rho}\right|\\ &\le C\big(|\langle A_{2^{kt+j-1}}(x'-y'),\xi\rangle|\big)^{-2\beta/m}\\ &= C\big(|\langle x'-y',\eta'\rangle||A_{2^{kt+j-1}}\xi|\big)^{-2\beta/m}\end{aligned}\tag{3.6}$$

Then by (3.6) we get

$$\begin{aligned}|\widehat{\tau}_{k,t}(\xi)|^2 &\le C\int_{S^{n-1}\times S^{n-1}} \Omega_k(y')\overline{\Omega_k(x')}\\ &\quad\times\left[\sum_{j=-\infty}^{0} 2^j\big(|A_{2^{kt+j-1}}\xi||\langle y'-x',\eta'\rangle|\big)^{-2\beta/m}\right] d\sigma(y')d\sigma(x')\\ &\le C\sum_{j=-\infty}^{0} 2^j 2^{-2j\alpha\beta/m}2^{2\alpha\beta/m}|A_{2^{kt}}\xi|^{-2\beta/m}\\ &\quad\times\int_{S^{n-1}\times S^{n-1}} \Omega_k(y')\overline{\Omega_k(x')}|\langle y'-x',\eta'\rangle|^{-2\beta/m}\,d\sigma(y')d\sigma(x').\end{aligned}$$

Since $4\beta/m < 1$ and $|\eta'| = 1$, there exists a constant $C > 0$, independent of η', such that

$$\begin{aligned}
&\int_{S^{n-1}\times S^{n-1}} \Omega_k(y')\overline{\Omega_k(x')}|\langle y'-x',\eta'\rangle|^{-2\beta/m}\,d\sigma(y')d\sigma(x') \\
\le\ & \|\Omega_k\|_2^2\Big(\int_{S^{n-1}\times S^{n-1}} |\langle y'-x',\eta'\rangle|^{-4\beta/m}\,d\sigma(y')d\sigma(x')\Big)^{1/2} \\
\le\ & C\|\Omega_k\|_2^2.
\end{aligned}$$

Thus, by $2\alpha\beta/m < 1$, we have

$$\begin{aligned}
|\widehat{\tau}_{k,t}(\xi)|^2 &\le C\sum_{j=-\infty}^{0} 2^j 2^{-2j\alpha\beta/m}2^{2\alpha\beta/m}|A_{2^{kt}}\xi|^{-2\beta/m}\|\Omega_k\|_2^2 \\
&\le C|A_{2^{kt}}\xi|^{-2\beta/m}\|\Omega_k\|_2^2.
\end{aligned}\tag{3.7}$$

Noting that if $k \in \Lambda$, then

$$\|\Omega_k\|_2 \le C2^{2k}2^k\sigma(E_k) = C2^{2k}B_k.\tag{3.8}$$

By (3.7) and (3.8), we have

$$|\widehat{\tau}_{k,t}(\xi)| \le C2^{2k}B_k|A_{2^{kt}}\xi|^{-\beta/m}.\tag{3.9}$$

On the other hand, by (3.5.i) we have

$$|\widehat{\tau}_{k,t}(\xi)| \le \|\tau_{k,t}\|_1 \le CB_k.\tag{3.10}$$

From (3.9) and (3.10), we get

$$\begin{aligned}
|\widehat{\tau}_{k,t}(\xi)| &\le (CB_k)^{(k-1)/k}[C2^{2k}B_k|A_{2^{kt}}\xi|^{-\beta/m}]^{1/k} \\
&\le CB_k|A_{2^{kt}}\xi|^{\frac{-\beta}{km}} = CB_k|A_{b_k^t}\xi|^{\frac{-\gamma}{\ln b_k}},
\end{aligned}$$

which proves (3.5.ii). As for (iii), using (3.1), (3.2) and $1 \le J(x') \le M$, we get

$$\begin{aligned}
|\widehat{\tau}_{k,t}(\xi)| &= 2^{-kt}\Big|\int_0^{2^{kt}}\int_{S^{n-1}} J(y')\Omega_k(y')[e^{-2\pi i\langle A_\rho y',\xi\rangle}-1]\,d\sigma(y')\,d\rho\Big| \\
&\le C2^{-kt}\int_0^{2^{kt}}\int_{S^{n-1}} |\Omega_k(y')||\langle A_\rho y',\xi\rangle|\,d\sigma(y')\,d\rho \\
&\le C\int_0^1\int_{S^{n-1}} |\Omega_k(y')||A_{2^{kt}}\xi||\langle A_\rho y',\frac{A_{2^{kt}}\xi}{|A_{2^{kt}}\xi|}\rangle|\,d\sigma(y')\,d\rho \\
&\le C|A_{2^{kt}}\xi|\int_{S^{n-1}} |\Omega_k(y')|\int_0^1 |A_\rho y'|\,d\rho\,d\sigma(y') \\
&\le C2^k\sigma(E_k)|A_{2^{kt}}\xi| = CB_k|A_{b_k^t}\xi|.
\end{aligned}$$

Using (3.10) again

$$|\widehat{\tau}_{k,t}(\xi)| \le (CB_k)^{(1-\frac{\beta}{km})}(CB_k|A_{2^{kt}}\xi|)^{\frac{\beta}{km}} \le CB_k|A_{2^{kt}}\xi|^{\frac{\beta}{km}} = CB_k|A_{b_k^t}\xi|^{\frac{\gamma}{\ln b_k}}.$$

It is just (3.5.iii). Finally, noting that $(\tau^{(k)})^*(f) = \sup_{t\in\mathbb{R}} |\tau_{k,t}| * |f|$, then

$$\begin{aligned}(\tau^{(k)})^*(f)(x) &= \sup_{t\in\mathbb{R}} 2^{-kt}\int_0^{2^{kt}}\int_{S^{n-1}} J(y')|\Omega_k(y')||f(x-A_\rho y')|\,d\sigma(y')\,d\rho \\ &\le C\int_{S^{n-1}}|\Omega_k(y')|\Big(\sup_{t\in\mathbb{R}} 2^{-kt}\int_0^{2^{kt}}|f(x-A_\rho y')|\,d\rho\Big)\,d\sigma(y') \\ &\le C\int_{S^{n-1}}|\Omega_k(y')|\mathcal{M}_\Gamma(f)(x)\,d\sigma(y'),\end{aligned}$$

where $\Gamma(\rho) = (y_1'\rho^{\alpha_1},\ldots,y_n'\rho^{\alpha_n})$ and $\mathcal{M}_\Gamma(f)(x)$ is defined in (2.2). By Lemma 2.3, $\|\mathcal{M}_\Gamma(f)\|_p \le C\|f\|_p$, where C is independent of $\{y_j'\}_{j=1}^n$ and f. Therefore, by the Minkowski inequality we have

$$\|(\tau^{(k)})^*(f)\|_p \le C\|f\|_p\|\Omega_k\|_1 \le C2^k\sigma(E_k)\|f\|_p = CB_k\|f\|_p.$$

Then (3.5.iv) is proved.

Let us now return to the proof of Theorem 1. By the Minkowski inequality, (3.11)

$$\begin{aligned}\mu_\Omega(f) &\le \mu_{\Omega_0}(f) + \sum_{k\in\Lambda}\Big(\int_0^\infty\Big|\int_{\rho(y)\le t}\frac{\Omega_k(y)}{\rho(y)^{\alpha-1}}f(x-y)\,dy\Big|^2\frac{dt}{t^3}\Big)^{1/2} \\ &= \mu_{\Omega_0}(f) + \sum_{k\in\Lambda}(k\ln 2)^{1/2}\Big(\int_{\mathbb{R}}\Big|\frac{1}{2^{kt}}\int_{\rho(y)\le 2^{kt}}\frac{\Omega_k(y)}{\rho(y)^{\alpha-1}}f(x-y)\,dy\Big|dt\Big)^{1/2} \\ &= \mu_{\Omega_0}(f) + \sum_{k\in\Lambda}(k\ln 2)^{1/2}\Delta_{\tau^{(k)}}(f).\end{aligned}$$

By (3.5), (3.11), Theorem A and Lemma 2.5, for $1<p<\infty$, we obtain

$$\begin{aligned}\|\mu_\Omega(f)\|_p &\le C\Big(1+\sum_{k\in\Lambda}(k\ln 2)^{1/2}B_k\Big)\|f\|_p \\ &\le C\Big(1+\sum_{k\in\Lambda}(k\ln 2)^{1/2}2^k\sigma(E_k)\Big)\|f\|_p \\ &\le C\Big(1+\|\Omega\|_{L(\log^+ L)^{1/2}}\Big)\|f\|_p \le C\|f\|_p.\end{aligned}$$

Thus we complete the proof of Theorem 1. □

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