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# Π-COHERENT DIMENSIONS AND Π-COHERENT RINGS

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Abstract. $R$ is called a right Π-coherent ring in case every finitely generated torsionless right $R$-module is finitely presented. In this paper, we define a dimension for rings, called Π-coherent dimension, which measures how far away a ring is from being Π-coherent. This dimension has nice properties when the ring in question is coherent. In addition, we study some properties of Π-coherent rings in terms of preenvelopes and precovers.

## 1. Introduction

Recall that $R$ is called a *right* Π-*coherent ring* [3] in case every finitely generated torsionless right $R$-module is finitely presented. This notion is also called *strong coherence* in [12]. $R$ is called a *right coherent ring* in case every finitely generated right ideal is finitely presented. Clearly, we have the following implications:

right Noetherian rings ⇒ right Π-coherent rings ⇒ right coherent rings.

But the converses do not hold generally (see [3]). Π-coherent rings have been studied by many authors (see, for example, [3, 4, 6, 12, 21]).

The concept of $FGT$-injective dimensions for modules and rings were introduced in [4]. The *$FGT$-injective dimension of a right $R$-module $M$*, denoted by $FGT-Id(M)$, is defined as the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(N,M) = 0$ for any finitely generated torsionless right $R$-module $N$. If no such $n$ exists, set $FGT-Id(M) = \infty$. Put $rFGT-I.\dim(R) = \sup\{FGT-Id(M)\colon M$ is any right $R$-module$\}$ and call $rFGT-I.\dim(R)$ the *right $FGT$-injective dimension of $R$*. It is known that $rFGT-I.\dim(R) = 0$ if and only if $R$ is a right Π-coherent left semihereditary ring (see [4]). So the right $FGT$-injective dimension of a ring $R$ measures how far away $R$ is from being a right Π-coherent left semihereditary ring.

Recall that a right $R$-module $M$ is called *$FP$-injective* (or *absolutely pure*) [19, 15] in case $\mathrm{Ext}_R^1(N,M) = 0$ for all finitely presented right $R$-modules $N$. In [16], Mao and Ding introduced the concept of $FP$-projective dimensions.

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The *FP-projective dimension of a right R-module M*, denoted by $fpd(M_R)$, is defined as the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for any $FP$-injective right $R$-module $N$. If no such $n$ exists, set $fpd(M_R) = \infty$. $M$ is called *FP-projective* if $fpd(M_R) = 0$, i.e., $\mathrm{Ext}_R^1(M, N) = 0$ for any $FP$-injective right $R$-module $N$.

In Section 2 of the present paper, we define a dimension for rings, which measures how far away a ring is from being Π-coherent. Put $r.\pi cD(R) = \sup\{fpd(M_R)$: $M_R$ is a finitely generated torsionless right $R$-module$\}$ and call $r.\pi cD(R)$ the *right* Π-*coherent dimension of* $R$. This dimension has nice properties when the ring in question is coherent. For example, it is true that $rFGT-I.\dim(R) \leq wD(R) + r.\pi cD(R)$ for a right coherent ring $R$. Let $R$ and $S$ be right coherent rings, we show that $r.\pi cD(R \oplus S) = \sup\{r.\pi cD(R), r.\pi cD(S)\}$. We also consider the Π-coherent dimensions under changes of rings, especially under excellent extensions of rings. It is proven that, if $R$ and $S$ is right coherent rings and $S$ is an excellent extension of $R$, then $r.\pi cD(S) = r.\pi cD(R)$.

In Section 3, we study some properties of Π-coherent rings in terms of preenvelopes and precovers. Let $R$ be a right Π-coherent ring. We first prove that every left $R$-module has an $FGT$-flat preenvelope, and every right $R$-module has an $FGT$-injective cover, where a right $R$-module $M$ (a left $R$-module $Q$) is called *FGT-injective* (*FGT-flat*) (see [4]) if $\mathrm{Ext}_R^1(N, M) = 0$ ($\mathrm{Tor}_1^R(N, Q) = 0$) for any finitely generated torsionless right $R$-module $N$. Next we show that the following are equivalent: (1) $R_R$ is $FGT$-injective. (2) Every left $R$-module has a monic $FGT$-flat preenvelope. (3) Every right $R$-module has an epic $FGT$-injective cover. It is also shown that the following are equivalent: (1) $rFGT - I.\dim(R) \leq 1$. (2) Every left $R$-module has an epic $FGT$-flat envelope. (3) Every right $R$-module has a monic $FGT$-injective cover. Finally we prove that the following are equivalent: (1) $rFGT - I.\dim(R) \leq 2$. (2) Every right $R$-module has an $FGT$-injective cover with the unique mapping property.

Throughout this paper, all rings are associative with identity and all modules are unitary. We write $M_R$ (${}_RM$) to indicate a right (left) $R$-module. $wD(R)$ stands for the weak global dimension of a ring $R$. For an $R$-module $M$, $pd(M)$ denotes the projective dimension of $M$, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by $M^*$, and the character module $M^+$ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. General background materials can be found in [10, 13, 18, 22].

## 2. Π-Coherent dimensions

We start with the following

**Definition 2.1.** Let $R$ be a ring. Put $r.\pi cD(R) = \sup\{fpd(M_R)$: $M_R$ is a finitely generated torsionless right $R$-module$\}$ and call $r.\pi cD(R)$ the *right* Π-*coherent dimension of* $R$. Similarly, we have $l.\pi cD(R)$.

**Proposition 2.2.** *The following are equivalent for a ring* $R$ :

(1) $r.\pi cD(R) = 0$.

(2) *$R$ is a right $\Pi$-coherent ring.*
(3) *Every $FP$-injective right $R$-module is $FGT$-injective.*

*Proof.* (1) $\Rightarrow$ (2). Let $M$ be a finitely generated torsionless right $R$-module. Then $\mathrm{Ext}_R^1(M,N)=0$ for any $FP$-injective right $R$-module $N$ by (1). Therefore $M$ is finitely presented by [8], and so $R$ is a right $\Pi$-coherent ring.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are clear by definition. □

*Remark* 2.3. (1) By Proposition 2.2, the right $\Pi$-coherent dimension $r.\pi cD(R)$ measures how far away a ring $R$ is from being right $\Pi$-coherent. It is known that right $\Pi$-coherent rings need not be left $\Pi$-coherent (see [13, Example 4.46 (e)]), so $r.\pi cD(R) \neq l.\pi cD(R)$ in general.

(2) If $R$ is a *left $FP$-injective ring* (i.e., ${}_RR$ is $FP$-injective), then every finitely presented right $R$-module is torsionless by [11, Theorem 2.3], and so every $FGT$-injective right $R$-module is $FP$-injective. In this case, $R$ is a right $\Pi$-coherent ring if and only if $FGT$-injective right $R$-modules coincide with $FP$-injective right $R$-modules by Proposition 2.2.

The next lemma will be used frequently in the sequel.

**Lemma 2.4.** [16, Proposition 3.1] *Let $R$ be a right coherent ring. For any right $R$-module $M$ and integer $n \geq 0$, the following are equivalent:*

(1) $fpd(M) \leq n$.
(2) $\mathrm{Ext}_R^{n+1}(M,N)=0$ *for any $FP$-injective right $R$-module $N$.*
(3) $\mathrm{Ext}_R^{n+j}(M,N)=0$ *for any $FP$-injective right $R$-module $N$ and $j \geq 1$.*
(4) *There exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each $P_i$ is $FP$-projective.*

**Proposition 2.5.** *The following are equivalent for a right coherent ring $R$:*

(1) $r.\pi cD(R) \leq 1$.
(2) *For any exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ with $A$ $FP$-injective and $B$ injective, $C$ is $FGT$-injective.*
(3) *For any pure submodule $N$ of an injective right module $M$, the quotient $M/N$ is $FGT$-injective.*

*Proof.* (1) $\Rightarrow$ (2). Let $0 \to A \to B \to C \to 0$ be an exact sequence of right $R$-modules with $A$ $FP$-injective and $B$ injective. If $M$ is a finitely generated torsionless right $R$-module, then $fpd(M) \leq r.\pi cD(R) \leq 1$ by (1). Thus $\mathrm{Ext}_R^1(M,C) \cong \mathrm{Ext}_R^2(M,A)=0$ by Lemma 2.4, and so $C$ is $FGT$-injective.

(2) $\Rightarrow$ (3) is obvious since $N$ is $FP$-injective.

(3) $\Rightarrow$ (1). Let $M$ be a finitely generated torsionless right $R$-module and $N$ an $FP$-injective right $R$-module. Then there exists an exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective. So $L$ is $FGT$-injective by (3) since $N$ is a pure submodule of $M$. Thus $\mathrm{Ext}_R^2(M,N) \cong \mathrm{Ext}_R^1(M,L)=0$, and hence $fpd(M) \leq 1$. It follows that $r.\pi cD(R) \leq 1$. □

**Proposition 2.6.** *Let $R$ be a right coherent ring. Then*

(1) $\sup\{FGT\!-\!Id(M)\!:\!M$ *is an* $FP$*-injective right* $R$*-module*$\} \leq r.\pi cD(R)$;
(2) $rFGT - I.\dim(R) \leq \sup\{pd(F)$: $F$ *is a finitely generated torsionless* $R$*-module*$\} \leq wD(R) + r.\pi cD(R)$.

*Proof.* (1). We may assume that $r.\pi cD(R) = n < \infty$. Let $M$ be an $FP$-injective right $R$-module and $N$ any finitely generated torsionless right $R$-module. Then $fpd(N) \leq n$. Thus $\mathrm{Ext}_R^{n+1}(N, M) = 0$ by Lemma 2.4, and so $FGT - Id(M) \leq n$. Consequently $\sup\{FGT - Id(M)$: $M$ is any $FP$-injective right $R$-module$\} \leq r.\pi cD(R)$.

(2). It is easy to check that $rFGT - I.\dim(R) \leq \sup\{pd(F)$: $F$ is a finitely generated torsionless $R$-module$\}$.

Now we assume that, without loss of the generality, both $r.\pi cD(R)$ and $wD(R)$ are finite. Let $r.\pi cD(R) = m < \infty$ and $wD(R) = n < \infty$. Suppose $M$ is a finitely generated torsionless right $R$-module, then $fpd(M) \leq m$. So by Lemma 2.4, $M$ admits an $FP$-projective resolution

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each $P_i$ is $FP$-projective. Note that $pd(P_i) \leq wD(R) = n$ by [16, Theorem 4.2], $i = 0, 1, 2, \ldots, m$. It follows that $pd(M) \leq m + n$ by dimension shifting, and hence $rFGT - I.\dim(R) \leq m + n$, as desired. □

**Theorem 2.7.** *Let $R$ and $S$ be right coherent rings. Then*

$$r.\pi cD(R \oplus S) = \sup\{r.\pi cD(R), r.\pi cD(S)\}.$$

*Proof.* We first show that $r.\pi cD(R \oplus S) \leq \sup\{r.\pi cD(R), r.\pi cD(S)\}$. We may assume that $r.\pi cD(R) = m < \infty, r.\pi cD(S) = n < \infty$, and $m \geq n$. Let $M$ be a finitely generated torsionless right $(R \oplus S)$-module. Then $M$ has a unique decomposition that $M = A \oplus B$, where $A = M(R, 0)$ is a right $R$-module and $B = M(0, S)$ is a right $S$-module via $xr = x(r, 0)$ for $x \in A$, $r \in R$, and $ys = y(0, s)$ for $y \in B$, $s \in S$. It is easy to verify that $A$ is a finitely generated torsionless right $R$-module and $B$ is a finitely generated torsionless right $S$-module. Thus $fpd(A_R) \leq m$ and $fpd(B_S) \leq n \leq m$. By Lemma 2.4, there exist two exact sequences $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ and $0 \to Q_m \to Q_{m-1} \to \cdots \to Q_1 \to Q_0 \to B \to 0$ of right $R$-modules and right $S$-modules respectively, where each $P_i$ is an $FP$-projective right $R$-module, and each $Q_i$ is an $FP$-projective right $S$-module. Regarding these as exact sequences of right $(R \oplus S)$-modules, we have an exact sequence of right $(R \oplus S)$-modules $0 \to P_m \oplus Q_m \to P_{m-1} \oplus Q_{m-1} \to \cdots \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to A \oplus B \to 0$. Note that each $P_i \oplus Q_i$ is an $FP$-projective right $(R \oplus S)$-module by [16, Lemma 3.15]. Thus $fpd(M_{R \oplus S}) \leq m$, and hence $r.\pi cD(R \oplus S) \leq \sup\{r.\pi cD(R), r.\pi cD(S)\}$.

Next we prove that $\sup\{r.\pi cD(R), r.\pi cD(S)\} \leq r.\pi cD(R \oplus S)$. We may assume that $r.\pi cD(R \oplus S) = k < \infty$. Let $M$ be a finitely generated torsionless right $R$-module. Note that $M$ may be regarded as a finitely generated torsionless right $(R \oplus S)$-module, so $fpd(M_{R \oplus S}) \leq r.\pi cD(R \oplus S) = k$. Thus there exists an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ of right $(R \oplus S)$-modules, where each $P_i$ is an $FP$-projective right $(R \oplus S)$-module. Let $P_i = A_i \oplus B_i$, where $A_i$ is a right $R$-module and $B_i$ is a right $S$-module, $i = 0, 1, \ldots, k$. Since $M$ is a right $R$-module, we have the exact sequence $0 \to A_k \to A_{k-1} \to \cdots \to A_1 \to A_0 \to M \to 0$ of right $R$-modules. Note that each $A_i$ is an $FP$-projective right $(R \oplus S)$-module, and so an $FP$-projective right $R$-module by [16, Lemma 3.15], whence $fpd(M_R) \leq k$, and so $r.\pi cD(R) \leq k$. Similarly $r.\pi cD(S) \leq k$. Thus $\sup\{r.\pi cD(R), r.\pi cD(S)\} \leq r.\pi cD(R \oplus S)$. The proof is complete. □

*Remark* 2.8. Theorem 2.7 shows that $r.\pi cD(\bigoplus_{i=1}^{n} R_i) = \sup_{1 \leq i \leq n} \{r.\pi cD(R_i)\}$ if each $R_i$ is right coherent. In particular, we get that $\bigoplus_{i=1}^{n} R_i$ is a right Π-coherent ring if and only if each $R_i$ is right Π-coherent, $i = 1, 2, \ldots, n$.

Next we investigate the Π-coherent dimensions under changes of rings.

**Proposition 2.9.** *Let $R$ and $S$ be right coherent rings. If $\varphi : R \to S$ is a surjective ring homomorphism with $S$ flat as a left $R$-module and projective as a right $R$-module, then $r.\pi cD(S) \leq r.\pi cD(R)$.*

*Proof.* We may assume that $r.\pi cD(R) = n < \infty$. Let $M_S$ be a finitely generated torsionless right $S$-module and $F_S$ an $FP$-injective right $S$-module. We claim that $F_R$ is an $FP$-injective right $R$-module. In fact, if $N$ is a finitely presented right $R$-module, then there is an exact sequence $0 \to K \to P \to N \to 0$ of right $R$-modules with $K$ finitely generated and $P$ finitely generated projective. Since ${}_RS$ is flat, we have the right $S$-module exact sequence $0 \to K \otimes_R S_S \to P \otimes_R S_S \to N \otimes_R S_S \to 0$. Note that $K \otimes_R S_S$ is a finitely generated right $S$-module, $P \otimes_R S_S$ is a finitely generated projective right $S$-module, and so $N \otimes_R S_S$ is a finitely presented right $S$-module. Therefore $\mathrm{Ext}_R^1(N_R, F_R) \cong \mathrm{Ext}_S^1(N \otimes_R S, F_S) = 0$ by [18, Theorem 11.65] since $F_S$ is $FP$-injective. So $F_R$ is $FP$-injective. On the other hand, $M_R$ is a finitely generated torsionless right $R$-module since $S_R$ is projective. Thus $fpd(M_R) \leq r.\pi cD(R) \leq n$, and hence $\mathrm{Ext}_S^{n+1}(M_S, \mathrm{Hom}_R(S, F_R)) \cong \mathrm{Ext}_R^{n+1}(M_R, F_R) = 0$ by [18, Theorem 11.66]. Note that $F_S \cong \mathrm{Hom}_R(S, F_R)$ (for $\varphi$ is surjective), so $\mathrm{Ext}_S^{n+1}(M_S, F_S) = 0$. Thus $fpd(M_S) \leq n$. It follows that $r.\pi cD(S) \leq r.\pi cD(R)$. □

Recall that a ring $S$ is said to be an *excellent extension of a subring $R$* [2] if the following conditions are satisfied:

(1) $R$ and $S$ have the same identity and $S$ is free with basis $s_1, \ldots, s_n$ as both a right and a left $R$-module, $s_1 = 1_R$, and $Rs_i = s_iR$ for all $i = 1, \ldots, n$;
(2) If $M_S$ is a submodule of $N_S$ and $M_R$ is a direct summand of $N_R$, then $M_S$ is a direct summand of $N_S$.

**Theorem 2.10.** *Let $R$ and $S$ be right coherent rings. If $S$ is an excellent extension of $R$, then $r.\pi cD(S) = r.\pi cD(R)$.*

*Proof.* Note that $R$ is an $R$-bimodule direct summand of $S$ since $S$ is an excellent extension of $R$. So we may let ${}_RS_R = R \oplus T$.

We first prove that $r.\pi cD(S) \leq r.\pi cD(R)$. We may assume that $r.\pi cD(R) = m < \infty$. Let $N_S$ be a finitely generated torsionless right $S$-module. It is easy to see that $N_R$ is a finitely generated torsionless right $R$-module. Thus $fpd(N_R) \leq m$, and so there exists an exact sequence $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ of right $R$-modules, where each $P_i$ is an $FP$-projective right $R$-module. Since ${}_RS$ is free, we have the exact sequence $0 \to P_m \otimes_R S \to P_{m-1} \otimes_R S \to \cdots \to P_1 \otimes_R S \to P_0 \otimes_R S \to N \otimes_R S \to 0$ of right $S$-modules. Note that each $P_i \otimes_R S$ is an $FP$-projective right $S$-module by [16, Lemma 3.18], and so $fpd(N \otimes_R S)_S \leq m$. Note that $(N \otimes_R S)_R \cong N_R \oplus (N \otimes_R T)$, and so we have $N_S$ is isomorphic to a direct summand of $(N \otimes_R S)_S$ since $S$ is an excellent extension of $R$. Thus $fpd(N_S) \leq fpd(N \otimes_R S)_S$, and hence $fpd(N_S) \leq m$. It follows that $r.\pi cD(S) \leq r.\pi cD(R)$.

Next we prove that $r.\pi cD(R) \leq r.\pi cD(S)$. We may assume that $r.\pi cD(S) = n < \infty$. Let $M_R$ be a finitely generated torsionless right $R$-module. It is easy to check that $M \otimes_R S$ is a finitely generated torsionless right $S$-module and hence $fpd(M \otimes_R S)_S \leq n$. By Lemma 2.4, there exists an exact sequence of right $S$-modules $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \otimes_R S \to 0$, where each $Q_i$ is an $FP$-projective right $S$-module. Thus each $Q_i$ is a direct summand in a right $S$-module $U_i$ such that $U_i$ is a union of a continuous chain, $(U_{\alpha,i} : \alpha < \lambda)$, for a cardinal $\lambda$, $U_{0,i} = 0$, and $U_{\alpha+1,i}/U_{\alpha,i}$ is a finitely presented right $S$-module for all $\alpha < \lambda$ (see [20, Definition 3.3]). It is easy to verify that $U_{\alpha+1,i}/U_{\alpha,i}$ is a finitely presented right $R$-module for all $\alpha < \lambda$ since $S$ is an excellent extension of $R$. Thus each $Q_i$ is an $FP$-projective right $R$-module, and so $fpd(M \otimes_R S)_R \leq n$. Since $(M \otimes_R S)_R \cong M_R \oplus (M \otimes_R T)$, we have $fpd(M_R) \leq fpd(M \otimes_R S)_R \leq n$ and hence $r.\pi cD(R) \leq r.\pi cD(S)$. So we have the desired equality. □

It has been proven that, if $S$ is an excellent extension of $R$, then $R$ is right coherent if and only if $S$ is right coherent (see [14, Lemma 8]). As an immediate consequence of Theorem 2.10, we have

**Corollary 2.11.** *Let $S$ be an excellent extension of $R$. Then $R$ is a right $\Pi$-coherent ring if and only if $S$ is right $\Pi$-coherent.*

## 3. Some properties of $\Pi$-coherent rings

Let $\mathcal{C}$ be a class of $R$-modules and $M$ an $R$-module. Following [9], we say that a homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-preenvelope of $M$* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism.

Dually we have the definitions of a $\mathcal{C}$*-precover* and a $\mathcal{C}$*-cover*. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let $R$ be a right coherent ring. It is known that every left $R$-module has a flat preenvelope (see [9]) and every right $R$-module has an $FP$-injective cover (see [17]). In this section, we will investigate the analogous properties of Π-coherent rings in terms of preenvelopes and precovers. The following lemmas will be needed.

**Lemma 3.1.** *Let $R$ be a ring. Then*

(1) *A left $R$-module $M$ is $FGT$-flat if and only if $M^+$ is $FGT$-injective.*
(2) *Pure submodules of $FGT$-flat left $R$-modules are $FGT$-flat.*
(3) *For a short exact sequence of left $R$-modules $0 \to M \to N \to L \to 0$, if $N$ and $L$ are $FGT$-flat, then $M$ is $FGT$-flat.*

*Proof.* (1) holds by the standard isomorphism $\mathrm{Ext}^1_R(N, M^+) \cong \mathrm{Tor}^R_1(N, M)^+$ for any right $R$-module $N$.

(2). Let $A$ be a pure submodule of an $FGT$-flat left $R$-module $B$, then the pure exact sequence $0 \to A \to B \to B/A \to 0$ induces the split exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Thus $A^+$ is $FGT$-injective since $B^+$ is $FGT$-injective by (1). So $A$ is $FGT$-flat.

(3). Let $H$ be a finitely generated torsionless right $R$-module. Then there is an exact sequence $0 \to K \to F \to H \to 0$ with $F$ finitely generated free. Note that $K = \varinjlim K_i$, where each $K_i$ is a finitely generated submodule of $K$ (see [10, Example 1.5.5 (2), p. 32]). Thus each $K_i$ is a finitely generated torsionless right $R$-module, and so $\mathrm{Tor}^R_2(H, L) \cong \mathrm{Tor}^R_1(K, L) = \mathrm{Tor}^R_1(\varinjlim K_i, L) \cong \varinjlim \mathrm{Tor}^R_1(K_i, L) = 0$ since $L$ is $FGT$-flat. On the other hand, we have the exact sequence $0 = \mathrm{Tor}^R_2(H, L) \to \mathrm{Tor}^R_1(H, M) \to \mathrm{Tor}^R_1(H, N) = 0$ since $N$ is $FGT$-flat. Thus $\mathrm{Tor}^R_1(H, M) = 0$, and so $M$ is $FGT$-flat. □

**Lemma 3.2.** *Let $R$ be a right Π-coherent ring. Then*

(1) *A right $R$-module $M$ is $FGT$-injective if and only if $M^+$ is $FGT$-flat.*
(2) *Any direct limit (direct sum) of $FGT$-injective right $R$-modules is $FGT$-injective.*
(3) *Any direct product of $FGT$-flat left $R$-modules is $FGT$-flat.*
(4) *Pure submodules of $FGT$-injective right $R$-modules are $FGT$-injective.*
(5) *For a short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$, if $A$ and $B$ are $FGT$-injective, then $C$ is $FGT$-injective.*
(6) *$FGT - Id(M) \le n$ if and only if $\mathrm{Ext}^{n+1}_R(N, M) = 0$ for any finitely generated torsionless right $R$-module $N$.*

*Proof.* (1) through (3) follow from [4].

(4). Let $N$ be a pure submodule of an $FGT$-injective right $R$-module $M$. For any finitely generated torsionless right $R$-module $L$, we have the exact sequence

$$\mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(L, M/N) \to \mathrm{Ext}^1_R(L, N) \to \mathrm{Ext}^1_R(L, M) = 0.$$

But the sequence $\mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(L, M/N) \to 0$ is exact since $L$ is finitely presented and $N$ is a pure submodule of $M$. Therefore $\mathrm{Ext}^1_R(L, N) = 0$ and so $N$ is $FGT$-injective.

(5). The exact sequence $0 \to A \to B \to C \to 0$ induces an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Note that $B^+$ and $A^+$ are $FGT$-flat by (1), so $C^+$ is $FGT$-flat by Lemma 3.1 (3), and hence $C$ is $FGT$-injective.

(6) holds by [4]. □

The next lemma is a special case of [1, Theorem 5].

**Lemma 3.3.** *Let $R$ be a ring. Then for each cardinal $\lambda$, there is a cardinal $\kappa$ such that for any $R$-module $M$ and for any $L \leq M$ such that* $\mathrm{Card}(M) \geq \kappa$ *and* $\mathrm{Card}(M/L) \leq \lambda$, *the submodule $L$ contains a non-zero submodule that is pure in $M$.*

Let $R$ be an arbitrary ring, it is easy to see that every left $R$-module has an $FGT$-flat cover by [20, Lemma 1.11 and Theorem 2.8], and every right $R$-module has an $FGT$-injective preenvelope by [7, Theorem 10]. If $R$ is a right $\Pi$-coherent ring, then we can obtain additional results as follows.

**Theorem 3.4.** *The following are true for a right $\Pi$-coherent ring $R$:*

(1) *Every left $R$-module has an $FGT$-flat preenvelope.*

(2) *Every right $R$-module has an $FGT$-injective cover.*

*Proof.* (1). Let $M$ be any left $R$-module. By [10, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any homomorphism $g : M \to L$ with $L$ $FGT$-flat, there is a pure submodule $Q$ of $L$ such that $\mathrm{Card}(Q) \leq \aleph_\alpha$ and $g(M) \subseteq Q$. Note that $Q$ is $FGT$-flat by Lemma 3.1 (2), and so $M$ has an $FGT$-flat preenvelope by [10, Proposition 6.2.1] since any direct product of $FGT$-flat left $R$-modules is $FGT$-flat by Lemma 3.2 (3).

(2). The proof is motivated by that of [17, Lemma 4.8]. Suppose that $N$ is a right $R$-module with $\mathrm{Card}(N) = \lambda$. Let $\kappa$ be a cardinal as in Lemma 3.3. By [10, Proposition 5.2.2], we only need to show that any homomorphism $f : D \to N$ with $D$ $FGT$-injective has a factorization $D \to C \to N$ with $C$ $FGT$-injective and $\mathrm{Card}(C) \leq \kappa$.

If $\mathrm{Card}(D) \leq \kappa$, then we have done. So we may assume that $\mathrm{Card}(D) > \kappa$.

Let $K = \ker(f)$. Note that $\mathrm{Card}(D/K) \leq \lambda$ since $D/K$ embeds in $N$. Thus $K$ contains a non-zero submodule $D_0$ which is pure in $D$ by Lemma 3.3. Since $D_0$ is $FGT$-injective by Lemma 3.2 (4), $D/D_0$ is $FGT$-injective by Lemma 3.2 (5).

If $\mathrm{Card}(D/D_0) \leq \kappa$, then we have done since $f$ factors through $D/D_0$.

Suppose that $\mathrm{Card}(D/D_0) > \kappa$. Note that there exists $g : D/D_0 \to N$ with $\ker(g) = K_1/D_0$. So $K_1/D_0$ contains a non-zero submodule $D_1/D_0$ which is pure in $D/D_0$ by Lemma 3.3. Therefore $D/D_1 \cong (D/D_0)/(D_1/D_0)$ is $FGT$-injective by Lemma 3.2 (5).

If $\mathrm{Card}(D/D_1) \leq \kappa$, then we have done since $f$ factors through $D/D_1$.

If $\mathrm{Card}(D/D_1) > \kappa$, then we can continue the process above. Ultimately we arrive at a direct limit $\varinjlim(D/D_i) = D/\varinjlim D_i$ such that $\mathrm{Card}(\varinjlim(D/D_i)) \leq \kappa$. Note that $\varinjlim(D/D_i)$ is $FGT$-injective by Lemma 3.2 (2), and $f$ factors through $\varinjlim(D/D_i)$. So $N$ has an $FGT$-injective precover, and hence has an $FGT$-injective cover by [10, Corollary 5.2.7]. □

In general, an $FGT$-flat preenvelope or an $FGT$-injective cover need not be a monomorphism or epimorphism. Next, we shall consider when every left $R$-module has a monic or epic $FGT$-flat preenvelope and when every right $R$-module has a monic or epic $FGT$-injective cover in case $R$ is a right Π-coherent ring.

**Theorem 3.5.** *The following are equivalent for a right* Π*-coherent ring $R$:*

(1) *$R_R$ is $FGT$-injective.*

(2) *Every left $R$-module has a monic $FGT$-flat preenvelope.*

(3) *Every right $R$-module has an epic $FGT$-injective cover.*

*If any of the above conditions holds, then $rFGT - I.\dim(R) = 0$ or $\infty$.*

*Proof.* (1) ⇒ (2). Let $M$ be any left $R$-module. Since $(R_R)^+$ is a cogenerator in the category of left $R$-modules, there is an exact sequence $0 \to M \to \Pi(R_R)^+$. Note that $(R_R)^+$ is $FGT$-flat by (1) and Lemma 3.2 (1), and so $\Pi(R_R)^+$ is $FGT$-flat by Lemma 3.2 (3). Thus $M$ embeds in an $FGT$-flat left $R$-module. But $M$ has an $FGT$-flat preenvelope $f : M \to F$ by Theorem 3.4 (1). So $f$ is a monomorphism.

(2) ⇒ (1). Note that the injective left $R$-module $(R_R)^+$ embeds in an $FGT$-flat left $R$-module by (2). Thus $(R_R)^+$ is $FGT$-flat, and so $R_R$ is $FGT$-injective.

(1) ⇒ (3). Let $M$ be a right $R$-module. Then there is an exact sequence $F \to M \to 0$ with $F$ free. Note that $F$ is $FGT$-injective by (1) and Lemma 3.2 (2). Since $M$ has an $FGT$-injective cover $f$ by Theorem 3.4 (2), $f$ is an epimorphism.

(3) ⇒ (1) is clear since $R_R$ has an epic $FGT$-injective cover.

Now suppose that any of the equivalent conditions above holds, and $rFGT - I.\dim(R) = n < \infty$. For any right $R$-module $M$, there exists an exact sequence $0 \to L \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$, where each $F_i$ is free. Thus each $F_i$ is $FGT$-injective by (1). So $M$ is $FGT$-injective by Lemma 3.2 (6) since $FGT - Id(L) \leq rFGT - I.\dim(R) \leq n$. It follows that $rFGT - I.\dim(R) = 0$. □

**Theorem 3.6.** *The following are equivalent for a right* Π*-coherent ring $R$:*

(1) *$rFGT - I.\dim(R) \leq 1$.*

(2) *Every left $R$-module has an epic $FGT$-flat envelope.*

(3) *Every right $R$-module has a monic $FGT$-injective cover.*

*Proof.* (1) $\Rightarrow$ (2). For any left $R$-module $M$, there is an $FGT$-flat preenvelope $f : M \to F$ by Theorem 3.4 (1). The exact sequence $0 \to \mathrm{im}(f) \to F \to L \to 0$ induces the exactness of the sequence $0 \to L^+ \to F^+ \to (\mathrm{im}(f))^+ \to 0$. Note that $F^+$ is $FGT$-injective, so $(\mathrm{im}(f))^+$ is $FGT$-injective by Lemma 3.2 (6) since $FGT - Id(L^+) \leq rFGT - I.\dim(R) \leq 1$. Thus $\mathrm{im}(f)$ is $FGT$-flat, and hence $M \to \mathrm{im}(f)$ is an epic $FGT$-flat preenvelope, equivalently, an epic $FGT$-flat envelope.

(2) $\Rightarrow$ (3). Let $M$ be a right $R$-module. Write $F = \sum\{N \leq M : N$ is $FGT$-injective$\}$ and $G = \oplus\{N \leq M : N$ is $FGT$-injective$\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$, which induces an exact sequence $0 \to F^+ \to G^+ \to K^+ \to 0$. Since $F^+$ has an epic $FGT$-flat envelope by (2) and $G^+$ is $FGT$-flat, we have $F^+$ is $FGT$-flat, and so $F$ is $FGT$-injective. Next we prove that the inclusion $i : F \to M$ is an $FGT$-injective precover of $M$. Let $\psi : F' \to M$ with $F'$ $FGT$-injective be an arbitrary homomorphism. Note that $\psi(F') \leq F$ by the proof above. Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is an $FGT$-injective precover of $M$. In addition, it is clear that the identity map $1_F$ of $F$ is the only homomorphism $g : F \to F$ such that $ig = i$, and hence $i$ is an $FGT$-injective cover of $M$.

(3) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with $E$ injective. Since $L$ has a monic $FGT$-injective cover by (3), we have $L$ is $FGT$-injective. Thus $FGT - Id(M) \leq 1$ and so $rFGT - I.\dim(R) \leq 1$. □

Recall that $\phi : M \to C$ is said to be a *$\mathcal{C}$-envelope with the unique mapping property* [5] if for any homomorphism $f$: $M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C \to C'$ such that $g\phi = f$. Dually we have the definition of a *$\mathcal{C}$-cover with the unique mapping property.*

**Theorem 3.7.** *The following are equivalent for a right $\Pi$-coherent ring $R$:*

(1) $rFGT - I.\dim(R) \leq 2$.

(2) *Every right $R$-module has an $FGT$-injective cover with the unique mapping property.*

*Moreover, the above conditions hold if $R$ satisfies that*

(3) *Every left $R$-module has an $FGT$-flat envelope with the unique mapping property.*

*Proof.* (1) $\Rightarrow$ (2). Let $M$ be a right $R$-module. Then $M$ has an $FGT$-injective cover $f : F \to M$ by Theorem 3.4 (2). It is enough to show that, for any $FGT$-injective right $R$-module $G$ and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\mathrm{im}(g)$ is the natural map. Note that $F/\mathrm{im}(g)$ is $FGT$-injective by Lemma 3.2 (6) since $FGT - Id(\ker(g)) \leq 2$. Thus there exists $\alpha : F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and so $f\alpha\pi = f$. Hence $\alpha\pi$ is an isomorphism since $f$ is a cover. Therefore $\pi$ is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then we have the exact sequence $0 \to M \to E^0 \xrightarrow{\varphi} E^1 \xrightarrow{\psi} N \to 0$, where $E^0$ and $E^1$ are injective. Let $\theta : H \to N$ be an $FGT$-injective cover with the unique mapping property. Then there exists $\delta : E^1 \to H$ such that $\psi = \theta\delta$. Thus $\theta\delta\varphi = \psi\varphi = 0 = \theta 0$, and hence $\delta\varphi = 0$, which implies that $\ker(\psi) = \mathrm{im}(\varphi) \subseteq \ker(\delta)$. Therefore there exists $\gamma : N \to H$ such that $\gamma\psi = \delta$, and so $\theta\gamma\psi = \psi$. Thus $\theta\gamma = 1_N$ since $\psi$ is epic. It follows that $N$ is isomorphic to a direct summand of $H$, and hence $N$ is $FGT$-injective. So $FGT - Id(M) \leq 2$, and hence $rFGT - I.\dim(R) \leq 2$.

(3) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then we have the exact sequence $0 \to M \to E^0 \to E^1 \to L \to 0$ with $E^0$ and $E^1$ injective. It gives rise to the exactness of the sequence

$$0 \to L^+ \xrightarrow{\psi} (E^1)^+ \xrightarrow{\varphi} (E^0)^+ \to M^+ \to 0.$$

Note that $(E^0)^+$ and $(E^1)^+$ are flat. Let $\theta : L^+ \to H$ be the $FGT$-flat envelope with the unique mapping property. Then there exists $\delta : H \to (E^1)^+$ such that $\psi = \delta\theta$. Thus $\varphi\delta\theta = \varphi\psi = 0$, and hence $\varphi\delta = 0$, which implies that $\mathrm{im}(\delta) \subseteq \ker(\varphi) = \mathrm{im}(\psi)$. So there exists $\gamma : H \to L^+$ such that $\psi\gamma = \delta$, and hence $\psi\gamma\theta = \psi$. Thus $\gamma\theta = 1_{L^+}$ since $\psi$ is monic. Consequently $L^+$ is isomorphic to a direct summand of $H$, and hence $L^+$ is $FGT$-flat. Thus $L$ is $FGT$-injective. It follows that $FGT - Id(M) \leq 2$, and so $rFGT - I.\dim(R) \leq 2$. □

Finally, we consider the special case that $R$ is a commutative Π-coherent ring.

**Proposition 3.8.** *Let $R$ be a commutative Π-coherent ring and $n$ a nonnegative integer. Then $wD(R) \leq n+1$ if and only if $FGT - I.\dim(R) \leq n$.*

*Proof.* It is easy to verify that $FGT - I.\dim(R) = \sup\{pd(F)$: $F$ is a finitely generated torsionless $R$-module$\}$ by Lemma 3.2 (6) since $R$ is a Π-coherent ring.

" $\Rightarrow$ ". Let $M$ be a finitely generated torsionless $R$-module. Then $M^*$ is finitely generated by [3, Theorem 1]. So there exists an exact sequence $R^m \to M^* \to 0$ with $m$ a nonnegative integer, which induces an exact sequence $0 \to M^{**} \to R^m$. Thus there exists an exact sequence $0 \to M \to R^m \to L \to 0$ since $M$ is torsionless. Note that $pd(L) \leq wD(R) \leq n+1$ by [19, Theorem 3.3] since $L$ is finitely presented. Hence $pd(M) \leq n$, and so $FGT - I.\dim(R) \leq n$.

" $\Leftarrow$ ". Let $N$ be a finitely presented $R$-module. Then there is an exact sequence $0 \to K \to F \to N \to 0$ with $F$ finitely generated free and $K$ finitely generated torsionless. Note that $pd(K) \leq FGT - I.\dim(R) \leq n$, and so $pd(N) \leq n+1$. Thus $wD(R) \leq n+1$ by [19, Theorem 3.3]. □

*Remark* 3.9. (1) Let $R$ be a commutative Π-coherent ring. Then Theorem 3.6 and 3.7 characterize respectively those rings such that $wD(R) \leq 2$ and $wD(R) \leq 3$ by Proposition 3.8.

(2) Let $R = F[x_1, x_2, \ldots, x_n]$, the ring of polynomials in $n$ indeterminates over a field $F$. Then $R$ is a commutative Noetherian ring, and hence a $\Pi$-coherent ring. Note that $wD(R) = n$, and so $FGT - I.\dim(R) = n - 1$ by Proposition 3.8. This fact also shows that the inequality $rFGT - I.\dim(R) \leq wD(R) + r.\pi cD(R)$ in Proposition 2.6 may be strict.

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