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CYCLES THROUGH A GIVEN SET OF VERTICES IN REGULAR MULTIPARTITE TOURNAMENTS

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ABSTRACT. A tournament is an orientation of a complete graph, and in general a multipartite or c-partite tournament is an orientation of a complete c-partite graph.

In a recent article, the authors proved that a regular c-partite tournament with $r \geq 2$ vertices in each partite set contains a cycle with exactly $r-1$ vertices from each partite set, with exception of the case that $c = 4$ and $r = 2$. Here we will examine the existence of cycles with $r-2$ vertices from each partite set in regular multipartite tournaments where the $r-2$ vertices are chosen arbitrarily. Let D be a regular c-partite tournament and let $X \subseteq V(D)$ be an arbitrary set with exactly 2 vertices of each partite set. For all $c \geq 4$ we will determine the minimal value $g(c)$ such that $D-X$ is Hamiltonian for every regular multipartite tournament with $r \geq g(c)$.

1. Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X,Y)$.

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and the *indegree* of x, respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ =$

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$\min\{d^+(x)|x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x)|x \in V(D)\}$ are the *minimum outdegree* and the *minimum indegree*, respectively.

For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. The length of a cycle C is denoted by $L(C)$. A cycle in a digraph D is *Hamiltonian*, if $L(C) = |V(D)|$. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles.

A digraph D is *strongly connected* or *strong*, if for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D-A$ obtained by deleting A is strong. The *connectivity*, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If $\kappa(D) = 1$ and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D.

There are several measures of how much a digraph differs from being regular. In [18], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)}\{d^+(x), d^-(x)\} - \min_{y \in V(D)}\{d^+(y), d^-(y)\}$$

and the *local irregularity* by $i_l(D) = \max\{|d^+(x) - d^-(x)|\,|x \in V(D)\}$. Clearly $i_l(D) \le i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \le 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \le |V_2| \le \cdots \le |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D.

Let $B = B(r_1, r_2, r_3, r_4)$ be the following bipartite tournament, which will be useful later. Let R_1, R_2, R_3, R_4 be pairwise disjoint independent sets of vertices with $|R_i| = r_i$ for $1 \le i \le 4$. Define $V(B) = R_1 \cup R_2 \cup R_3 \cup R_4$ such that $R_i \to R_{i+1}$ for $i = 1, 2, 3$ and $R_4 \to R_1$.

There is an extensive literature on cycles in multipartite tournaments, see e.g., Bang-Jensen and Gutin [1], Guo [2], Gutin [3], Volkmann [11], Winzen [15] and Yeo [17]. A new approach on cycles was presented by the authors in [12]:

Problem 1.1 (Volkmann, Winzen [12]). Which conditions have to be fulfilled in order that a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ contains a cycle with exactly r_i vertices of V_i for all $1 \le i \le c$ and given integers $0 \le r_i \le |V_i|$?

The most interesting part of Problem 1.1 is the case that $r_i = r_j$ for all $1 \leq i, j \leq c$. In 1997, A. Yeo [16] gave a solution of this problem for regular c-partite tournaments in the case that $r_i = |V_i|$ for all $1 \leq i \leq c$.

Theorem 1.2 (Yeo [16]). *Every regular multipartite tournament D is Hamiltonian.*

Since, according to the well known result of Moon [5] that every strongly connected tournament is vertex-pancyclic, a strongly connected tournament is Hamiltonian, we note that the next theorem also treats Problem 1.1 (especially in the case that $r_i = 1$ for all i).

Theorem 1.3 (Volkmann, Winzen [14]). *Let D be an almost regular c-partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$.*

In a recent article, Volkmann and Winzen [12] solved Problem 1.1 in the case that $r_i = r - 1$ for all $1 \leq i \leq c$.

Theorem 1.4 (Volkmann, Winzen [12]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $|V_1| = |V_2| = \cdots = |V_c| = r \geq 2$. If $c \geq 5$ or $c = 4$ and $r \geq 4$ or $c = 3$ or $c = 2$ and D is not isomorphic to $B\left(\frac{r}{2}, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}\right)$, then D contains a cycle with exactly $r - 1$ vertices from each partite set.*

For the case that $c = 4$ and $r = 2$, Theorem 1.4 is not true in general as the following example (see also [10, 11]) demonstrates.

Example 1.5. Let $V_i = \{v_i', v_i''\}$ for $i = 1, 2, 3, 4$ be the partite sets of a 4-partite tournament such that $v_1' \to v_2' \to v_3' \to v_1'$, $v_1'' \to v_2'' \to v_3'' \to v_1''$,

$$\{v_1', v_2', v_3'\} \to v_4' \to \{v_1'', v_2'', v_3''\} \to v_4'' \to \{v_1', v_2', v_3'\},$$

$$v_1' \to v_3'' \to v_2' \to v_1'' \to v_3' \to v_2'' \to v_1'$$

(see also Figure 1). Now it is a simple matter to check that the resulting 4-partite tournament is 3-regular without a cycle containing exactly $r - 1 = 1$ vertices of every partite set.

The complexity of the proof of Theorem 1.4 shows that the effort of an analysis of Problem 1.1 for the case $r_i = r - 2$ for all i would exceed the value of the result. Thus, in this paper we examine Problem 1.1 from another point of view. First, let us give a reformulation of Problem 1.1.

Problem 1.6. Let D be a c-partite tournament with the partite sets V_1, V_2, $\ldots$, V_c. Which conditions have to be fulfilled in order that for given integers $0 \leq s_i \leq |V_i|$ there is a set $X \subseteq V(D)$ with the property $|X \cap V_i| = s_i$ $(1 \leq i \leq c)$ such that the multipartite tournament $D - X$ is Hamiltonian?

If we replace the condition that there exists a set X with the properties mentioned above by the condition that for every choice of s_i vertices of V_i $D - X$ is Hamiltonian, then we arrive at the following new problem.

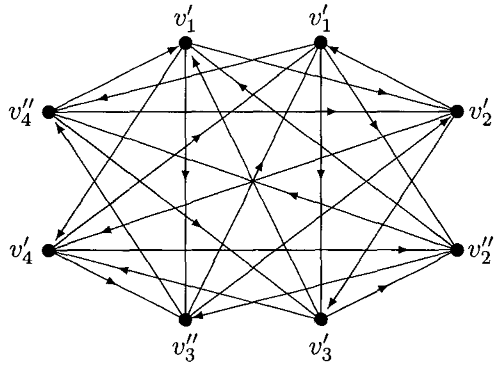

Figure 1: A regular 4-partite tournament without a strong subtournament of order 4

Problem 1.7. Let D be a c-partite tournament with the partite sets V_1, V_2, $\dots$, V_c. Which conditions have to be fulfilled in order that for given integers $0 \le s_i \le |V_i|$ and for every choice of the set $X \subseteq V(D)$ with $|X \cap V_i| = s_i$ $(1 \le i \le c)$ the multipartite tournament $D - X$ is Hamiltonian?

In this article, we investigate the special case that D is regular and that $s_i = s_j$ for $1 \le i, j \le c$.

Problem 1.8. Let D be a regular c-partite tournament with $c \ge 4$ and exactly r vertices in each partite set. Furthermore let $X \subseteq V(D)$ be an arbitrary set with exactly $k < r$ vertices of each partite set. For all given integers $k \ge 1$ and $c \ge 4$ find the minimal value $g(k, c)$ such that $D - X$ is Hamiltonian for every regular multipartite tournament with $r \ge g(k, c)$.

The condition $c \ge 4$ is important as we can see in the following two examples.

Example 1.9. Let D be the regular bipartite tournament $B\left(\frac{r}{2}, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}\right)$ with $r \ge 2$ even. If X contains exactly k vertices of R_1 and k vertices of R_2 (if $k \le \frac{r}{2}$) or if X contains all vertices of R_1 and R_2 and $k - \frac{r}{2}$ vertices of R_3 and R_4 (if $k > \frac{r}{2}$), then obviously $D - X$ is not Hamiltonian.

Example 1.10. Let k and r be positive integers such that $k < r$. Then we define the 3-partite tournament D with the partite sets $V_1 = \{u_1, u_2, \dots, u_r\}$, $V_2 = \{v_1, v_2, \dots, v_r\}$ and $V_3 = \{w_1, w_2, \dots, w_r\}$ by $Y = \{u_1, u_2, \dots, u_{r-k}\} \to Q_2 = \{v_1, v_2, \dots, v_{r-k}\} \to Z = \{w_1, w_2, \dots, w_{r-k-1}\} \to Y$, $Y \to \{w_{r-k}, w_{r-k+1}, \dots, w_{r-1}\} \to Q_2 \to w_r \to Y$, $Q_2 \to U = \{u_{r-k+1}, u_{r-k+2}, \dots, u_r\} \to (Z \cup \{w_{r-k}\}) \to V = \{v_{r-k+1}, v_{r-k+2}, \dots, v_r\} \to W = \{w_{r-k+1}, w_{r-k+2}, \dots, w_r\} \to U \to V \to Y$. It is easy to see that D is a regular 3-partite tournament such that $D - (U \cup V \cup W)$ is not Hamiltonian.

In [12], the authors showed that $g(1,4) = 3$ and $g(1,c) = 4$, if $c \geq 4$ is odd and $g(1,c) \leq 3$, if $c \geq 4$ is even. In this paper, we will determine $g(2,c)$ for all $c \geq 4$.

2. Preliminary results

To decide whether a multipartite tournament is Hamiltonian or not, the connectivity of this digraph is important as we can see in the following result of Yeo [16].

Theorem 2.1 (Yeo [16]). *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected c-partite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

He also gave a sharp bound for the connectivity of a multipartite tournament.

Theorem 2.2 (Yeo [17]). *Let D be a c-partite tournament. Then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil .$$

Nevertheless, in some cases Yeo's bound can be improved, as the next theorem for example demonstrates.

Theorem 2.3 (Volkmann, Winzen [13]). *Let S be a separating set of a multipartite tournament D with $\kappa(D) = \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3}$ and $|S| = \kappa(D)$. Then there is no partite set V_i of D such that $V_i \cap (V(D) - S) \neq \emptyset$ and $V_i \cap S \neq \emptyset$.*

A characterization whether a digraph D has a cycle-factor or not was given by Ore [6].

Theorem 2.4 (Ore [6]). *A digraph D has a cycle-factor if and only if $|N_D^+(S)| \geq |S|$ for each subset $S \subseteq V(D)$.*

In 1999, Yeo [18] (see also Gutin and Yeo [4]) rewrote Theorem 2.4 in the following useful form.

Theorem 2.5 (Yeo [18], Gutin, Yeo [4]). *A digraph D has no cycle-factor if and only if $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that*

(1) $\qquad R_1 \leadsto Y,\ (R_1 \cup Y) \leadsto R_2,\ Y$ *is an independent set*

and $|Y| > |Z|$.

This result leads to conditions for the global irregularity of a multipartite tournament D without a cycle-factor.

Theorem 2.6 (Stella, Volkmann, Winzen [7]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. Assume that D has no cycle-factor. According to Theorem 2.5, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + 1 \leq |Y| \leq |V_c| - t$ with an integer $t \geq 0$. Let V_i be the partite set with the property that*

$Y \subseteq V_i$. Let $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$, $Q_2 = Q \cap R_2$, $Y_1 = R_1 \cap V_i$ and $Y_2 = R_2 \cap V_i$. Then

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3 + |Y_2|}{2},$$

if $Q_1 = \emptyset$,

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3 + |Y_1|}{2},$$

if $Q_2 = \emptyset$, and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

An analysis of the proof of the last theorem yields the following.

Corollary 2.7 (Stella, Volkmann, Winzen [7]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the multipartite tournament D such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. Assume that D does not contain a cycle-factor. Let $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, Y_1$ and Y_2 be defined as in Theorem 2.6.*

If $Q_1 = \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3+|Y_2|}{2}$, then the following holds.

i) $\min\{d^-(w)|w \in V_i\} = |Z| = |Y| - 1$.
ii) $|Y| = |V_i| - |Y_2|$, *which means that* $|Y_1| = 0$ *and* $|V_i \cap Z| = 0$.
iii) $Y \rightarrow Q_2 \rightarrow (Y_2 \cup Z)$.
iv) $d^-(q_2) = d^+(q_2) - |Y_2| + 1$ *for all* $q_2 \in Q_2$.
v) $\max\{d^+(w), d^-(w)|w \in V(D) - V_i\} = d^-(q)$ *for a vertex* $q \in Q_2$ *such that* $|V(q)| = |V_{c-1}|$.
vi) $i_g(D) = \max\{d^-(q)|q \in Q_2\} - \min\{d^-(w)|w \in V_i\}$.
vii) $|V_i| = |V_c|$.
viii) $|V(D)| - |V_{c-1}| - 2|V_c| + 3 + |Y_2|$ *is even.*

Let $j = c - 1$, if $i = c$ and $j = c$, if $i < c$. If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3+t}{2}$, then we conclude that

a) $i_g(D) = i_l(D)$.
b) $\{|V_i|, |V_j|\} = \{|V_c|, |V_{c-1}|\}$.
c) $V_i \cap Z = \emptyset$, $|Z| = |Y| - 1$, $|Y| = |V_c| - t$.
d) $|V_m \cap Q_1| = |V_l \cap Q_1|$ *and* $|V_m \cap Q| = |V_l \cap Q|$ *for all* $1 \leq l, m \leq c$ *such that* $V_m \cap Q \neq \emptyset$ *and* $V_l \cap Q \neq \emptyset$.
e) $V_j \subseteq Q$.
f) $\frac{d(Q_1,Q_2)}{|Q_1|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+t}{2} - |Y_2| + |Y_1|$ *and* $\frac{d(Q_1,Q_2)}{|Q_2|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+t}{2} + |Y_2| - |Y_1|$.
g) $d^+(q_1) = d^-(q_1) + i_g(D)$ *for all* $q_1 \in Q_1$ *and* $d^-(q_2) = d^+(q_2) + i_g(D)$ *for all* $q_2 \in Q_2$.
h) $Q_2 \rightarrow (Z \cup Y_2)$, $(Z \cup Y_1) \rightarrow Q_1$.
j) $|V(D)| - |V_{c-1}| - 2|V_c| + 3 + t$ *is even.*

If the global irregularity differs slightly from the lower bound of Theorem 2.6, then some results of Corollary 2.7 are still valid. The proof of the next corollary is omitted here. It is a simple consequence of the proof of Theorem 2.6, which can be found in [7].

Corollary 2.8. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the multipartite tournament D such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. Assume that D does not contain a cycle-factor. Let Y, Z, Q, Q_1 and Q_2 be defined as in Theorem 2.6.*

If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ and $i_g(D) \leq \frac{|V(D)|-|V_{c-1}|-2|V_c|+4+t}{2}$, then we conclude that $i_g(D) = i_l(D)$ and $|Y| = |Z| + 1$.

The next result is a well-known theorem of Turán [8] (see also [9], p. 212) and is a good help to give an estimation for the number of arcs in a digraph.

Theorem 2.9 (Turán [8]). *Let D be a digraph without 2-cycles. If the underlying graph of D has no clique of order $p+1$, then*

$$|E(D)| \leq \frac{p-1}{2p}|V(D)|^2.$$

The following remark concerning regular multipartite tournaments is well-known but important for this article.

Remark 2.10. Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament. Then it follows that $r = |V_1| = |V_2| = \cdots = |V_c|$ and

$$d^+(x), d^-(x) = \frac{(c-1)r}{2}$$

for all $x \in V(D)$. That means especially that r is even, if c is even.

3. Main results

The following theorem of Volkmann and Winzen [12] presents a first estimation for the value of $g(k,c)$ of Problem 1.8.

Theorem 3.1 (Volkmann, Winzen [12]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \cdots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of exactly k vertices of each partite set for $1 \leq k \leq r-1$. If*

$$r \geq \left\lceil \frac{2k(c-1)-2}{c-3} \right\rceil + k = 3k + \left\lceil \frac{4k-2}{c-3} \right\rceil,$$

then D contains a cycle C such that $V(C) = V(D) - X$.

Hence, we have $g(k,c) \leq 3k + \left\lceil \frac{4k-2}{c-3} \right\rceil$ and because of $c \geq 4$ it follows that $g(k,c) \leq 7k-2$ and especially $g(2,c) \leq 12$. The following example demonstrates that $g(2,c) \geq 5$ for each $c \geq 4$.

Example 3.2. Let $V_1', V_2', \ldots, V_c'$ with $|V_i'| = 2$ for all $1 \le i \le c$ be the partite sets of a multipartite tournament D' such that $V_i' \to V_j'$, if $1 \le i < j \le c$. Then let the c-partite tournament D consist of D', a copy D'' of D' (with the partite sets $V_1'', V_2'', \ldots, V_c''$) such that $V_j'' \to V_i'$ and $V_j' \to V_i''$ for all $1 \le i < j \le c$ (see Figure 2 for $c = 4$). Then we observe that D is a regular c-partite tournament with exactly 4 vertices in each partite set such that $D - V(D')$ is not Hamiltonian.

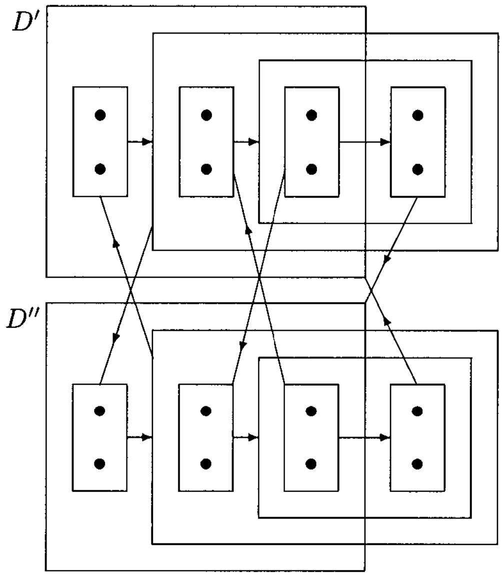

Figure 2: A regular 4-partite tournament D with the property that $D - V(D')$ is not Hamiltonian

With a little change of the last example, we can prove that $g(1, c) = 3$, if $c \ge 4$ is even.

Remark 3.3. If we remove one vertex from each set V_i' and V_i'' for all $1 \le i \le c$, then the resulting c-partite tournament D_1 is regular such that D' without the c vertices of the sets V_i' $(1 \le i \le c)$ is not Hamiltonian. Since, according to the results in [12], $g(1, c) \le 3$, if $c \ge 4$ is even it follows that $g(1, c) = 3$ in this case.

The next example shows that $g(2, 4) \ge 9$.

Example 3.4. Let $V_1 = \{u_1, u_2, \ldots, u_8\}$, $V_2 = \{v_1, v_2, \ldots, v_8\}$, $V_3 = \{w_1, w_2, \ldots, w_8\}$ and $V_4 = \{a_1, a_2, \ldots, a_8\}$ be the partite sets of a multipartite tournament D such that $R_1 = \{u_1, u_2, u_3, v_1, v_2, v_3\} \to Y = \{a_1, a_2, \ldots, a_6\} \to R_2 = \{u_4, u_5, u_6, v_4, v_5, v_6, w_6\} \leadsto Z = \{w_1, w_2, \ldots, w_5\} \to R_1$, $Y \to Z$,

$R_1 \rightsquigarrow R_2$, $v_3 \to u_1 \to \{v_1, v_2\}$, $v_1 \to u_2 \to \{v_2, v_3\}$, $\{v_2, v_3\} \to u_3 \to v_1$, $v_4 \to u_4 \to \{v_5, v_6\}$, $\{v_4, v_5\} \to u_5 \to v_6$, $\{v_5, v_6\} \to u_6 \to v_4$, $w_6 \to (R_1 - \{w_6\})$, $(\{w_6\} \cup Z) \rightsquigarrow X = \{u_7, u_8, v_7, v_8, w_7, w_8, a_7, a_8\} \rightsquigarrow Y$, $v_1 \to u_7 \rightsquigarrow (R_1 - \{v_1\})$, $v_2 \to u_8 \rightsquigarrow (R_1 - \{v_2\})$, $u_3 \to v_7 \rightsquigarrow (R_1 - \{u_3\})$, $(R_2 - \{v_4\}) \rightsquigarrow u_7 \to v_4$, $(R_2 - \{v_5\}) \rightsquigarrow u_8 \to v_5$, $(R_2 - \{u_4\}) \rightsquigarrow v_7 \to u_4$, $R_2 \rightsquigarrow \{v_8, w_7, w_8, a_7, a_8\} \rightsquigarrow R_1$, $\{a_7, a_8\} \to \{u_7, u_8, v_7, v_8\} \to \{w_7, w_8\}$, $\{a_7, a_8\} \to \{w_7, w_8\}$ and $u_7 \to v_7 \to u_8 \to v_8 \to u_7$. Then we observe that D is regular, and $D - X$ consists of the sets Y, Z, R_1 and R_2 satisfying (1). According to Theorem 2.5 it follows that $D - X$ is not Hamiltonian.

To give a complete analysis of the case $c = 4$, we will also give an example for a regular 4-partite tournament D with 6 vertices in each partite set such that there is a subset $X \subset V(D)$ with exactly 2 vertices from each partite set and with the property that $D - X$ is not Hamiltonian. (Remember that, according to Remark 2.10, the cardinality of each partite set has to be even.)

Example 3.5. Let $V_1 = \{u_1, u_2, \ldots, u_6\}$, $V_2 = \{v_1, v_2, \ldots, v_6\}$, $V_3 = \{w_1, w_2, \ldots, w_6\}$ and $V_4 = \{a_1, a_2, \ldots, a_6\}$ be the partite sets of a multipartite tournament D such that $R_1 = \{u_1, u_2, v_1, v_2\} \to Y = \{a_1, a_2, a_3, a_4\} \to R_2 = \{u_3, u_4, v_3, v_4, w_4\} \rightsquigarrow Z = \{w_1, w_2, w_3\} \to R_1$, $R_1 \rightsquigarrow R_2$, $Y \to Z$, $u_1 \to v_1 \to u_2 \to v_2 \to u_1$, $u_3 \to v_3 \to u_4 \to v_4 \to u_3$, $w_4 \to (R_2 - \{w_4\})$, $u_1 \to v_5 \to u_3$, $\{u_4, w_4\} \to v_5 \to u_2$, $u_2 \to v_6 \to u_4$, $\{u_3, w_4\} \to v_6 \to u_1$, $v_1 \to u_5 \to v_3$, $\{v_4, w_4\} \to u_5 \to v_2$, $v_2 \to u_6 \to v_4$, $\{v_3, w_4\} \to u_6 \to v_1$, $a_1 \to u_5 \to w_1$, $(Z - \{w_1\}) \to u_5 \to (Y - \{a_1\})$, $a_2 \to u_6 \to w_2$, $(Z - \{w_2\}) \to u_6 \to (Y - \{a_2\})$, $a_3 \to w_5 \to (Y - \{a_3\})$, $a_4 \to w_6 \to (Y - \{a_4\})$, $R_2 \rightsquigarrow \{w_5, w_6\} \to R_1$, $Z \to a_5 \to w_4$, $((Z - \{w_3\}) \cup \{w_4\}) \to a_6 \to w_3$, $(R_2 - \{w_4\}) \to \{a_5, a_6\} \to R_1$, $\{a_5, a_6\} \to \{u_5, u_6\} \to \{w_5, w_6\} \to \{v_5, v_6\} \to \{a_5, a_6\} \to \{w_5, w_6\}$ and $u_5 \to v_5 \to u_6 \to v_6 \to u_5$. Let $X = \{u_5, u_6, v_5, v_6, w_5, w_6, a_5, a_6\}$. Then we observe that D is regular, and $D - X$ consists of the sets Y, Z, R_1 and R_2 satisfying (1). According to Theorem 2.5 it follows that $D - X$ is not Hamiltonian.

In the case of a regular 5-partite tournament D with exactly 8 vertices in each partite set, there is no guarantee that D contains a cycle through any set of 6 vertices from each partite set.

Example 3.6. Let $V_1 = \{u_1, u_2, \ldots, u_8\}$, $V_2 = \{v_1, v_2, \ldots, v_8\}$, $V_3 = \{w_1, w_2, \ldots, w_8\}$, $V_4 = \{a_1, a_2, \ldots, a_8\}$ and $V_5 = \{b_1, b_2, \ldots, b_8\}$ be the partite sets of a multipartite tournament D such that $R_1 = \{u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3\} \to Y = \{b_1, b_2, \ldots, b_6\} \to R_2 = \{u_4, u_5, u_6, v_4, v_5, v_6, w_4, w_5, w_6, a_6\} \rightsquigarrow Z = \{a_1, a_2, \ldots, a_5\} \to R_1$, $R_1 \rightsquigarrow R_2$, $Y \to Z$, $\{u_1, u_2, u_3\} \to \{v_1, v_2, v_3\} \to \{w_1, w_2, w_3\} \to \{u_1, u_2, u_3\}$, $\{u_4, u_5, u_6\} \to \{v_4, v_5, v_6\} \to \{w_4, w_5, w_6\} \to \{u_4, u_5, u_6\}$, $a_6 \to (R_2 - \{a_6\}) \rightsquigarrow X = (V(D) - (Y \cup Z \cup R_1 \cup R_2)) \rightsquigarrow R_1$, $b_1 \to u_7 \to a_1$, $((Z - \{a_1\}) \cup \{a_6\}) \to u_7 \to (Y - \{b_1\})$, $b_2 \to u_8 \to a_2$, $((Z - \{a_2\}) \cup \{a_6\}) \to u_8 \to (Y - \{b_2\})$, $b_3 \to v_7 \to a_3$, $((Z - \{a_3\}) \cup \{a_6\}) \to v_7 \to (Y - \{b_3\})$, $b_4 \to v_8 \to a_4$, $((Z - \{a_4\}) \cup \{a_6\}) \to v_8 \to (Y - \{b_4\})$,

$b_5 \to a_7 \to (Y - \{b_5\})$, $b_6 \to a_8 \to Y - \{b_6\}$, $((Z - \{a_5\}) \cup \{a_6\}) \to b_7 \to a_5$, $Z \to b_8 \to a_6$, $(Z \cup \{a_6\}) \to \{w_7, w_8\} \to Y$, $\{a_7, a_8\} \to \{b_7, b_8\} \to (X - \{a_7, a_8, b_7, b_8\}) \to \{a_7, a_8\}$, $\{u_7, u_8\} \to \{v_7, v_8\} \to \{w_7, w_8\} \to \{u_7, u_8\}$. Then we observe that D is regular, and $D - X$ consists of the sets Y, Z, R_1 and R_2 satisfying (1). According to Theorem 2.5 it follows that $D - X$ is not Hamiltonian.

The following example deals with the case of a c-partite tournament D with $c \geq 6$ and exactly 6 vertices in each partite set.

Example 3.7. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,6}\}$ for $1 \leq i \leq c$ be the partite sets of a c-partite tournament D with $c \geq 6$ such that $R_1 = \{v_{i,j} \mid 1 \leq i \leq c-2,\ 1 \leq j \leq 2\} \to Y = \{v_{c,1}, v_{c,2}, v_{c,3}, v_{c,4}\} \to R_2 = \{v_{i,j} \mid 1 \leq i \leq c-2,\ 3 \leq j \leq 4 \ \vee\ (i = c-1 \ \wedge\ j = 4)\} \leadsto Z = \{v_{c-1,1}, v_{c-1,2}, v_{c-1,3}\} \to R_1$, $R_1 \leadsto R_2$, $Y \to Z$ and $v_{c-1,4} \to (R_2 - \{v_{c-1,4}\})$. The vertices in R_1 as well as the vertices in $R_2 - \{v_{c-1,4}\}$ let be connected such that $D[R_1]$ ($D[R_2 - \{v_{c-1,4}\}]$, respectively) is $(c-3)$-regular. Moreover, let $X = (V(D) - (Y \cup Z \cup R_1 \cup R_2)) \leadsto R_1$ except the arcs $\{v_{i,1} \to v_{i+1,5}, v_{i,2} \to v_{i+1,6}, v_{c-2,1} \to v_{1,5}, v_{c-2,2} \to v_{1,6} \mid 1 \leq i \leq c-3\}$. Analogously, let $(R_2 - \{v_{c-1,4}\}) \leadsto X$ except the arcs $\{v_{i+1,5} \to v_{i,3}, v_{i+1,6} \to v_{i,4}, v_{1,5} \to v_{c-2,3}, v_{1,6} \to v_{c-2,4} \mid 1 \leq i \leq c-3\}$. Furthermore let $V_c \to X' = \{v_{i,j} \mid (1 \leq i \leq \lfloor \frac{c-5}{2} \rfloor) \ \wedge\ 5 \leq j \leq 6) \ \vee\ (i = \frac{c-4}{2} \ \wedge\ j = 5 \ \wedge\ c \ \text{ is even})\} \to V_{c-1} \to (X - (X' \cup \{v_{c-1,5}, v_{c-1,6}, v_{c,5}, v_{c,6}\})) \to V_c$, $(Y \cup \{v_{c,5}, v_{c,6}\}) \to \{v_{c-1,5}, v_{c-1,6}\}$ and $\{v_{c,5}, v_{c,6}\} \to (Z \cup \{v_{c-1,4}\})$. If finally the vertices of $X - \{v_{c-1,5}, v_{c-1,6}, v_{c,5}, v_{c,6}\}$ are connected in a regular way, then we observe that D is regular, but $D - X$ consists of the sets Y, Z, R_1 and R_2 satisfying (1). According to Theorem 2.5 it follows that $D - X$ is not Hamiltonian.

Example 3.7 shows that $g(2, c) \geq 7$, if $c \geq 6$. If $c \geq 8$, then this estimation is sharp as we will see in the main result of this paper. But for $c = 7$ this bound is not sharp.

Example 3.8. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,7}\}$ for $1 \leq i \leq 7$ be the partite sets of a 7-partite tournament D such that $V(D)$ can be partitioned in the sets $R_1 = \{v_{i,j} \mid (1 \leq i \leq 5 \wedge 1 \leq j \leq 2) \vee (i \in \{2,4,5\} \wedge j = 3)\}$, $Y = V_7 - \{v_{7,6}, v_{7,7}\}$, $R_2 = \{v_{i,j} \mid (1 \leq i \leq 5 \wedge 4 \leq j \leq 5) \vee (i \in \{1,3\} \wedge j = 3) \vee (i = 6 \wedge j = 5)\}$, $Z = \{v_{6,1}, v_{6,2}, v_{6,3}, v_{6,4}\}$ and $X = \{v_{i,j} \mid 1 \leq i \leq 7, 6 \leq j \leq 7\}$. Furthermore let us define $R_{1,i} = R_1 \cap V_i$, $R_{2,i} = R_2 \cap V_i$ and $X_i = X \cap V_i$ for all $1 \leq i \leq 7$. If $R_1 \to Y \to R_2 \leadsto Z \to R_1 \leadsto R_2$, $Y \to Z$, $R_{1,1} \to R_{1,2} \to R_{1,3} \to R_{1,4} \to R_{1,5} \to R_{1,1} \to R_{1,3} \to R_{1,5} \to R_{1,2} \to R_{1,4} \to R_{1,1}$, $R_{2,1} \to R_{2,2} \to R_{2,3} \to R_{2,4} \to R_{2,5} \to R_{2,1} \to R_{2,3} \to R_{2,5} \to R_{2,2} \to R_{2,4} \to R_{2,1}$, $R_{2,3} \to v_{6,5} \to R_2 - (R_{2,3} \cup R_{2,6})$, $R_2 \leadsto X$, $X \leadsto R_1$ except the arcs $\{v_{1,1} \to v_{3,6}, v_{1,2} \to v_{3,7}\}$, $(X - (X_1 \cup X_6 \cup X_7)) \to Y \to X_1 \to Z \to (X - (X_1 \cup X_6 \cup X_7))$, $Y \to X_6$, $X_7 \to Z$, $X_6 \to (X - (X_1 \cup X_6 \cup X_7)) \to X_7 \to X_1 \to X_6$, $X_7 \to X_6$ and $X_1 \to X_2 \to X_3 \to X_4 \to X_5 \to X_1 \to X_3 \to X_5 \to X_2 \to X_4 \to X_1$, then we observe that D is regular, and $D - X$ consists of the sets Y, Z, R_1

and R_2 satisfying (1). According to Theorem 2.5 it follows that $D - X$ is not Hamiltonian.

The next theorem, the main result of this paper, determines $g(2, c)$ for all $c \geq 4$.

Theorem 3.9. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D such that $|V_1| = |V_2| = \cdots = |V_c| = r$. If $g(k, c)$ is defined as in Problem 1.8, then it follows that*

$$g(2,4) = g(2,5) = 9, \quad g(2,6) = 7, \quad g(2,7) = 8 \quad \textit{and} \quad g(2,c) = 7 \quad \textit{if } c \geq 8.$$

Proof. Let D be a regular c-partite tournament with the partite sets $V_1, V_2, \ldots,$ V_c each of the cardinality r. To prove this theorem we distinguish different cases.

Case 1. Let $c = 4$. Theorem 3.1 yields $g(2,4) \leq 12$ and Example 3.4 implies $g(2,4) \geq 9$. According to Remark 2.10 there is no regular 4-partite tournament with exactly 11 or exactly 9 vertices in each partite set. Hence, it remains to consider the case $r = 10$.

If $X \subseteq V(D)$ is an arbitrary set with exactly two vertices from each partite set of D, then for the multipartite tournament $D' := D - X$ it follows that $i_l(D') \leq 6$. Let $V_1', V_2', \ldots, V_c'$ be the partite sets of D'. It follows that $|V_i'| = 8$ for all $1 \leq i \leq c$. Theorem 2.2 implies that

$$\kappa(D') \geq \left\lceil \frac{|V(D')| - \alpha(D') - 2i_l(D')}{3} \right\rceil \geq \left\lceil \frac{32 - 8 - 12}{3} \right\rceil = 4.$$

Suppose that $\kappa(D') = 4$. Then Theorem 2.3 with $|S| = 4$ yields that there is at least one partite set V_i' of D' such that $V_i' \subseteq S$, a contradiction. Hence, we have $\kappa(D') \geq 5 = \lfloor \frac{\alpha(D')}{2} \rfloor + 1$. Applying Theorem 2.1 we see that D' is Hamiltonian, if it contains a cycle-factor. Hence, it remains to consider the case that D' has no cycle-factor. Then $V(D')$ can be partitioned into subsets Y, Z, R_1 and R_2 satisfying (1). Furthermore let Q_1, Q_2, Y_1, Y_2, V_i' and t be defined as in Theorem 2.6.

If $Q_1 = \emptyset$, then because of $|V_i'| + |Z| \leq 15$ it follows that $d^+(y) \geq |Q_2| \geq |V(D')| - |V_i'| - |Z| \geq 32 - 15 = 17$ for every $y \in Y$, a contradiction to $d^+(y) = \frac{(c-1)r}{2} = 15$.

Analogously the case $Q_2 = \emptyset$ is impossible.

Finally, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Since D is regular together with Theorem 2.6 we arrive at

$$6 \geq i_g(D') \geq \frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 3 + t}{2} = \frac{11 + t}{2},$$

and thus $i_g(D') = 6$ and $t \leq 1$.

First let $t = 1$. Then Corollary 2.7 $c)$ yields $|Y| = 7 = |Z| + 1$. Let $y \in V_i' - Y \neq \emptyset$. Since, according to Corollary 2.7 $c)$, $V_i' \cap Z = \emptyset$, it follows that $y \in (Y_1 \cup Y_2)$. If $y \in Y_1$, then Corollary 2.7 $h)$ implies $y \rightarrow Q_1$ and thus

$y \to Q$, because of $|Q| = |V(D')| - |Y| - |Z| - 1 = 18$ a contradiction to the regularity of D. If $y \in Y_2$, then analogously we arrive at a contradiction.

Second, let $t = 0$. Then Corollary 2.8 implies that $8 = |Y| = |Z| + 1$ and $i_g(D') = i_l(D')$. Without loss of generality, we may suppose that $Y = V_4'$. It follows that $|Q| = |Q_1| + |Q_2| = 17$ and $Y_1 = Y_2 = \emptyset$. Without loss of generality, let $|Q_1| \leq |Q_2|$. If Q_1 is bipartite, then for every vertex $q_1 \in Q_1$ we observe that

$$d^-_{D'[Q_1]}(q_1) \geq d^-_{D'}(q_1) - |Z| \geq 9 - 7 = 2.$$

Combining this together with Theorem 2.9, we arrive at

$$2|Q_1| \leq \sum_{q_1 \in Q_1} d^-_{D'[Q_1]}(q_1) \leq \frac{1}{4}|Q_1|^2 \quad \Rightarrow \quad |Q_1| \geq 8,$$

and thus $|Q_1| = 8$ and $|Q_2| = 9$. To avoid a contradiction, we may suppose that $Z \to Q_1$ and thus, without loss of generality, $Z \subseteq V_3'$. Now, for a vertex $q_2 \in (Q_2 \cap V_3') \neq \emptyset$ we have the contradiction $d^-(q_2) \geq |Q_1| + |Y| = 16$. In the remaining case that Q_1 is 3-partite we deduce that for every vertex $z \in Z$ there is a vertex $q_1 \in Q_1$ such that $q_1 \in V(z)$. Since $d^-_{D'}(q_1) \geq 9$ and $|Z| = 7$ a combination with Theorem 2.9 yields

$$7 + 2|Q_1| \leq \sum_{q_1 \in Q_1} d^-_{D'[Q_1]}(q_1) \leq \frac{1}{3}|Q_1|^2,$$

a contradiction to $|Q_1| \leq |Q_2|$ and thus $|Q_1| \leq 8$.

Case 2. Let $c = 5$. Theorem 3.1 implies that $g(2,5) \leq 9$ and Example 3.6 leads to $g(2,5) \geq 9$, and thus the desired result $g(2,5) = 9$.

Case 3. Let $c = 6$. According to Theorem 3.1, we have $g(2,6) \leq 8$. Since Remark 2.10 yields that $r = 8$ is impossible and thus we even have $g(2,6) \leq 7$. Now because of Example 3.7 it follows that $g(2,6) = 7$.

Case 4. Let $c = 7$. Theorem 3.1 implies $g(2,7) \leq 8$ and according to Example 3.8 we have $g(2,7) \geq 8$. Hence, we conclude that $g(2,7) = 8$.

Case 5. Let $c = 8$. With Theorem 3.1 we conclude that $g(2,8) \leq 8$ and Remark 2.10 implies that $r = 8$ is impossible such that $g(2,8) \leq 7$. Finally Example 3.7 demonstrates that $g(2,8) = 7$.

Case 6. Let $c \geq 9$. With a combination of Theorem 3.1 and Example 3.7 it follows that $g(2,c) = 7$ in this case.

This completes the proof of the theorem. □

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