J. Korean Math. Soc. **44** (2007), No. 3, pp. 673–682

# EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A KIND OF RAYLEIGH EQUATION WITH A DEVIATING ARGUMENT

QIYUAN ZHOU, BING XIAO, YUEHUA YU, BINGWEN LIU, AND LIHONG HUANG

ABSTRACT. In this paper, we use the coincidence degree theory to establish new results on the existence and uniqueness of $T$-periodic solutions for a kind of Rayleigh equation with a deviating argument of the form

$$x'' + f(x'(t)) + g(t, x(t - \tau(t))) = p(t).$$

## 1. Introduction

Consider the Rayleigh equation with a deviating argument of the form

$$x'' + f(x'(t)) + g(t, x(t - \tau(t))) = p(t), \tag{1.1}$$

where $f,\ \tau,\ p : R \to R$ and $g : R \times R \to R$ are continuous functions, $\tau$ and $p$ are $T$-periodic, $g$ is $T$-periodic in its first argument, $f(0) = 0$ and $T > 0$. In recent years, the problem of the existence of periodic solutions of Eq. (1.1) has been extensively studied in the literature. We refer the reader to [1, 3-8] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Eq. (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of $T$-periodic solutions of Eq. (1.1). The results of this paper are new and they complement previously known results.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = (\int_0^T |x(t)|^k dt)^{1/k}, \quad |x|_\infty = \max_{t \in [0,T]} |x(t)|.$$

Let

$$X = \{x | x \in C^1(R,\ R),\ x(t+T) = x(t) \ \text{ for all } t \in R\}$$

Received December 1, 2005.

2000 *Mathematics Subject Classification.* 34C25, 34D40.

*Key words and phrases.* Rayleigh equation, deviating argument, periodic solution, coincidence degree.

This work was supported by the NNSF (10371034) of China and the Project supported by Hunan Provincial Natural Science Foundation of China (05JJ40009).

and

$$Y = \{x | x \in C(R,\ R),\ x(t+T) = x(t) \text{ for all } t \in R\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty,\ |x'|_\infty\} \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define a linear operator $L : D(L) \subset X \longrightarrow Y$ by setting

$$D(L) = \{x | x \in X,\ x'' \in C(R,\ R)\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{1.2}$$

We also define a nonlinear operator $N : X \longrightarrow Y$ by setting

$$Nx = -f(x'(t)) - g(t, x(t - \tau(t))) + p(t). \tag{1.3}$$

It is easy to see that

$$\ker L = R, \quad \text{and } \operatorname{Im} L = \{x | x \in Y,\ \int_0^T x(s)ds = 0\}.$$

Thus the operator $L$ is a Fredholm operator with index zero.

Define the continuous projectors $P : X \longrightarrow \ker L$ and $Q : Y \longrightarrow Y$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)ds.$$

Hence, $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L$. Denoting by $L_P^{-1} : \operatorname{Im} L \longrightarrow D(L) \cap \ker P$ the inverse of $L|_{D(L)\cap\ker P}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)ds + \int_0^t (t-s)y(s)ds. \tag{1.4}$$

It is convenient to introduce the following assumptions.

$(A_0)$ assume that there exists a nonnegative constant $C_1$ such that

$$|f(x_1) - f(x_2)| \le C_1|x_1 - x_2| \text{ for all } x_1, x_2 \in R.$$

$(\widetilde{A_0})$ assume that there exists a nonnegative constant $C_2$ such that

$$C_2|x_1 - x_2|^2 \le (x_1 - x_2)(f(x_1) - f(x_2)) \text{ for all } x_1, x_2 \in R.$$

## 2. Preliminary results

In view of (1.2) and (1.3), the operator equation $Lx = \lambda Nx$ is equivalent to the following equation

$$x'' + \lambda[f(x'(t)) + g(t, x(t - \tau(t)))] = \lambda p(t), \tag{2.1$_\lambda$}$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the Continuation Theorem [4] as follows.

**Lemma 2.1.** *Let $X$ and $Y$ be two Banach spaces. Suppose that $L : D(L) \subset X \longrightarrow Y$ is a Fredholm operator with index zero and $N : X \longrightarrow Y$ is $L-$compact on $\overline{\Omega}$, where $\Omega$ is an open bounded subset of $X$. Moreover, assume that all the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0,\ 1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(3) *The Brower degree*

$$\deg\{QN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0.$$

*Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega}$.*

The following lemmas will be useful to prove our main results in Section 3.

**Lemma 2.2.** *If $x \in C^2(R, R)$ with $x(t + T) = x(t)$, then*

$$|x'(t)|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |x''(t)|_2^2. \tag{2.2}$$

*Proof.* Lemma 2.2 is known as Wirtinger inequality, and see [2, 3] for its proof. □

**Lemma 2.3.** *Suppose that there exists a constant $d > 0$ such that one of the following conditions holds:*

($A_1$) *$x(g(t, x) - p(t)) < 0$ for all $t \in R$, $|x| \geq d$;*
($A_2$) *$x(g(t, x) - p(t)) > 0$ for all $t \in R, |x| \geq d$.*

*If $x(t)$ is a $T$-periodic solution of $(2.1)_\lambda$, then*

$$|x|_\infty \leq d + \sqrt{T}|x'|_2. \tag{2.3}$$

*Proof.* Let $x(t)$ be a $T$-periodic solution of $(2.1)_\lambda$. Set

$$x(t_1) = \max_{t \in R} x(t), \qquad x(t_2) = \min_{t \in R} x(t), \quad \text{where } t_1,\ t_2 \in R.$$

Then we have

$$x'(t_1) = 0,\ x''(t_1) \leq 0 \quad \text{and} \quad x'(t_2) = 0,\ x''(t_2) \geq 0.$$

In view of ($A_1$) and ($A_2$), we shall consider two cases as follows.

Case (i). If ($A_1$) holds, it follows from $f(0) = 0$, $(2.1)_\lambda$ and ($A_1$) that

$$g(t_1, x(t_1 - \tau(t_1))) - p(t_1) \geq 0 \quad \text{and} \quad g(t_2, x(t_2 - \tau(t_2))) - p(t_2) \leq 0.$$

Thus,

$$x(t_1 - \tau(t_1)) < d \quad \text{and} \quad x(t_2 - \tau(t_2)) > -d.$$

Since $x(t - \tau(t))$ is a continuous function on $R$, it follows that there exists a constant $\xi \in R$ such that

$$|x(\xi - \tau(\xi))| \leq d.$$

Let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$, and $m$ is an integer. Then, using Schwarz inequality and the following relation

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \leq d + \int_{0}^{T} |x'(s)|ds, \ t \in [0,\ T],$$

we have

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \leq d + \sqrt{T}|x'|_2.$$

Case (ii). If $(A_2)$ holds, then using a similar argument as that in the proof of case (i), we see that (2.3) holds. This completes the proof of Lemma 2.3. □

**Lemma 2.4.** *Assume that one of the following conditions is satisfied:*

$(A_3)$ *Suppose that* $(A_0)$ *hold,* $g(t, x)$ *is a strictly monotone function in* $x$*, and there exists a nonnegative constant* $b$ *such that*

$$C_1 \frac{T}{2\pi} + b\frac{T^2}{2\pi} < 1, \quad \textit{and } |g(t,x_1) - g(t,x_2)| \leq b|x_1 - x_2| \quad \textit{for all } t, x_1, x_2 \in R;$$

$(A_4)$ *Suppose that* $(\widetilde{A_0})$ *hold,* $g(t, x)$ *is a strictly monotone function in* $x$*, and there exists a constant* $b$ *such that*

$$0 \leq b < \frac{C_2}{T}, \quad \textit{and } |g(t,x_1) - g(t,x_2)| \leq b|x_1 - x_2| \quad \textit{for all } t, x_1, x_2 \in R.$$

*Then Eq.* (1.1) *has at most one* $T$*-periodic solution.*

*Proof.* Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of Eq. (1.1). Then, we have

$$x_1''(t) + f(x_1'(t)) + g(t, x_1(t - \tau(t))) = p(t)$$

and

$$x_2''(t) + f(x_2'(t)) + g(t, x_2(t - \tau(t))) = p(t).$$

This implies that

(2.4)
$$(x_1(t) - x_2(t))'' + (f(x_1'(t)) - f(x_2'(t))) + (g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t)))) = 0.$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from (2.4), we obtain

(2.5) $$Z''(t) + (f(x_1'(t)) - f(x_2'(t))) + (g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t)))) = 0.$$

Set

$$Z(\bar{t}_1) = \max_{t\in R} Z(t), \qquad Z(\bar{t}_2) = \min_{t\in R} Z(t), \quad \text{where } \bar{t}_1,\ \bar{t}_2 \in R.$$

Then, we have

$$Z'(\bar{t}_1) = x_1'(\bar{t}_1) - x_2'(\bar{t}_1) = 0, \quad Z''(\bar{t}_1) \leq 0 \qquad \text{and}$$
$$Z'(\bar{t}_2) = x_1'(\bar{t}_2) - x_2'(\bar{t}_2) = 0, \quad Z''(\bar{t}_2) \geq 0.$$

In view of (2.5), we obtain

$$g(\bar{t}_1, x_1(\bar{t}_1 - \tau(\bar{t}_1))) - g(\bar{t}_1, x_2(\bar{t}_1 - \tau(\bar{t}_1))) \geq 0 \qquad \text{and}$$
$$g(\bar{t}_2, x_1(\bar{t}_2 - \tau(\bar{t}_2))) - g(\bar{t}_2, x_2(\bar{t}_2 - \tau(\bar{t}_2))) \leq 0.$$

Since $g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))$ is a continuous function on $R$, it follows that there exists a constant $\bar{\xi} \in R$ such that

$$g(\bar{\xi}, x_1(\bar{\xi} - \tau(\bar{\xi}))) - g(\bar{\xi}, x_2(\bar{\xi} - \tau(\bar{\xi}))) = 0. \tag{2.6}$$

Let $\bar{\xi} - \tau(\bar{\xi}) = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0,\ T]$ and $n$ is an integer. Then, (2.6), together with $(A_3)$ (or $(A_4)$), implies that

$$Z(\widetilde{\gamma}) = x_1(\widetilde{\gamma}) - x_2(\widetilde{\gamma}) = x_1(\bar{\xi} - \tau(\bar{\xi})) - x_2(\bar{\xi} - \tau(\bar{\xi})) = 0. \tag{2.7}$$

Hence,

$$|Z(t)| = |Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^{t} Z'(s)ds| \leq \int_0^T |Z'(s)|ds, \ t \in [0,\ T]$$

and

$$|Z|_\infty \leq \sqrt{T}|Z'|_2. \tag{2.8}$$

Now suppose that $(A_3)$ (or $(A_4)$) holds, we shall consider two cases as follows.

**Case (i)** If $(A_3)$ holds, multiplying $Z''(t)$ and (2.5) and then integrating it from 0 to $T$, we have

$$\begin{aligned}
|Z''|_2^2 &= \int_0^T |Z''(t)|^2 dt \\
&= -\int_0^T (f(x_1'(t)) - f(x_2'(t)))Z''(t)dt \\
&\quad -\int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))Z''(t)dt \\
&\leq C_1 \int_0^T |x_1'(t) - x_2'(t)||Z''(t)|dt \\
&\quad + b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z''(t)|dt.
\end{aligned} \tag{2.9}$$

From (2.2), (2.8) and Schwarz inequality, (2.9) implies that

$$\begin{aligned}
|Z''|_2^2 &\leq C_1|Z'|_2|Z''|_2 + b|Z|_\infty\sqrt{T}|Z''|_2 \\
&\leq C_1\frac{T}{2\pi}|Z''|_2^2 + b\sqrt{T}|Z'|_2\sqrt{T}|Z''|_2 \\
&\leq (C_1\frac{T}{2\pi} + b\frac{T^2}{2\pi})|Z''|_2^2.
\end{aligned} \tag{2.10}$$

Since $Z(t), Z'(t)$ and $Z''(t)$ are $T$-periodic and continuous functions, in view of $(A_3)$, (2.7) and (2.10), we have

$$Z(t) \equiv Z'(t) \equiv Z''(t) \equiv 0 \quad \text{for all} \quad t \in R.$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in R$. Therefore, Eq. (1.1) has at most one $T$-periodic solution.

**Case (ii)** If $(A_4)$ holds, multiplying $Z'(t)$ and (2.5) and then integrating it from 0 to $T$, together with (2.8), we obtain

$$\begin{aligned} C_2|Z'|_2^2 &= \int_0^T C_2|x_1'(t) - x_2'(t)|^2 dt \\ &\le \int_0^T (f(x_1'(t) - f(x_2'(t)))(x_1'(t) - x_2'(t))dt \\ &= -\int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))Z'(t)dt \\ &\le b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z'(t)|dt \\ &\le bT|Z'|_2^2. \end{aligned} \tag{2.11}$$

From (2.7) and $(A_4)$, (2.11) implies that

$$Z(t) \equiv Z'(t) \equiv 0 \quad \text{for all} \quad t \in R.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one $T$-periodic solution. The proof of Lemma 2.4 is now complete. □

## 3. Main results

**Theorem 3.1.** *Let* $(A_1)$ (or $(A_2)$) *hold. Assume that either the condition* $(A_3)$ *or the condition* $(A_4)$ *is satisfied. Then Eq.* (1.1) *has a unique $T$-periodic solution.*

*Proof.* By Lemma 2.4, together with $(A_3)$ and $(A_4)$, it is easy to see that Eq. (1.1) has at most one $T$-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one $T$-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible $T$-periodic solutions of Eq. $(2.1)_\lambda$ are bounded. In view of $(A_3)$ and $(A_4)$, we consider two cases as follows.

**Case (1)** If $(A_3)$ holds. Let $x(t)$ be a $T$-periodic solution of Eq. $(2.1)_\lambda$. Multiplying $x''(t)$ and Eq. $(2.1)_\lambda$ and then integrating it from 0 to $T$, in view of (2.2), (2.3), $(A_3)$ and the inequality of Schwarz, we have

$$
\begin{aligned}
|x''|_2^2 &= \int_0^T |x''(t)|^2 dt \\
&= -\lambda \int_0^T f(x'(t))x''(t)dt - \lambda \int_0^T g(t, x(t-\tau(t)))x''(t)dt \\
&\quad + \lambda \int_0^T p(t)x''(t)dt \\
&= -\lambda \int_0^T g(t, x(t-\tau(t)))x''(t)dt + \lambda \int_0^T p(t)x''(t)dt \\
&\leq \int_0^T |g(t, x(t-\tau(t)))| \cdot |x''(t)|dt + \int_0^T |p(t)| \cdot |x''(t)|dt \\
&\leq \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x''(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x''(t)|dt \\
&\leq b \int_0^T |x(t-\tau(t))| \cdot |x''(t)|dt + \int_0^T |g(t,0)| \cdot |x''(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x''(t)|dt \\
&\leq b\frac{T^2}{2\pi}|x''|_2^2 + [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}|x''|_2,
\end{aligned}
\tag{3.1}
$$

which, together with $(A_3)$, implies that there exist positive constants $D_1$ and $D_2$ such that

$$|x''|_2 < D_1, \tag{3.2}$$

and

$$|x'|_2 < D_2, \quad |x|_\infty < D_2. \tag{3.3}$$

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that $x'(\zeta) = 0$, and

$$|x'(t)| = |x'(\zeta) + \int_\zeta^t x''(s)ds| \leq \sqrt{T}|x''|_2 < \sqrt{T}D_1 \text{ for all } t \in [0,\ T]. \tag{3.4}$$

**Case (2)** If $(A_4)$ holds. Let $x(t)$ be a $T$-periodic solution of Eq. $(2.1)_\lambda$. Multiplying $x'(t)$ and Eq. $(2.1)_\lambda$ and then integrating it from 0 to $T$, by $(A_4)$, (2.3) and the inequality of Schwarz, we have

$$
\begin{aligned}
C_2|x'|_2^2 &= \int_0^T C_2 x'(t)x'(t)dt \\
&\leq \int_0^T f(x'(t))x'(t)dt
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&= -\int_0^T g(t, x(t-\tau(t)))x'(t)dt + \int_0^T p(t)x'(t)dt \\
&\le \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x'(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x'(t)|dt \\
&\le b\int_0^T |x(t-\tau(t))| \cdot |x'(t)|dt + \int_0^T |g(t,0)| \cdot |x'(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x'(t)|dt \\
&\le bT|x'|_2^2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x'|_2.
\end{aligned}
$$

This implies that there exists a constant $\overline{D_2} > 0$ such that

$$|x'|_2 < \overline{D_2}, \quad |x|_\infty < \overline{D_2}. \tag{3.6}$$

Multiplying $x''(t)$ and Eq. $(2.1)_\lambda$ and then integrating it from 0 to $T$, by $(A_4)$, (2.3), (3.1) and the inequality of Schwarz. Therefore, from (3.4), we obtain

$$
\begin{aligned}
&|x''|_2^2 \\
&= \int_0^T |x''(t)|^2 dt \\
&\le \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x''(t)|dt + \int_0^T |p(t)| \cdot |x''(t)|dt \\
&\le b\int_0^T |x(t-\tau(t))| \cdot |x''(t)|dt + \int_0^T |g(t,0)| \cdot |x''(t)|dt + \int_0^T |p(t)| \cdot |x''(t)|dt \\
&\le b\sqrt{T}|x'|_2\sqrt{T}|x''|_2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x''|_2 \\
&\le bT\overline{D_2}|x''|_2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x''|_2,
\end{aligned}
$$

it follows from (3.4) that there exists a positive constant $\overline{D_1}$

$$|x'(t)| \le \sqrt{T}|x''|_2 \le \overline{D_1}. \tag{3.7}$$

Therefore, in view of (3.3), (3.4), (3.6) and (3.7), there exists a positive constant $M_1 > \max\{\sqrt{T}D_1 + D_2, \overline{D_1} + \overline{D_2}\}$ such that

$$\|x\|_X \le |x|_\infty + |x'|_\infty < M_1.$$

If $x \in \Omega_1 = \{x | x \in \ker L \cap X \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant $M_2$ such that

$$x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g(t,\ M_2) - p(t)]dt = 0. \tag{3.8}$$

Thus,

$$|x(t)| \equiv |M_2| < d \quad \text{for all } x(t) \in \Omega_1. \tag{3.9}$$

Let $M = M_1 + d + 1$. Set $\Omega = \{x|x \in X,\ |x|_\infty < M,\ \ |x'|_\infty < M\}$. It is easy to see from (1.3) and (1.4) that $N$ is $L-$compact on $\overline{\Omega}$. We have from (3.8), (3.9) and the fact $M > \max\{M_1,\ d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions $H_1(x,\mu)$ and $H_2(x,\mu)$ by setting

$$H_1(x,\mu) = (1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(t,x) - p(t)]dt;\ \mu \in [0\ \ 1],$$

$$H_2(x,\mu) = -(1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(t,x) - p(t)]dt;\ \mu \in [0\ \ 1].$$

If $(A_1)$ holds, then

$$xH_1(x,\mu) \neq 0 \ \text{ for all } x \in \partial\Omega \cap \ker L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN,\ \Omega\cap\ker L,\ 0\} &= \deg\{-\frac{1}{T}\int_0^T [g(t,x)-p(t)]dt,\ \Omega\cap\ker L,\ 0\}\\ &= \deg\{x,\ \Omega\cap\ker L,\ 0\} \neq 0.\end{aligned}$$

If $(A_2)$ holds, then

$$xH_2(x,\mu) \neq 0 \text{ for } \ \text{all } x \in \partial\Omega \cap \ker L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN,\ \Omega\cap\ker L,\ 0\} &= \deg\{-\frac{1}{T}\int_0^T [g(t,x)-p(t)]dt,\ \Omega\cap\ker L,\ 0\}\\ &= \deg\{-x,\ \Omega\cap\ker L,\ 0\} \neq 0.\end{aligned}$$

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

## 4. Example and Remark

**Example 4.1.** Let $g(t,x) = \frac{1}{6\pi}x$, for all $t \in R$, $x > 0$, and $g(t,x) = \arctan x$, for all $t \in R$, $x \leq 0$. Then the Rayleigh equation

$$x''(t) + \frac{1}{8}x'(t) + \frac{1}{8}\sin x'(t) + g(t, x(t-\sin^2 t)) = \frac{1}{40}e^{\cos t} \tag{4.1}$$

has a unique $2\pi$-periodic solution.

*Proof.* By (4.1), we have $b = \frac{1}{6\pi}, C_1 = \frac{1}{4}$, $\tau(t) = \sin^2 t$ and $p(t) = \frac{1}{40}e^{\cos t}$. It *is obvious that the assumptions* $(A_1)$ and $(A_3)$ hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique $2\pi$-periodic solution. □

*Remark* 4.1. Eq. (4.1) is a very simple version of Rayleigh equation. Since $f(x) = \frac{1}{8}\dot{x} + \frac{1}{8}\sin x$ and $\tau(t) = \sin^2 t$, all the results in [1, 3-8] and the references therein are can not be applicable to Eq. (4.1) to obtain the existence and uniqueness of $2\pi$-periodic solutions. This implies that the results of this paper are essentially new.

## References


[1] T. A. Burton, *Stability and Periodic Solution of Ordinary and Functional Differential Equations*, Academic Press, Orland, FL., 1985.

[2] R. E. Gaines and J. Mawhin, *Coincidence degree, and nonlinear differential equations*, Lecture Notes in Math., 568, Spring-Verlag, 1977.

[3] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, London: Cambridge Univ. Press, 1952.

[4] X. Huang and Z. G. Xiang, *On existence of $2\pi$−periodic solutions for delay duffing equation $x'' + g(t, x(t - \tau(t))) = p(t)$*, Chin. Sci. Bull. **39** (1994), no. 3, 201–203.

[5] S. Lu and W. Ge, *Periodic solutions for a kind of Liéneard equations with a deviating argument*, J. Math. Anal. Appl. **289** (2004), no. 1, 231–243.

[6] ———, *Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument*, Nonlinear Analy. **56** (2004), no. 4, 501–514.

[7] J. Mawhin, *Periodic solutions of some vector retarded functional differential equations*, J. Math. Anal. Appl. **45** (1974), 588–603.

[8] G. Wang, *A priori bounds for periodic solutions of a delay Rayleigh equation*, Appl. Math. Lett. **12** (1999), 41–44.



Qiyuan Zhou
Department of Mathematics
Hunan University of Arts and Science
Changde 415000, P. R. China

Bing Xiao
Department of Mathematics
Hunan University of Arts and Science
Changde 415000, P. R. China

Yuehua Yu
Department of Mathematics
Hunan University of Arts and Science
Changde 415000, P. R. China

Bingwen Liu
Department of Mathematics
Hunan University of Arts and Science
Changde 415000, P. R. China
*E-mail address*: liubw007@yahoo.com.cn

Lihong Huang
College of Mathematics and Econometrics
Hunan University
Changsha 410082, P. R. China