J. Korean Math. Soc. **44** (2007), No. 3, pp. 661–671

KNOTS AND LINKS IN LINEAR EMBEDDINGS OF K_6

YOUNGSIK HUH AND CHOON BAE JEON

ABSTRACT. We investigate the number of knots and links in linear embeddings of K_6, the complete graph with 6 vertices. Concretely, we show that any linear embedding of K_6 contains either only one Hopf link, or three Hopf links and one trefoil knot.

1. Introduction

Throughout this paper we consider a *graph* as a 1-dimensional simplicial complex. Each 0-simplex and 1-simplex of a graph are called a *vertex* and an *edge*, respectively. The *complete graph* K_n is a graph with n vertices such that any two vertices are joined by one edge. We call an embedding of a graph G into $\mathbf{R}^3$ a *spatial embedding* of G and the image of the embedding a *spatial graph*. Especially, a *linear embedding* of G is a spatial embedding such that the image of each edge is a line segment. Two spatial embeddings $\mathcal{G}_1$ and $\mathcal{G}_2$ of the same graph G are said to be *equivalent* if there exists a homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(\mathcal{G}_1) = \mathcal{G}_2$.

An *m-component link* is a disjoint union of m simple closed curves in $\mathbf{R}^3$. Especially, a 1-component link is called a *knot*. The definition of the equivalence of spatial graphs is also applied to knots and links. A link is said to be *trivial* if it is equivalent to a link in a plane of $\mathbf{R}^3$. The knot 3_1 and the link 2_1^2 in Figure 1 are non-trivial. A knot and a link which are equivalent to 3_1 and 2_1^2 are called a *trefoil knot* and a *Hopf link*, respectively. If a link k consists of finite number of line segments, then we say that k is a *polygonal link*. The *polygonal index* of k, denoted by $p(k)$, is defined to be the minimal number of line segments to constitute a polygonal link which is equivalent to k.

If a graph G contains a cycle or more than one disjoint cycles as a subgraph, then a knot or a link exists in any spatial embeddings of G as a subspace. In [2] Conway and Gordon showed that every spatial embedding of K_6 contains odd number of non-trivial links and every spatial embedding of K_7 contains at least one non-trivial knot. Negami generalized Conway-Gordon's work. He

Received November 24, 2005.

2000 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M15, 05C10.

Key words and phrases. trefoil knot, Hopf link, complete graph, linear embedding.

This work was supported by a research fund from Research Institute for Natural Sciences at Hanyang University in 2004.

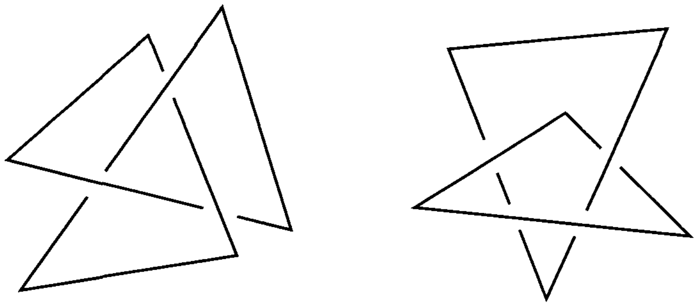

FIGURE 1. 3_1 and 2_1^2

proved that given a link k there is a finite number $r(k)$ such that every linear embedding of K_n with $n \geq r(k)$ contains a link equivalent to k [5]. For example, $r(\text{trefoil knot}) = 7$ [6].

In this paper we investigate the number of non-trivial knots and links in linear embeddings of K_6. If a polygonal knot k is non-trivial, then $p(k) \geq 6$. Furthermore, k is a trefoil knot if $p(k) = 6$ [1, 3, 5]. It follows that if k is contained in a linear embedding of K_6, then k is a trefoil knot. Also it's easy to prove that the polygonal index is more than 5 for every 2-component link and Hopf link is the only non-trivial 2-component link with $p = 6$, which implies that every non-trivial link in any linear embedding of K_6 is a Hopf link. The following theorems are the main results of this paper.

Theorem 1. *A linear embedding of K_6 can contain at most one trefoil knot.*

Theorem 2. *A linear embedding of K_6 contains a trefoil knot if and only if it contains three Hopf links.*

By combination of Theorems 1, 2 and the result of Conway-Gordon [2], we have

Corollary 3. *The number of Hopf links in any linear embedding of K_6 is either* 1 *or* 3.

Therefore we can conclude that if a linear embedding of K_6 contains a trefoil knot, then only one trefoil knot and three Hopf links exist in the embedding. Otherwise, the embedding contains only one Hopf link.

The rest of the paper is devoted to the proofs of Theorems 1 and 2. In Section 2, we investigate the relative positions of line segments in a polygonal trefoil knot, which are necessary for the proofs in Sections 3 and 4. In Section 5, a generalization of our result is discussed.

2. Polygonal trefoil knots

A set of points of $\mathbf{R}^3$ is said to be in *general position* if no three of its elements are collinear and no four of them are coplanar. Given a set $\{x_1, x_2, \ldots, x_n\}$ of

FIGURE 2. Reduction of a triangle

points in $\mathbf{R}^3$ and given $\delta_1 > 0$, there exists a set $\{y_1, y_2, \ldots, y_n\}$ of points in general position such that $|x_i - y_i| < \delta_1$ for all i [4].

A linear embedding of a graph is determined by the positions of its vertices in $\mathbf{R}^3$. Consider a linear embedding $\mathcal{G}_1$ of a graph G whose vertex set is, say, $\{x_1, x_2, \ldots, x_n\}$. It is clear that there exists $\delta_2 > 0$ such that if $|x_i - y_i| < \delta_2$ for each corresponding vertex y_i of another linear embedding $\mathcal{G}_2$ of G, then $\mathcal{G}_1$ is equivalent to $\mathcal{G}_2$. Note that the aim of this paper is to count the number of non-trivial knots and links in linear embeddings of K_6. Therefore we may assume that the vertex set of any linear spatial graph we consider is in general position.

Let $\mathcal{T}$ be a polygonal trefoil knot consisting of six line segments. We consider $\mathcal{T}$ as a linear embedding of a cycle with 6 vertices and label the vertices by $v_1, v_2, \ldots, v_6$ along an orientation of $\mathcal{T}$. For convenience, let v_i and v_j denote the same vertex, if $i \equiv j \pmod 6$. $e_{i,i+1}$ and $e_{i+1,i}$ denote the same line segment (or the embedded edge) between v_i and v_{i+1}. As mentioned in the above, we assume that the vertex set of $\mathcal{T}$ is in general position. For each a, a *triangle* $\triangle_a$ *of* $\mathcal{T}$ is defined to be the triangle which is determined by v_{a-1}, v_a and v_{a+1}. If $\triangle_a \cap \mathcal{T} = e_{a-1,a} \cup e_{a,a+1}$, we say that $\triangle_a$ is *reducible*. Otherwise, *irreducible*. Note that the line segments of $\mathcal{T}$ which may penetrate $\triangle_a$ are $e_{a+2,a+3}$ and $e_{a+3,a+4}$.

Lemma 4. *Every triangle of* $\mathcal{T}$ *is irreducible.*

Proof. Suppose $\triangle_a$ is reducible. Then, by replacing $e_{a-1,a}$, $e_{a,a+1}$ with $e_{a-1,a+1}$ (see Figure 2), we get another trefoil knot consisting of five line segments, which is contradictory to $p(\text{trefoil}) = 6$. □

Lemma 5. *For each* $a \in \{1, 2, \ldots, 6\}$, $\triangle_a$ *is penetrated by only one edge of* $\mathcal{T}$.

Proof. If $\triangle_a$ is penetrated by both of $e_{a+2,a+3}$ and $e_{a+3,a+4}$, then $\triangle_{a+3}$ is reducible (See Figure 3). □

Lemma 5 implies that only one of $e_{a+2,a+3}$ and $e_{a+3,a+4}$ can penetrate $\triangle_a$. By labeling the vertices of $\mathcal{T}$ in the other direction if necessary, we may assume that only $e_{a+2,a+3}$ penetrates $\triangle_a$ for some a.

Lemma 6. $e_{a+4,a+5}$ *penetrates* $\triangle_{a+1}$ *and* $\triangle_{a+2}$.

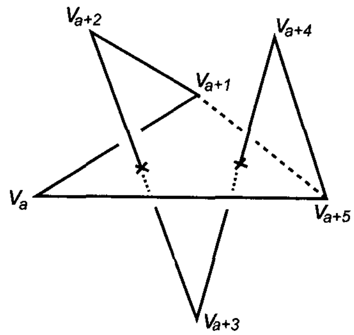

FIGURE 3

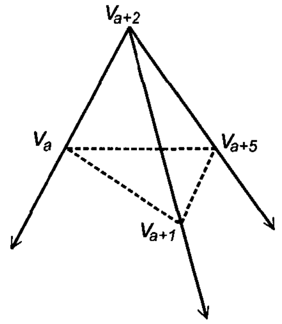

FIGURE 4. T^∞

Proof. Let P be the plane which contains $\triangle_a$, and let E^+ and E^- be the connected components of $\mathbf{R}^3 - P$. Since $e_{a+2,a+3}$ penetrates $\triangle_a$, we may assume that $v_{a+2} \in E^+$ and $v_{a+3} \in E^-$. Let T be the tetrahedron determined by $\triangle_a$ and v_{a+2}, and let T^∞ be the union of all half lines each of which starts at v_{a+2} and passes through a point of $\triangle_a$ (See Figure 4).

Note that $v_{a+3} \in T^\infty - T$. Since $e_{a+3,a+4}$ can't penetrate $\triangle_{a+1}$ without any intersection with $\triangle_a$, Lemma 5 implies that only $e_{a+4,a+5}$ penetrates $\triangle_{a+1}$. Furthermore, $e_{a+4,a+5}$ should penetrate $\triangle_{a+2}$ because $\triangle_{a+2} \subset T^\infty$ and $e_{a+5,a} \subset \partial T^\infty$. □

Proposition 7. *It is possible to label the vertices of $\mathcal{T}$ so that*

- $e_{1,2}$ *penetrates* $\triangle_4$ *and* $\triangle_5$,
- $e_{3,4}$ *penetrates* $\triangle_6$ *and* $\triangle_1$,
- $e_{5,6}$ *penetrates* $\triangle_2$ *and* $\triangle_3$.

Proof. Label the vertices of $\mathcal{T}$ so that $\triangle_1$ is pierced by $e_{3,4}$. Then, by Lemma 6, we know that $e_{5,6}$ penetrates $\triangle_2$ and $\triangle_3$. By applying Lemma 6 to $\triangle_3$

triangles of $\mathcal{T}$	edges of $\mathcal{K}_6$
$\triangle_1$	$e_{3,4}$
$\triangle_2$	$e_{5,6}$
$\triangle_3$	$e_{5,6}$
$\triangle_4$	$e_{1,2}$
$\triangle_5$	$e_{1,2}$
$\triangle_6$	$e_{3,4}$

TABLE 1. Edges of $\mathcal{K}_6$ penetrating the triangles of $\mathcal{T}$

and $e_{5,6}$, we know that $e_{1,2}$ penetrates $\triangle_4$ and $\triangle_5$. Similarly $e_{3,4}$ penetrates $\triangle_6$. □

3. Proof of theorem 1

In this section we prove Theorem 1. Let $\mathcal{K}_6$ be a linear embedding of K_6 which contains a trefoil knot $\mathcal{T}$. Label the vertices of $\mathcal{K}_6$ along $\mathcal{T}$ so that $\mathcal{T}$ satisfies Proposition 7.

We extend the mathematical notations in Section 2. P_a will denote the plane containing $\triangle_a$. E_a^+ and E_a^- are the connected components of $\mathbf{R}^3 - P_a$. $e_{i,j}$ will denote the embedded edge of $\mathcal{K}_6$ between v_i and v_j.

We observe which embedded edges of $\mathcal{K}_6$ penetrate $\triangle_a$ for each $a \in \{1, 2, \ldots, 6\}$. By Proposition 7, we know that if a is an even number then $e_{a-3,a-2}$ (i.e., $e_{a+3,a+4}$) penetrates $\triangle_{a+1}$. Other possible edges of $\mathcal{K}_6$ piercing $\triangle_{a+1}$ are $e_{a+4,a+5}$ and $e_{a+3,a+5}$. Since $e_{a+4,a+5} \subset \mathcal{T}$, Lemma 5 enables us to exclude it from the candidates. $\triangle_{a+2}$ is pierced by $e_{a+5,a}$, for $a+2$ is even. If we assume $v_{a+3} \in E_{a+1}^+$, then $\triangle_{a+2} \subset E_{a+1}^+ \cup P_{a+1}$, which implies $v_{a+5} \in E_{a+1}^+$. So $e_{a+3,a+5}$ can not penetrate $\triangle_{a+1}$. If a is an odd number, $\triangle_{a+1}$ is penetrated by $e_{a+4,a+5}$. Other possible edges are $e_{a+3,a+4}$ and $e_{a+3,a+5}$. Since $e_{a+3,a+4} \subset \mathcal{T}$, it's enough to consider $e_{a+3,a+5}$. Note that $\triangle_a$ is penetrated by $e_{a+2,a+3}$. If $v_{a+5} \in E_{a+1}^-$, then $\triangle_a \subset E_{a+1}^- \cup P_{a+1}$. Hence $v_{a+3} \in E_{a+1}^-$ and $e_{a+3,a+5}$ can not penetrate $\triangle_{a+1}$. What we have observed are summarized in Table 1.

Now we consider other triangles. Let $\triangle_{a,b,c}$ be the triangle which is determined by v_a, v_b and v_c, and let $P_{a,b,c}$ be the plane containing $\triangle_{a,b,c}$. Since we are assuming that the vertex set of $\mathcal{K}_6$ is in general position, $\triangle_{a,b,c}$ and $P_{a,b,c}$ are well-defined for every distinct $a, b, c \in \{1, 2, \ldots, 6\}$. The connected components of $\mathbf{R}^3 - P_{a,b,c}$ are denoted by $E_{a,b,c}^+$ and $E_{a,b,c}^-$. It would be observed which triangles among $\{\triangle_{6,a,b} | 1 \le a < b \le 5\}$ are penetrated by the embedded edges of $\mathcal{K}_6$.

It follows from Table 1 that $e_{5,6}$ and $e_{3,4}$ penetrate $\triangle_{1,2,3}$ and $\triangle_{6,1,2}$, respectively. Assume that $v_3 \in E_{6,1,2}^+$ and $v_4 \in E_{6,1,2}^-$. The interior of $e_{1,2}$ is contained in a component C of $\mathbf{R}^3 - (P_{6,1,3} \cup P_{6,2,3})$, which implies that the interiors of $\triangle_{6,1,2}$ and $\triangle_{1,2,3}$ are subsets of C. Hence $\{v_4, v_5\} \subset C$. The other

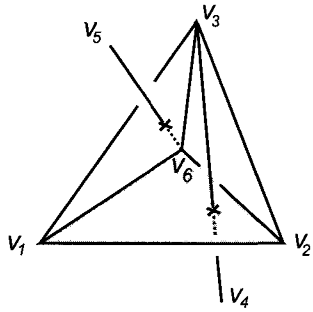

FIGURE 5

vertices are in ∂C (See Figure 5). Because $\triangle_{6,1,3}$ and $\triangle_{6,2,3}$ are contained in ∂C, they are not penetrated by any edge of $\mathcal{K}_6$. Note that v_4 is the only vertex in $E^-_{6,1,2}$. Therefore $\triangle_{6,1,4}$ and $\triangle_{6,2,4}$ are not penetrated. Similarly, we also know that $\triangle_{6,3,5}$ is not penetrated.

The remaining triangles to check are $\triangle_{6,3,4}$ and $\triangle_{6,2,5}$. Let T be the tetrahedron which is determined by $\triangle_{6,1,2}$ and v_3. If $\triangle_{6,3,4}$ is penetrated, the intersection point should be in $T \cap \triangle_{6,3,4}$. But it can't happen because $e_{1,2} \subset \partial T$ and $(e_{1,5} \cup e_{2,5}) \cap T = \{v_1, v_2\}$. For $\triangle_{6,2,5}$, the edges to consider are $e_{1,3}$, $e_{1,4}$ and $e_{3,4}$. Since $e_{1,3} \subset \partial C$ and $\triangle_{6,2,5} \cap \partial C = e_{6,2}$, $e_{1,3}$ can't meet $\triangle_{6,2,5}$. Also $e_{1,4}$ can be excluded from our consideration because $e_{1,4} \subset P_{6,1,2} \cup E^-_{6,1,2}$. Suppose $e_{3,4}$ penetrates $\triangle_{6,2,5}$. Then $\triangle_{6,2,5} \cap \triangle_{2,3,4}$ is a line segment between v_2 and an interior point x of $e_{3,4}$. On the other hand, considering that $e_{5,6}$ penetrates $\triangle_{2,3,4}$, $\triangle_{6,2,5} \cap \triangle_{2,3,4}$ is a line segment between v_2 and an interior point y of $e_{5,6}$. It follows that $x = y$ and $e_{3,4} \cap e_{5,6} \neq \emptyset$. Hence we get a contradiction.

From the observation in the above, we can conclude that among the ten triangles containing v_6, only $\triangle_{6,1,2}, \triangle_{6,1,5}$ and $\triangle_{6,4,5}$ are penetrated. To complete the proof, we check whether $\mathcal{K}_6$ contains any trefoil knot other than $\mathcal{T}$. A knot with 6 edges in $\mathcal{K}_6$ can be represented by a sequence of the vertices. For example, according to our labeling, $\mathcal{T}$ can be expressed as

$$\langle 123456 \rangle.$$

Suppose there exists another trefoil knot $\mathcal{T}'$ in $\mathcal{K}_6$. By Lemma 5, every triangle determined by three consecutive vertices along an orientation of $\mathcal{T}'$ should be irreducible. Hence $\mathcal{T}'$ corresponds to one of the following sequences.

$$\begin{array}{lll} S_1 = \langle \cdots 612 \cdots \rangle & S_2 = \langle \cdots 615 \cdots \rangle & S_3 = \langle \cdots 645 \cdots \rangle \\ S_4 = \langle \cdots 621 \cdots \rangle & S_5 = \langle \cdots 651 \cdots \rangle & S_6 = \langle \cdots 654 \cdots \rangle \\ S_7 = \langle \cdots 261 \cdots \rangle & S_8 = \langle \cdots 561 \cdots \rangle & S_9 = \langle \cdots 564 \cdots \rangle. \end{array}$$

Since S_7 dose not contain $e_{1,2}$, $\triangle_{3,4,5}$ is reducible in S_7. Similarly $\triangle_{2,3,4}$ and $\triangle_{1,2,3}$ are also reducible in S_2 and S_3, respectively. Therefore, the three sequences are excluded from our consideration. Now we try to guess the remaining parts of the five sequences in left direction. For example, only the number 5 can be put into the left of the number 6 in S_1 because $\triangle_{5,6,1}$ and $\triangle_{6,1,2}$ are the only irreducible triangles which contain both v_6 and v_1. On the other hand, S_4 can't be extended because $\triangle_{6,2,1}$ is the only irreducible triangle which contains both v_6 and v_2. In this way, we get the five possible sequences in the below.

$$\langle \cdots 5612 \cdots \rangle \quad \langle \cdots 4651 \cdots \rangle \quad \langle \cdots 4561 \cdots \rangle \quad \langle \cdots 1654 \cdots \rangle \quad \langle \cdots 1564 \cdots \rangle.$$

By filling the remaining parts completely, we get only three possible sequences.

$$\langle 4\underline{356}12 \rangle(\mathrm{X}) \quad \langle 3\underline{246}51 \rangle(\mathrm{X}) \quad \langle 345612 \rangle(\mathrm{O}) \quad \langle 321654 \rangle(\mathrm{O}) \quad \langle 315\underline{642} \rangle(\mathrm{X})$$
$$\langle 345612 \rangle(\mathrm{O}) \quad \langle 2\underline{346}51 \rangle(\mathrm{X}) \quad \langle 245\underline{613} \rangle(\mathrm{X}) \quad \langle 2\underline{316}54 \rangle(\mathrm{X}) \quad \langle 215\underline{643} \rangle(\mathrm{X}).$$

All of the three sequences correspond to $\mathcal{T}$. Therefore, $\mathcal{T}$ is the only nontrivial knot in $\mathcal{K}_6$.

Remark. From the proof we also know that if $\mathcal{K}_6$ contains a trefoil knot, then it contains only three Hopf links.

4. Proof of Theorem 2

We give a proof of Theorem 2. By the remark at the end of section 3, we only have to prove the "if" part of the theorem. Suppose that $\mathcal{K}_6$ contains three Hopf links. Then, without loss of generality, we may accept the following assumptions. The situation is depicted in Figure 6.

(1) the vertex set of $\mathcal{K}_6$ is in general position.
(2) $\partial\triangle_{1,2,3} \cup \partial\triangle_{4,5,6}$ is a Hopf link.
(3) $e_{4,5}$ penetrates $\triangle_{1,2,3}$.
(4) $e_{1,3}$ penetrates $\triangle_{4,5,6}$.
(5) $v_4, v_6 \in E^+_{1,3,2}$ and $v_5 \in E^-_{1,3,2}$.
(6) $v_6 \in E^+_{1,3,4}$ and $v_2, v_5 \in E^-_{1,3,4}$.

Let T be the tetrahedron determined by v_1, v_2, v_3, v_4, and let T^∞ be the union of all half lines starting at v_5 and intersecting $\triangle_{1,2,3}$.

Claim. $T \cap e_{5,6} = \emptyset$ *and* $\mathrm{Int}(T) \cap (e_{1,6} \cup e_{3,6}) = \emptyset$.

Proof of the Claim. Since $\partial\triangle_{1,2,3}$ is a component of a Hopf link, $e_{5,6}$ doesn't penetrate $\triangle_{1,2,3}$ and hence $e_{5,6} \cap T^\infty = \{v_5\}$. Considering that $T \subset T^\infty$ and $T \subset P_{1,3,4} \cup E^-_{1,3,4}$, we know that the statement holds. □

We will say that a triangle $\triangle_{a,b,c}$ is *reducible* if $\mathrm{Int}(\triangle_{a,b,c}) \cap \mathcal{K}_6 = \emptyset$. We proceed the proof of the theorem by considering two cases.

Case 1: $e_{2,6}$ doesn't penetrate $\triangle_{1,3,4}$. We observe the triangles containing v_1 as a vertex. In this case, $e_{5,6}$ is the only possible edge which may penetrate $\triangle_{1,3,4}$ because $e_{2,5} \subset P_{1,3,2} \cup E^-_{1,3,2}$. Therefore, by the claim in the above, $\triangle_{1,3,4}$

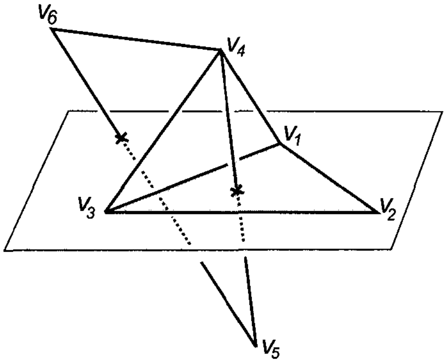

FIGURE 6

is reducible. Since $\triangle_{1,2,5}, \triangle_{1,3,5} \subset P_{1,3,2} \cup E^-_{1,3,2}$, they are reducible. Similarly $\triangle_{1,3,6}$ and $\triangle_{1,4,6}$ are reducible. Considering $T \cap e_{5,6} = \emptyset$ and $\mathrm{Int}(\triangle_{1,2,4}) \subset E^-_{1,3,4} \cap E^+_{1,3,2}$, we know that $\triangle_{1,2,4}$ is also reducible.

Let $x = P_{1,3,2} \cap e_{5,6}$ and $y = \triangle_{1,3,2} \cap e_{4,5}$. Then $\triangle_{1,5,6} = \triangle_{1,5,x} \cup \triangle_{x,1,6}$. Note that $e_{1,3} \cap \triangle_{4,5,6}$ is a point in the line segment between x and y. So $x \in E^+_{1,3,4}$. Hence $\triangle_{x,1,6} \subset P_{1,3,4} \cup E^+_{1,3,4}$ and $\triangle_{1,5,x} \subset P_{1,3,2} \cup E^-_{1,3,2}$, which implies that $\triangle_{1,5,6}$ is reducible. For $\triangle_{1,4,5}$, the possible edges to penetrate are $e_{3,6}$, $e_{2,3}$ and $e_{2,6}$. $e_{2,3}$ is removed from the candidates by the fact that $\triangle_{1,4,5} \subset T^\infty$. $e_{3,6}$ is also excluded because $e_{3,6} \subset P_{1,3,4} \cup E^+_{1,3,4}$. By the hypothesis of the case 1, $e_{2,6} \cap \mathrm{T} = \{v_2\}$. Considering that $\triangle_{1,4,y} \subset T$ and $\triangle_{1,4,5} = \triangle_{1,4,y} \cup \triangle_{y,5,1}$, we know that $e_{2,6}$ doesn't penetrate $\triangle_{1,4,5}$.

To summarize what we have observed, among the triangles containing v_1 as a vertex, at most two triangles are components of Hopf links in $\mathcal{K}_6$. Therefore, the case 1 can't happen.

Case 2: $e_{2,6}$ penetrates $\triangle_{1,3,4}$. We choose an orthogonal coordinate system of $\mathbf{R}^3$ so that the third axis has the same direction with $\overrightarrow{v_1v_3}$. Let $p : \mathbf{R}^3 \to \mathbf{R}^2$ be a projection map defined by $p(x,y,z) = (x,y)$ with respect to the chosen coordinate system. The piecewise linear curves in the left hand side of Figure 7 show the images of the cycles, $\langle 132654\rangle$ and $\langle 126543\rangle$, under p. After perturbing the cycles in $\mathbf{R}^3$ slightly enough and expressing the relative heights of double points with respect to the third axis, except for $p(e_{4,5}) \cap p(e_{2,6})$, we obtain the diagrams in the right hand side of Figure 7. If $e_{4,5}$ passes over $e_{2,6}$ at $p(e_{4,5}) \cap p(e_{2,6})$, then $\langle 132654\rangle$ is a trefoil knot. Otherwise, $\langle 126543\rangle$ is a trefoil knot. Therefore, $\mathcal{K}_6$ contains a trefoil knot.

5. Knots in linear embeddings of K_n

Let $c(\mathcal{K}_n)$ be the number of knots with polygonal index n in a linear embedding $\mathcal{K}_n$ of the complete graph K_n. Define $M(n)$ and $m(n)$ to be the maximum and the minimum of $c(\mathcal{K}_n)$ over all linear embeddings of K_n. For $n = 3, 4, 5$

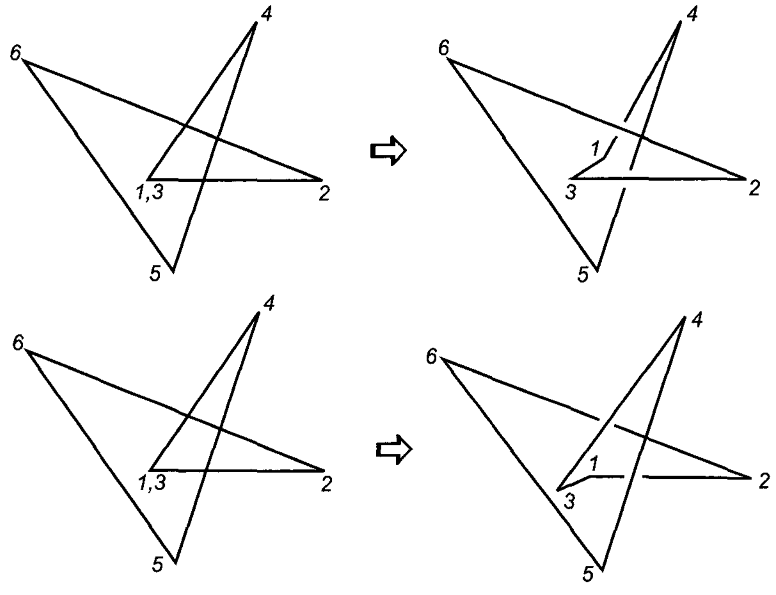

FIGURE 7

these numbers are meaningless because there is no non-trivial knot with polygonal index n. In Section 3 we proved that $M(6) = 1$. As a further study, we may try to find $M(n)$ and $m(n)$ for $n \geq 7$.

Let V_n be a set of points in $\mathbf{R}^3$ such that

$$V_n = \{v_i \mid v_i = (i, i^2, i^3),\ i = 0, 1, 2, \ldots, n-1\} .$$

Since V_n is in general position, it gives a linear embedding of K_n. Consider two line segments e_{i_1,j_1} and e_{i_2,j_2}. If $i_1 < i_2 < j_1 < j_2$, then

$$| \, p(e_{i_1,j_1}) \cap p(e_{i_2,j_2}) \, | = 1, \text{ where } p(x,y,z) = (x,y) .$$

Note that e_{i_1,j_1} passes under e_{i_2,j_2} at the double point, as seen in Figure 8. Now suppose that the embedding determined by V_n contains a knot k with polygonal index n. Then, for some i and j, $e_{i,0}$ and $e_{0,j}$ are line segments of k. Because $p(v_0)$ is the leftmost point among $p(V_n)$, the triangle determined by the two consecutive line segments is reducible, which is contradictory to $p(k) = n$. It follows that $m(n) = 0$ for $n \geq 6$.

The Figure-eight knot in Figure 9 is the only knot with polygonal index 7 [1]. So, to find $M(7)$, we have only to find the maximal number of Figure-eight knots in linear embeddings of K_7. For an arbitrary n, it is hard to count the number of specific knots with polygonal index n in linear embeddings of K_n, because the polygonal indices are known only for relatively small number of knots among all knot equivalence classes.

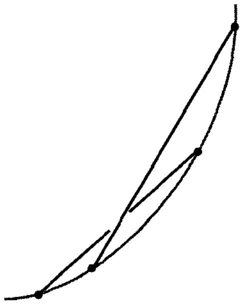

FIGURE 8

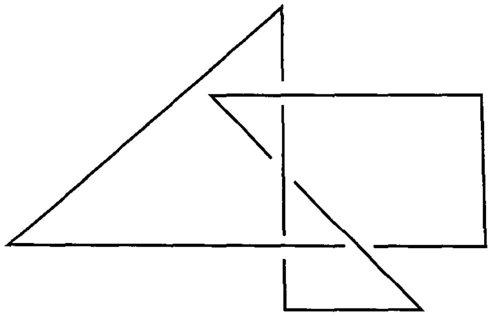

FIGURE 9. Figure-eight knot

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Youngsik Huh
Department of Mathematics
School of Natural Sciences
Hanyang University
Seoul 133-791, Korea
E-mail address: yshuh@hanyang.ac.kr

Choon Bae Jeon
General Education
Daeduk College
Daejeon 305-715, Korea
E-mail address: cbjeon@ddc.ac.kr