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EIGENVALUES FOR THE SEMI-CIRCULANT PRECONDITIONING OF ELLIPTIC OPERATORS WITH THE VARIABLE COEFFICIENTS

HOI SUB KIM, SANG DONG KIM, AND YONG HUN LEE

ABSTRACT. We investigate the eigenvalues of the semi-circulant preconditioned matrix for the finite difference scheme corresponding to the second-order elliptic operator with the variable coefficients given by $L_v u := -\Delta u + a(x,y)u_x + b(x,y)u_y + d(x,y)u$, where a and b are continuously differentiable functions and d is a positive bounded function. The semi-circulant preconditioning operator $L_c u$ is constructed by using the leading term of $L_v u$ plus the constant reaction term such that $L_c u := -\Delta u + d_c u$. Using the field of values arguments, we show that the eigenvalues of the preconditioned matrix are clustered at some number. Some numerical evidences are also provided.

1. Introduction

Consider the second-order elliptic boundary value problem given by

$$L_v u \quad := \quad -\Delta u + a(x,y)u_x + b(x,y)u_y + d(x,y)u = f \quad \text{in } \Omega := (0,1)\times(0,1) \tag{1.1}$$

with the homogeneous Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.2}$$

The usual central finite-difference scheme [16] to find an approximated solution for (1.1) is given by

$$\begin{aligned} & -\frac{1}{h_2^2}\{C_{k-1,j}u_{k-1,j} + Au_{k,j} + B_{k+1,j}u_{k+1,j} + \gamma_{k,j-1}u_{k,j-1} \\ & +\alpha_{k,j}u_{k,j} + \beta_{k,j+1}u_{k,j+1}\} \\ = \ & f_{k,j}, \quad 1 \le k \le m_1, 1 \le j \le m_2, \end{aligned} \tag{1.3}$$

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where

$$A = -2\varphi^2, \tag{1.4}$$

$$B_{k+1,j} = \varphi^2\Big(1 - \frac{a_{k,j}h_1}{2}\Big), \quad C_{k-1,j} = \varphi^2\Big(1 + \frac{a_{k,j}h_1}{2}\Big), \tag{1.5}$$

$$\alpha_{k,j} = -2 - d_{k,j}h_2^2, \quad \beta_{k,j+1} = 1 - \frac{1}{2}b_{k,j}h_2, \quad \gamma_{k,j-1} = 1 + \frac{1}{2}b_{k,j}h_2 \tag{1.6}$$

and, m_1 and m_2 are positive integers such that

$$h_1 = \frac{1}{m_1+1}, \ h_2 = \frac{1}{m_2+1}, \ \varphi = \frac{h_2}{h_1}. \tag{1.7}$$

Here $u_{k,j}$ is the approximated value of $u(x,y)$ at the point (x_k, y_j) and $g_{k,j}$ is the value of a $g(x,y)$ at the point (x_k, y_j), where $g = f, a, b$ or d, and

$$(x_k, y_j) := (kh_1, jh_2), \qquad 1 \le k \le m_1, 1 \le j \le m_2,$$

are interior nodal points in the domain Ω.

It is known that the semi-circulant preconditioner can be constructed by adding circulant elements to the matrix induced by finite difference discretization. In many literature, constructing semi-circulant and circulant preconditioner has been studied widely (see [1], [6], [7], [8], [9], [10], [11], [12] and [19]). These preconditioning strategy has been extended to HSS and PSS preconditioners (see [2] and [3]). Because the finite difference matrix $\hat{L}_v$ which represents the finite difference discretization corresponding to the operator L_v on the left-hand side of (1.3) can not be represented as tensor product form, it may be required to set up a semi-circulant preconditioner matrix corresponding to L_c given by

$$L_c u := -\Delta u + a_c u_x + b_c u_y + d_c u = f \qquad \text{in } \Omega \tag{1.8}$$

with zero boundary condition, where a_c, b_c and d_c are constants. We also note that the idea of optimal preconditioning in [17] is to consider a preconditioner by the leading term $-\Delta u + d_c u$, where d_c is chosen as $d_c = \sqrt{a_c^2 + b_c^2}$ so that L_c remains an optimal preconditioning operator. Adopting this optimal choice, one may construct an optimal semi-circulant preconditioner for the case of variable coefficients. But in this paper, we choose

$$d_c := \min_{(x,y)\in\bar{\Omega}} d(x,y) \ge 0 \quad \text{and} \quad a_c = b_c = 0. \tag{1.9}$$

The aim of this paper is to provide the distribution of eigenvalues and behavior of the extreme eigenvalues of the preconditioned matrix for the case that the coefficients $a(x,y)$ and $b(x,y)$ are continuously differentiable functions on Ω, and $d(x,y)$ is a positive bounded function. These will be assumed throughout this paper. This analysis is based on the way setting up a semi-circulant preconditioner and analyzing the finite difference preconditioner in terms of coefficient functions even if the finite difference preconditioning is well known using the leading terms of the given operator (1.1) (for example, see [17], [18]).

The usual central finite-difference scheme which approximates (1.8) with (1.9) is given by

$$(1.10)\qquad \begin{aligned} &-\frac{1}{h_2^2}\{\hat{C}u_{k-1,j} + \hat{A}u_{k,j} + \hat{B}u_{k+1,j} + u_{k,j-1} + \hat{\alpha}u_{k,j} + u_{k,j+1}\} \\ &= f_{k,j}, \quad 1 \le k \le m_1,\ 1 \le j \le m_2, \end{aligned}$$

where

$$(1.11)\qquad \hat{A} = -2\varphi^2, \quad \hat{B} = \hat{C} = \varphi^2,$$

and

$$(1.12)\qquad \hat{\alpha} = -2 - d_c h_2^2.$$

We assume that

$$(1.13)\qquad 0 \le \frac{\|a\|_\infty h_1}{2} < 1, \quad 0 \le \frac{\|b\|_\infty h_2}{2} < 1.$$

Note that the coefficients in (1.9) trivially satisfy the above assumption (1.13). While the matrix $\hat{L}_v$ can not be expressed as a tensor product form, the matrix $\hat{L}_c$ which represents the finite difference discretization corresponding to the operator L_c on the left-hand side of (1.10) is easily described as a tensor product form:

$$(1.14)\qquad \hat{L}_c := -\frac{1}{h_2^2}\{T_2 \otimes I_{m_1} + I_{m_2} \otimes T_1\},$$

where I_{m_k} are the identity matrices with the size m_k, and T_1 and T_2 are the tridiagonal matrices of order m_1 and m_2 respectively. In particular,

$$(1.15)\qquad T_1 = \mathrm{tridiag}[\hat{C}, \hat{A}, \hat{B}] \quad \text{and} \quad T_2 = \mathrm{tridiag}[1, \hat{\alpha}, 1].$$

The semi-circulant preconditioner for (1.14) is given by

$$(1.16)\qquad \hat{S}_c := -\frac{1}{h_2^2}\{\overline{C} \otimes I_{m_1} + I_{m_2} \otimes T_1\},$$

where $\overline{C}$ is the circulant matrix corresponding to T_2 given by

$$\overline{C} = \begin{pmatrix} \hat{\alpha} & 1 & 0 & \cdots & 0 & 1 \\ 1 & \hat{\alpha} & 1 & 0 & \cdots & 0 \\ 0 & 1 & \hat{\alpha} & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \hat{\alpha} & 1 \\ 1 & 0 & \cdots & 0 & 1 & \hat{\alpha} \end{pmatrix}.$$

In this paper we will propose $\hat{S}_c$ in (1.16) as a semi-circulant preconditioner for $\hat{L}_v$ which is the matrix representation of (1.3). The semi-circulant preconditioning for the equation (1.1) with constant coefficients was analyzed in [5], [14] and [15], for example, where more general cases including convection-diffusion dominated flows were discussed. Hence our concern is to analyze the clustering situations of eigenvalues λ and is to approximate extreme eigenvalues λ of the semi-circulant preconditioned matrix $\hat{S}_c^{-1}\hat{L}_v$ which satisfy

$$\lambda \hat{S}_c U = \hat{L}_v U, \tag{1.17}$$

where U is the eigenvector corresponding to λ. This analysis can be done by using the derived formula of eigenvalues $\hat{\lambda}$ of the matrix $\hat{S}_c^{-1}\hat{L}_c$ satisfying

$$\hat{\lambda} \hat{S}_c V = \hat{L}_c V, \tag{1.18}$$

which was analyzed in [15]. In this purpose, we represent the eigenvalues λ in (1.17) as follows:

$$\lambda = \frac{(\hat{L}_v U, U)}{(\hat{S}_c U, U)} = \frac{(\hat{L}_v U, U)}{(\hat{L}_c U, U)} \cdot \frac{(\hat{L}_c U, U)}{(\hat{S}_c U, U)}, \tag{1.19}$$

where the l^2 inner product is defined by

$$(U, V) = \sum_{k=1}^{N} u_k \bar{v}_k,$$

for complex vectors $U = (u_1, \ldots, u_N)$ and $V = (v_1, \ldots, v_N)$. Hence, we first investigate the field of values:

$$\frac{(\hat{L}_v U, U)}{(\hat{L}_c U, U)},$$

which means the preconditioning $\hat{L}_v$ by finite difference matrix $\hat{L}_c$.

This paper consists as follows: In section 2, the finite difference preconditioning for one and two dimensional cases are discussed in terms of coefficients of given differential equations. Finally, the semi-circulant preconditioner is discussed in section 3 with some numerical evidences.

2. Preconditioning by finite difference

The aim of this section is to get the bounds of eigenvalues of the preconditioned finite difference matrix in terms of coefficients functions given in (1.1) even if a finite difference preconditioner is well known (see, for example, [17] and [18]). Because the analysis for the $2D$ case can be derived by the way of analyzing the $1D$ case, we first consider the following one-dimensional elliptic operators L_v^1 and L_c^1 defined by

$$L_v^1 u := -u'' + a(x)u' + d(x)u \quad \text{in} \quad (0,1) \tag{2.1}$$

and

$$L_c^1 u := -u'' + d_c u \quad \text{in} \quad (0,1), \tag{2.2}$$

where $d_c = \min d(x)$ is a positive constant. Further we assume in this section that $a'(x)$ is a bounded function on $[0,1]$,

$$\|a'\|_\infty := \max_{x\in[0,1]} |a'(x)| < \infty \quad \text{and} \quad \max_{x\in[0,1]} \left|\frac{a(x)h}{2}\right| < 1. \tag{2.3}$$

Now let $\{x_i\}_{i=0}^{N+1}$ be the uniform knots such that

$$x_i := ih, \qquad \text{where } h = \frac{1}{N+1}.$$

Denote $a_i = a(x_i)$ and $d_i = d(x_i)$ for each $i = 1, \ldots, N$. Then the usual central finite-difference approximation to (2.1) at each knot x_i, $i = 1, \ldots, N$ is given by

$$(2.4) \qquad (L_v^1 u)_i = -\frac{1}{h^2}[\alpha_i u_{i+1} - 2u_i + \beta_i u_{i-1}] + d_i u_i, \quad i = 1, \ldots, N,$$

where

$$(2.5) \qquad \alpha_i = 1 - \frac{a_i h}{2}, \qquad \beta_i = 1 + \frac{a_i h}{2}.$$

The change of indices with the boundary conditions $u_0 = u_{N+1} = 0$ yields that

$$\begin{aligned}
&\sum_{i=1}^{N}(L_v^1 u)_i \bar{u}_i \\
&= -\frac{1}{h^2}\sum_{i=1}^{N}[\alpha_i(u_{i+1} - u_i) - \beta_i(u_i - u_{i-1})]\bar{u}_i + \sum_{i=1}^{N} d_i |u_i|^2 \\
&= -\frac{1}{h^2}\Big[\sum_{i=1}^{N}\alpha_i(u_{i+1} - u_i)\bar{u}_i - \sum_{i=0}^{N-1}\beta_{i+1}(u_{i+1} - u_i)\bar{u}_{i+1}\Big] + \sum_{i=1}^{N} d_i |u_i|^2 \\
&= \frac{1}{h^2}\sum_{i=0}^{N}(u_{i+1} - u_i)(\beta_{i+1}\bar{u}_{i+1} - \alpha_i \bar{u}_i) + \sum_{i=1}^{N} d_i |u_i|^2.
\end{aligned}$$

Since

$$\beta_{i+1}\bar{u}_{i+1} - \alpha_i \bar{u}_i = (\bar{u}_{i+1} - \bar{u}_i) + \frac{h}{2}(a_{i+1}\bar{u}_{i+1} + a_i \bar{u}_i),$$

we have

$$(2.6) \qquad \begin{aligned}
&\sum_{i=1}^{N}(L_v^1 u)_i \bar{u}_i \\
&= \frac{1}{h^2}\sum_{i=0}^{N}\Big[|u_{i+1} - u_i|^2 + \frac{h}{2}(u_{i+1} - u_i)(a_{i+1}\bar{u}_{i+1} + a_i \bar{u}_i) + h^2 d_i |u_i|^2\Big].
\end{aligned}$$

As a result of (2.6), by taking $a(x) = 0$ and $d(x) = d_c$, we have

$$(2.7) \qquad \sum_{i=1}^{N}(L_c^1 u)_i \bar{u}_i = \frac{1}{h^2}\sum_{i=0}^{N}\Big[|u_{i+1} - u_i|^2 + h^2 d_c |u_i|^2\Big],$$

which has real valued for any complex numbers $u_1, \ldots, u_N$.

For convenience, put

$$x := \sum_{k=0}^{N}|u_{k+1} - u_k|^2, \quad y_v := h^2 \sum_{k=0}^{N} d_k |u_k|^2, \quad y_c := h^2 \sum_{k=0}^{N} d_c |u_k|^2$$

and

$$z_v := \frac{h}{2}\sum_{k=0}^{N}(u_{k+1}-u_k)(a_{k+1}\bar{u}_{k+1}+a_k\bar{u}_k).$$

Note that for any complex numbers $u_1,\ldots,u_N$, the new variables x, y_v and y_c are positive real values, but z_v is complex value. First of all, let us estimate z_v in the following lemma.

Lemma 2.1. *The following estimates hold:*

$$\left|\text{Re } z_v\right| \le \frac{\|a'\|_\infty}{2d_c}y_c, \tag{2.8}$$

and

$$\left|z_v\right| \le \frac{\|a\|_\infty}{2\sqrt{d_c}}(x+y_c). \tag{2.9}$$

Proof. Since $u_0 = u_{N+1} = 0$ and $a_{k+1} - a_k = ha'(\xi_k)$ for some $\xi_k \in (x_k, x_{k+1})$, z_v can be written as

$$\begin{aligned} z_v &= \frac{h}{2}\sum_{k=0}^{N}(u_{k+1}-u_k)(a_{k+1}\bar{u}_{k+1}+a_k\bar{u}_k) \\ &= \frac{h}{2}\sum_{k=0}^{N}(a_k u_{k+1}\bar{u}_k - a_{k+1}u_k\bar{u}_{k+1}) \\ &= \frac{h}{2}\sum_{k=0}^{N}\Big\{a_k(u_{k+1}\bar{u}_k - u_k\bar{u}_{k+1}) - (a_{k+1}-a_k)u_k\bar{u}_{k+1}\Big\} \\ &= \frac{h}{2}\sum_{k=0}^{N}\Big\{a_k(u_{k+1}\bar{u}_k - u_k\bar{u}_{k+1}) - ha'(\xi_k)u_k\bar{u}_{k+1}\Big\}. \end{aligned} \tag{2.10}$$

Since $u_{k+1}\bar{u}_k - u_k\bar{u}_{k+1}$ is pure imaginary number,

$$\text{Re } z_v = -\frac{h^2}{2}\sum_{k=0}^{N}a'(\xi_k)\text{Re }(u_k\bar{u}_{k+1}). \tag{2.11}$$

Using the fact $|\text{Re } z| \le |z|$ and Cauchy-Schwarz inequality, we have

$$|\text{Re } z_v| \le \frac{h^2\|a'\|_\infty}{2}\sum_{k=0}^{N}|u_k||u_{k+1}| \le \frac{h^2\|a'\|_\infty}{2}\sum_{k=0}^{N}|u_k|^2 = \frac{\|a'\|_\infty}{2d_c}y_c.$$

For the estimation of $|z_v|$, Cauchy-Schwarz inequality applied to the first term of (2.10) and boundary conditions yield that

$$\begin{aligned} |z_v| &\le \frac{h}{2}\sum_{k=0}^{N}|u_{k+1}-u_k|(|a_{k+1}||u_{k+1}|+|a_k||u_k|) \\ &\le \frac{\|a\|_\infty h}{2}\sum_{k=0}^{N}|u_{k+1}-u_k||u_{k+1}| + \frac{\|a\|_\infty h}{2}\sum_{k=0}^{N}|u_{k+1}-u_k||u_k| \qquad (2.12) \\ &\le \|a\|_\infty\Big(\sum_{k=0}^{N}|u_{k+1}-u_k|^2\Big)^{1/2}\Big(h^2\sum_{k=0}^{N}|u_k|^2\Big)^{1/2}. \end{aligned}$$

Using the so called ε-inequality:

$$ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2,$$

we have

$$\begin{aligned} |z_v| &\le \frac{\|a\|_\infty}{2}\Big(\frac{1}{\sqrt{d_c}}\sum_{k=0}^{N}|u_{k+1}-u_k|^2 + \sqrt{d_c}h^2\sum_{k=0}^{N}|u_k|^2\Big) \\ &= \frac{\|a\|_\infty}{2\sqrt{d_c}}(x+y_c). \end{aligned}$$

These arguments complete the proof. □

The above lemma will be used to estimate the preconditioning results for L_v. The point of the following proposition is to estimate the lower and upper bounds of field values of preconditioned system $L_c^{-1}L_v$ in terms of the coefficient functions $a(x), d(x)$ and d_c as possible as we can. The choice of preconditioner using leading term was analyzed in detail in [18].

Proposition 2.1. *Assume that* (2.3) *holds. Let $\hat{L}_v^1$ and $\hat{L}_c^1$ be the matrix of the finite difference approximation corresponding to the operators L_v^1 and L_c^1 respectively. For any nonzero complex vector $U = (u_1, \dots, u_N)^t$, the following estimates hold:*

$$1 - \frac{\|a'\|_\infty}{2d_c} \le \mathrm{Re}\left(\frac{(\hat{L}_v^1 U, U)}{(\hat{L}_c^1 U, U)}\right) \le 1 + \frac{\|a'\|_\infty}{2d_c} + \frac{d_{\max}}{d_c} \qquad (2.13)$$

and

$$\left|\frac{(\hat{L}_v^1 U, U)}{(\hat{L}_c^1 U, U)}\right| \le 1 + \frac{d_{\max}}{d_c} + \frac{\|a\|_\infty}{2\sqrt{d_c}} < \infty, \qquad (2.14)$$

where $d_{\max}$ is a nonnegative constant such that

$$0 \le d_k - d_c \le d_{\max}, \quad \text{for all} \quad k = 0, 1, \dots, N.$$

Proof. Let

$$W := \frac{(\hat{L}_v^1 U, U)}{(\hat{L}_c^1 U, U)}.$$

Then

$$W = \frac{x + y_v + z_v}{x + y_c} = \frac{x + y_v + \text{Re } z_v}{x + y_c} + i\frac{\text{Im } z_v}{x + y_c}.$$

Note that

$$0 \le d_k - d_c \le d_{\max}, \quad \text{for } k = 0, 1, \ldots, N, \tag{2.15}$$

for some constant $d_{\max}$. From Lemma 2.1, we have

$$\left|\frac{\text{Re } z_v}{x + y_c}\right| \le \frac{\|a'\|_\infty}{2d_c}\frac{y_c}{x + y_c} \le \frac{\|a'\|_\infty}{2d_c}, \tag{2.16}$$

and from (2.15)

$$0 \le \frac{y_v - y_c}{x + y_c} \le \frac{d_{\max}}{d_c}. \tag{2.17}$$

Then the real part of W

$$\text{Re } (W) = \frac{x + y_v + \text{Re } z_v}{x + y_c} = 1 + \frac{y_v - y_c}{x + y_c} + \frac{\text{Re } z_v}{x + y_c},$$

can be estimated as

$$1 - \frac{\|a'\|_\infty}{2d_c} \le \text{Re } (W) \le 1 + \frac{\|a'\|_\infty}{2d_c} + \frac{d_{\max}}{d_c},$$

and the absolute value of W is

$$\begin{aligned} |W| \le \frac{x + y_v + |z_v|}{x + y_c} &\le 1 + \frac{y_v - y_c}{x + y_c} + \frac{\|a\|_\infty}{2\sqrt{d_c}}\frac{x + y_c}{x + y_c} \\ &\le 1 + \frac{d_{\max}}{d_c} + \frac{\|a\|_\infty}{2\sqrt{d_c}}. \end{aligned}$$

These arguments complete the proof. □

Now, let's consider the operators L_v and L_c in (1.1) and (1.8) with (1.9). Throughout this paper, we assume that $d(x, y)$ in (1.1) is a positive bounded function with $d_c = \min d(x, y) > 0$ and further assume that $a_x(x, y)$ and $b_y(x, y)$ are bounded functions on Ω, that is,

$$\begin{aligned} \|a_x\|_\infty &:= \max_{(x,y)\in\Omega} |a_x(x, y)| < \infty \quad \text{and} \\ \|b_y\|_\infty &:= \max_{(x,y)\in\Omega} |b_y(x, y)| < \infty. \end{aligned} \tag{2.18}$$

Let us rewrite (1.3) as

$$
\begin{aligned}
&(L_v u)_{k,j} \\
(2.19)\quad &= -\frac{1}{h_1^2}[(1+\frac{a_{k,j}h_1}{2})u_{k-1,j} - 2u_{k,j} + (1-\frac{a_{k,j}h_1}{2})u_{k+1,j}] + \frac{d_{k,j}}{2}u_{k,j} \\
&\quad -\frac{1}{h_2^2}[(1+\frac{b_{k,j}h_2}{2})u_{k,j-1} - 2u_{k,j} + (1-\frac{b_{k,j}h_2}{2})u_{k,j+1}] + \frac{d_{k,j}}{2}u_{k,j}.
\end{aligned}
$$

First we note that a similar one dimensional argument yields that

$$
\sum_{k,j=1}^{m_1,m_2} (L_v u)_{k,j}\bar{u}_{k,j} = I_v + II_v, \tag{2.20}
$$

where

$$
\begin{aligned}
I_v = \frac{1}{h_1^2}\sum_{k,j=0}^{m_1,m_2}\Big[&|u_{k+1,j}-u_{k,j}|^2 + \frac{h_1^2}{2}d_{k,j}|u_{k,j}|^2 \\
&+\frac{h_1}{2}(u_{k+1,j}-u_{k,j})(a_{k+1,j}\bar{u}_{k+1,j}+a_{k,j}\bar{u}_{k,j})\Big]
\end{aligned}
\tag{2.21}
$$

and

$$
\begin{aligned}
II_v = \frac{1}{h_2^2}\sum_{k,j=0}^{m_1,m_2}\Big[&|u_{k,j+1}-u_{k,j}|^2 + \frac{h_2^2}{2}d_{k,j}|u_{k,j}|^2 \\
&+\frac{h_2}{2}(u_{k,j+1}-u_{k,j})(b_{k,j+1}\bar{u}_{k,j+1}+b_{k,j}\bar{u}_{k,j})\Big].
\end{aligned}
\tag{2.22}
$$

For the preconditioning operator L_c, we have

$$
\sum_{k,j=1}^{m_1,m_2} (L_c u)_{k,j}\bar{u}_{k,j} = I_c + II_c, \tag{2.23}
$$

where

$$
I_c = \frac{1}{h_1^2}\sum_{k,j=0}^{m_1,m_2}\Big[|u_{k+1,j}-u_{k,j}|^2 + \frac{h_1^2}{2}d_c|u_{k,j}|^2\Big] \tag{2.24}
$$

and

$$
II_c = \frac{1}{h_2^2}\sum_{k,j=0}^{m_1,m_2}\Big[|u_{k,j+1}-u_{k,j}|^2 + \frac{h_2^2}{2}d_c|u_{k,j}|^2\Big]. \tag{2.25}
$$

For the convenience, put
(2.26)

$$
I_x := \sum_{k,j=0}^{m_1,m_2}|u_{k+1,j}-u_{k,j}|^2,\ I_{y,v} := \frac{h_1^2}{2}\sum_{k,j=0}^{m_1,m_2}d_{k,j}|u_{k,j}|^2,\ I_{y,c} := \frac{h_1^2}{2}\sum_{k,j=0}^{m_1,m_2}d_c|u_{k,j}|^2,
$$

and

(2.27)

$$II_x := \sum_{k,j=0}^{m_1,m_2} |u_{k,j+1} - u_{k,j}|^2, \; II_{y,v} := \frac{h_2^2}{2} \sum_{k,j=0}^{m_1,m_2} d_{k,j}|u_{k,j}|^2, \; II_{y,c} := \frac{h_2^2}{2} \sum_{k,j=0}^{m_1,m_2} d_c|u_{k,j}|^2,$$

and

$$I_{z,v} := \frac{h_1}{2} \sum_{k,j=0}^{m_1,m_2} (u_{k+1,j} - u_{k,j})(a_{k+1,j}\bar{u}_{k+1,j} + a_{k,j}\bar{u}_{k,j}), \tag{2.28}$$

and

$$II_{z,v} := \frac{h_2}{2} \sum_{k,j=0}^{m_1,m_2} (u_{k,j+1} - u_{k,j})(b_{k,j+1}\bar{u}_{k,j+1} + b_{k,j}\bar{u}_{k,j}) \tag{2.29}$$

Note that for any complex numbers $u_{1,1}, \ldots, u_{N,N}$, all these $I_x, II_x, I_{y,v}, II_{y,v}$ and $I_{y,c}, II_{y,c}$ are positive real values, but $I_{z,v}$ and $II_{z,v}$ only are complex values. First let us start with estimations for $I_{z,v}$ and $II_{z,v}$ in the following lemma.

Lemma 2.2. *The following estimates hold.*

$$\left|\text{Re } I_{z,v}\right| \le \frac{\|a_x\|_\infty}{d_c} I_{y,c}, \tag{2.30}$$

and

$$\left|I_{z,v}\right| \le \frac{\|a\|_\infty}{\sqrt{2d_c}}(I_x + I_{y,c}). \tag{2.31}$$

Also, it follows that

$$\left|\text{Re } II_{z,v}\right| = \frac{\|b_y\|_\infty}{d_c} II_{y,c}, \tag{2.32}$$

and

$$\left|II_{z,v}\right| \le \frac{\|b\|_\infty}{\sqrt{2d_c}}(II_x + II_{y,c}). \tag{2.33}$$

Proof. The similar arguments in the proof of Lemma 2.1 lead to the conclusions. □

Let $\hat{L}_v$ and $\hat{L}_c$ denote the matrices of the finite difference approximations corresponding to the operators L_v and L_c, respectively.

Proposition 2.2. *Assume that* (2.18) *and* (1.13) *hold. For any nonzero complex vector* $U = (u_{1,1}, \ldots, u_{m_1,m_2})^t$, *there are constants* c_1, c_2 *and* C *independent of* N *such that*

$$c_1 := 1 - \frac{\Gamma_1}{d_c} \le \text{Re}\left(\frac{(\hat{L}_vU, U)}{(\hat{L}_cU, U)}\right) \le c_2 := 1 + \frac{\Gamma_1}{d_c} + \frac{2d_{\max}}{d_c} \tag{2.34}$$

and

$$\left|\frac{(\hat{L}_v U, U)}{(\hat{L}_c U, U)}\right| \leq C := 1 + \frac{2d_{\max}}{d_c} + \frac{2M}{\sqrt{2d_c}} < \infty, \tag{2.35}$$

where $d_{\max}$ *is a positive constant such that*

$$0 \leq d_{k,j} - d_c \leq d_{\max} \quad \text{for } 1 \leq k \leq m_1, 1 \leq j \leq m_2, \tag{2.36}$$

and

$$\Gamma_1 := \max\{\|a_x\|_\infty, \|b_y\|_\infty\}, \quad \text{and} \quad M := \max\{\|a\|_\infty, \|b\|_\infty\}. \tag{2.37}$$

Proof. Let

$$W := \frac{(\hat{L}_v U, U)}{(\hat{L}_c U, U)},$$

which can be rewritten as

$$\begin{aligned} W &= \frac{\varphi^2(I_x + I_{y,v} + I_{z,v}) + II_x + II_{y,v} + II_{z,v}}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} \\ &= \frac{\varphi^2(I_x + I_{y,v}) + II_x + II_{y,v} + \text{Re}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} \\ &\quad + i\frac{\text{Im}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}}. \end{aligned} \tag{2.38}$$

Note that the real part of W can be written as

$$\begin{aligned} \text{Re}\,(W) &= \frac{\varphi^2(I_x + I_{y,v}) + II_x + II_{y,v} + \text{Re}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} \\ &= 1 + \frac{\varphi^2(I_{y,v} - I_{y,c}) + II_{y,v} - II_{y,c}}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} + \frac{\text{Re}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} \\ &\leq 1 + \frac{I_{y,v} - I_{y,c}}{I_{y,c}} + \frac{II_{y,v} - II_{y,c}}{II_{y,c}} + \frac{\text{Re}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}}. \end{aligned} \tag{2.39}$$

Using (2.30) and (2.32), we have

$$\left|\frac{\text{Re}\,(\varphi^2 I_{z,v} + II_{z,v})}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}}\right| \leq \frac{\Gamma_1}{d_c}\frac{\varphi^2 I_{y,c} + II_{y,c}}{\varphi^2(I_x + I_{y,c}) + II_x + II_{y,c}} \leq \frac{\Gamma_1}{d_c}. \tag{2.40}$$

Hence combining (2.39) with (2.40) yields that

$$1 - \frac{\Gamma_1}{d_c} \leq \text{Re}\,(W) \leq 1 + \frac{\Gamma_1}{d_c} + \frac{2d_{\max}}{d_c}.$$

Using (2.31) and (2.33), the estimates of absolute value of W is as follows:

$$\begin{aligned} |W| &\le \frac{\varphi^2(I_x+I_{y,v})+II_x+II_{y,v}+|\varphi^2 I_{z,v}+II_{z,v}|}{\varphi^2(I_x+I_{y,c})+II_x+II_{y,c}} \\ &\le 1+\frac{\varphi^2(I_{y,v}-I_{y,c})+II_{y,v}-II_{y,c}}{\varphi^2(I_x+I_{y,c})+II_x+II_{y,c}} \\ &\quad +\frac{\frac{\|a\|_\infty}{\sqrt{2d_c}}\varphi^2(I_x+I_{y,c})+\frac{\|b\|_\infty}{\sqrt{2d_c}}(II_x+II_{y,c})}{\varphi^2(I_x+I_{y,c})+II_x+II_{y,c}} \\ &\le 1+\frac{I_{y,v}-I_{y,c}}{I_{y,c}}+\frac{II_{y,v}-II_{y,c}}{II_{y,c}}+\frac{\|a\|_\infty}{\sqrt{2d_c}}+\frac{\|b\|_\infty}{\sqrt{2d_c}} \\ &\le 1+\frac{2d_{\max}}{d_c}+\frac{2M}{\sqrt{2d_c}}, \end{aligned}$$

where $M=\max\{\|a\|_\infty,\|b\|_\infty\}$. □

3. Semi-circulant preconditioning

In this section, we will provide the behaviors of eigenvalues of the semi-circulant preconditioned matrix $\hat{S}_c^{-1}\hat{L}_v$ with some numerical evidences. First of all, let us recall the results in [15] for the semi-circulant preconditioned matrix $\hat{S}_c^{-1}\hat{L}_c$ for (1.8). We note that the eigenvalues of (1.18) has the $(m_1m_2-2m_1)$-number of eigenvalues which are exactly 1. The remaining $2m_1$-number of the eigenvalues λ are also real-valued. These results can be described as follows (see Theorem 5.3 and 5.5 in [15]):

Theorem 3.1. *Let $\hat{d}_j=d_c-\frac{\tau_j}{h_2^2}$, where τ_j is the eigenvalues of T_1. Let*

$$M_1(j)=\sqrt{\hat{d}_j},\quad M_2(j)=-\sqrt{\hat{d}_j},$$

and

$$K_0(j)=-\frac{[M_1(j)e^{M_1(j)}+M_2(j)e^{M_2(j)}]}{(e^{M_1(j)}-1)(1-e^{M_2(j)})},\ K_1(j)=\frac{M_1(j)e^{M_1(j)}-M_2(j)e^{M_2(j)}}{e^{M_1(j)}-e^{M_2(j)}},$$

and

$$\bar{s}=\bar{s}(j)=-2\sqrt{\hat{d}_j}\Big(\frac{e^{M_1}+e^{M_2}-2}{e^{M_1}-e^{M_2}}\Big).$$

Let $\hat{\lambda}_i$, $i=1,2,\ldots,m_1m_2$, be the eigenvalues of (1.18) satisfying

$$\hat{\lambda}_1\le\hat{\lambda}_2\le\cdots\le\hat{\lambda}_{m_1m_2}.$$

(a) *The m_1-smallest eigenvalues $\hat{\lambda}_j$ satisfy*

$$.38<\frac{3-\sqrt{5}}{2}\le\hat{\lambda}_j\le 1,\qquad j=1,\ldots,m_1.$$

(b) *There are the $(m_1m_2-2m_1)$-number of eigenvalues $\hat{\lambda}_j$ which are exactly 1, i.e.,*

$$\hat{\lambda}_j = 1, \qquad j = m_1+1, \ldots, m_1m_2 - m_1.$$

(c) *For the other m_1-number of the eigenvalues $\hat{\lambda}_j$, we have, as $h_1, h_2 \to 0$,*

$$\hat{\lambda}_j \approx \frac{1}{2} + \frac{2[K_0(j)+K_1(j)]}{\bar{s}(j)} + \frac{1}{h_2}\frac{2}{|\bar{s}(j)|}, \quad j = m_1m_2 - m_1 + 1, \ldots, m_1m_2.$$

Now, let us recall that the field of values $\mathcal{W}(A)$ of a given $n \times n$ matrix A is defined as

$$\mathcal{W}(A) = \left\{ \frac{(AU,U)}{(U,U)} \mid \mathbf{0} \neq U \in \mathbb{C}^n \right\}.$$

Proposition 3.1. *Assume that* (2.18) *and* (1.13) *hold. Suppose that the reaction coefficient $d(x,y) = d_c$ in* (1.1) *is constant which is chosen as sufficiently large enough so that*

$$\frac{\Gamma_1}{d_c} \quad \text{and} \quad \frac{M}{\sqrt{d_c}} \tag{3.1}$$

are small enough, where Γ_1 and M are defined in (2.37)*. Then the field values $\frac{(\hat{L}_vU,U)}{(\hat{L}_cU,U)}$ are clustered at number* 1 *. Hence the eigenvalues of $\hat{L}_c^{-1}L_v$ are also clustered at number* 1.

Proof. This comes from Proposition 2.2. □

Theorem 3.2. *Assume that* (2.18) *and* (1.13) *hold. Suppose that $\{\hat{\lambda}_i\}_{i=1}^{m_1m_2}$ are the eigenvalues of $\hat{S}_c^{-1}\hat{L}_c$. Let $\{\lambda_i\}_{i=1}^{m_1m_2}$ be the eigenvalues of the semi-circulant preconditioned matrix $\hat{S}_c^{-1}\hat{L}_v$. Then we have*

$$\mathrm{Re}(w_1)\hat{\lambda}_{\min} \leq \mathrm{Re}(\lambda_i) \leq \mathrm{Re}(w_2)\hat{\lambda}_{\max}$$

and

$$|\lambda_i| \leq |w_3|\hat{\lambda}_{\max},$$

as $h_1, h_2 \to 0$, where w_i's are constants in $\mathcal{W}(\hat{L}_c^{-1}\hat{L}_v)$ which satisfy, due to Proposition 2.2,

$$\mathrm{Re}(w_i) \geq c_1, \quad \text{and} \quad |w_i| \leq C, \qquad i = 1,2,3,$$

where c_1 and C are independent of h_1 and h_2, and $\hat{\lambda}_{\min}$ and $\hat{\lambda}_{\max}$ are minimum and maximum of $\{\hat{\lambda}_i\}$, respectively.

Proof. First note that $\hat{L}_c^{-1}\hat{S}_c$ is symmetric and positive definite. Note that

$$\frac{(\hat{L}_vU,U)}{(\hat{S}_cU,U)} = \frac{(\hat{L}_cU,U)}{(\hat{S}_cU,U)}\frac{(\hat{L}_vU,U)}{(\hat{L}_cU,U)}. \tag{3.2}$$

Now the conclusions come from combining Theorem 3.1 and Proposition 2.2. □

Let us point out some special case of the above theorem. Assume that $a(x,y) = f(y)$, $b(x,y) = g(x)$ and $d(x,y) = d_c$, so that $a_x = b_y = 0$ and $d(x,y)$ is constant. Then we have $\Gamma_1 = 0$ and $d_{\max} = 0$. Then, as $h_1, h_2 \to 0$ we have from Proposition 2.2, for all $w \in \mathcal{W}(\hat{L}_c^{-1}\hat{L}_v)$,

$$\text{Re}\,(w) \;=\; 1 \quad \text{and} \quad |w| \;\le\; 1 + \frac{2M}{\sqrt{2d_c}}.$$

Hence if we choose d_c so that

$$\lim \frac{M}{d_c} \to 0,$$

then

$$\lim_{\frac{M}{d_c} \to 0} w = 1.$$

These situations yield that all field values of $\hat{L}_c^{-1}\hat{L}_v$ cluster around 1, so that all its eigenvalues cluster around 1. Hence one may prove that $\lambda_i \sim \hat{\lambda}_i$. Summarizing the above notice, we will put as a theorem here

Theorem 3.3. *Under assumption of Proposition 3.1, the eigenvalues* $\{\hat{\lambda}_i\}_{i=1}^{m_1 m_2}$ *of* $\hat{S}_c^{-1}\hat{L}_c$ *are clustered at number* 1 *as* h_1 *and* h_2 *approach to* 0.

Proof. This result comes from combining Proposition 3.1 and Theorem 3.2. ☐

Example 1. Consider

$$L_v u = -\Delta u + y^2 u_x + x^3 u_y + (d_c + e^{x+y})u$$

with

$$L_c u = -\Delta u + (d_c + 1)u.$$

We compute the eigenvalues $\hat{S}_c^{-1}\hat{L}_v$ for the cases $d_c = 10$ and 10000 with mesh size $h_1 = h_2 = \frac{1}{40}$, which shows that its eigenvalues are clustering at a number 1 in Figure 3.1 as a constant d_c is chosen large enough. These phenomena support Theorem 3.3.

Example 2. Consider

$$L_v u = -\Delta u + e^{xy} u_x + e^{(x+y)} u_y + (d_c + x)(d_c + y)u$$

with

$$L_c u = -\Delta u + d_c^2 u.$$

We plot the eigenvalues of in Figure 3.2 for $h_1 = h_2 = \frac{1}{40}$. From these computations, we see that the eigenvalues of $\hat{S}_c^{-1}\hat{L}_v$ become clustering at a number 1 as d_c increases. These phenomena support also Theorem 3.3.

Example 3. Consider the boundary value problem

$$L_v u = -\Delta u + \cos(8xy)u_x + \sin(x+y)u_y + d_c u = f(x,y) \quad \text{in } \Omega$$

and

$$L_c u = -\Delta u + d_c u$$

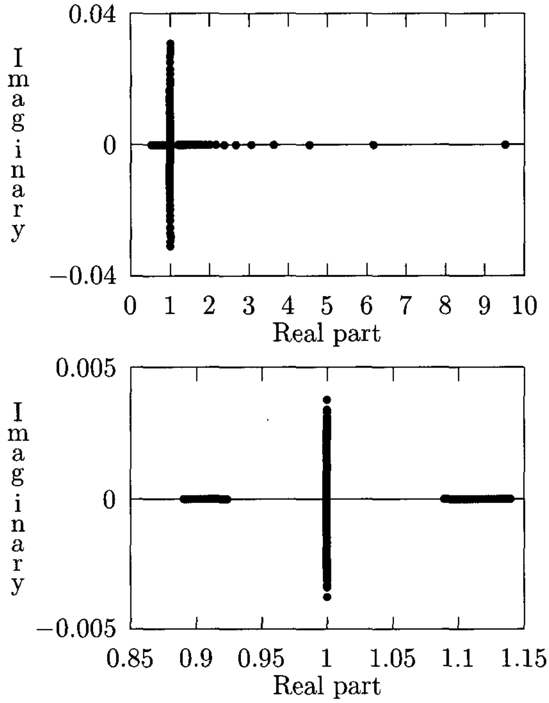

FIGURE 3.1. Distribution of the eigenvalues of $\hat{S}_c^{-1}\hat{L}_v$ for $d_c = 10$(Left) and $d_c = 10000$(Right)

with homogeneous Dirichlet boundary conditions, where the function $f(x, y)$ is chosen as the exact solution to be $u(x, y) = \sin(2\pi x)\sin(2\pi y)$. We compare the semi-circulant preconditioning system with no-preconditioning system in the view of the iteration number of the iterative method and maximum error of the exact solution in Table 3.1 and 3.2. We use the Bi-CGSTAB method.

We plot the eigenvalues of in Figure 3.3 for $h_1 = h_2 = \frac{1}{40}$. From these computations, we see that the eigenvalues of $\hat{S}_c^{-1}\hat{L}_v$ become clustering at a number 1 as d_c increases as we have shown in Example 1 and 2.

4. Concluding remark

In the present paper, we analyzed the distribution of eigenvalues of a semi-circulant preconditioned matrix based on the leading term and reaction term of the target elliptic operators (1.1) in which the coefficient of reaction term is chosen as the minimum of the given reaction coefficient function. Because of the equivalence of two finite difference matrices in the sense of l^2-inner product, it may be enough to set up a semi-circulant preconditioner having constant

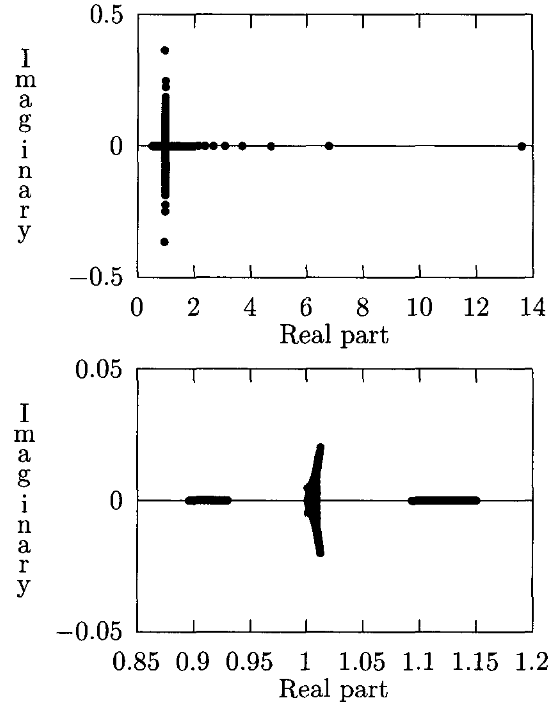

FIGURE 3.2. Distribution of the eigenvalues of $\hat{S}_c^{-1}\hat{L}_v$ for $d_c = 1$(Left) and $d_c = 100$(Right)

	Preconditioning			nopreconditioning		
N	No. of Iter	Maximum Error	CPU time	No. of Iter	Maximum Error	CPU time
4	7	0.242661	1	8	0.242661	1
8	10	0.055644	1	21	0.055644	1
12	12	0.024340	1	36	0.024340	2
16	13	0.013615	1	48	0.013615	2
24	15	0.006027	4	73	0.006027	15
32	17	0.003386	13	96	0.003386	66
40	18	0.002165	33	126	0.002165	208

TABLE 3.1. Number of iteration and maximum error for $d_c = 1$

coefficients of the appropriate second-order elliptic operator. For several computational results, we certify that the eigenvalues of the preconditioned matrix are clustering at a number 1 and the preconditioned system are applicable to

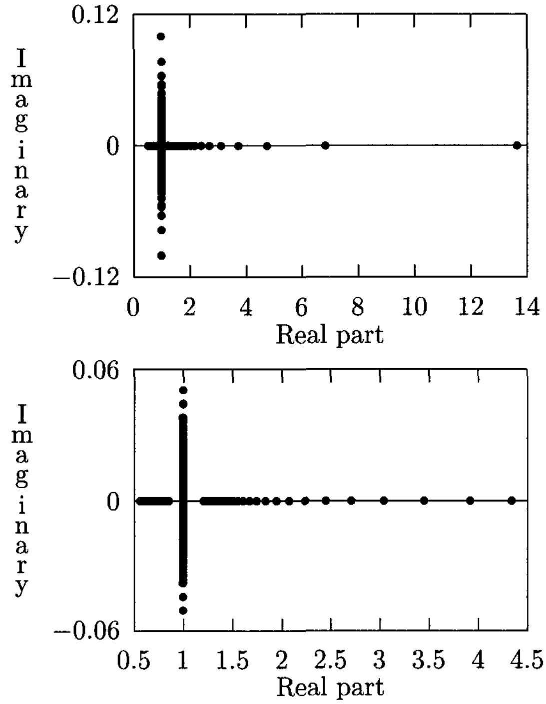

FIGURE 3.3. Distribution of the eigenvalues of $\hat{S}_c^{-1}\hat{L}_v$ for $d_c = 1$(Left) and $d_c = 100$(Right)

	Preconditioning			nopreconditioning		
N	No. of Iter	Maximum Error	CPU time	No. of Iter	Maximum Error	CPU time
4	4	0.093259	1	7	0.093259	1
8	6	0.023517	1	14	0.023517	1
12	7	0.010464	1	22	0.010464	2
16	9	0.005888	2	30	0.005888	2
24	11	0.002618	3	45	0.006027	10
32	12	0.001473	9	69	0.001473	48
40	13	0.000943	24	80	0.000943	133

TABLE 3.2. Number of iteration and maximum error for $d_c = 100$

the Bi-CGSTAB iterative method. Further, we hope to investigate the easy method to find the inverse of the semi-circulant preconditioning matrix.

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Hoi Sub Kim
Department of Mathematics and Information
Kyungwon University
Songnam 461-701, Korea
E-mail address: hskimm@kyungwon.ac.kr

Sang Dong Kim
Department of Mathematics
Kyungpook National University
Daegu 702-701, Korea
E-mail address: skim@knu.ac.kr

Yong Hun Lee
Division of Mathematics and Statistical Informatics
(Institute of Applied Statistics)
Chonbuk National University
Chonju 561-756, Korea
E-mail address: lyh229@chonbuk.ac.kr