퍼지 및 지능시스템학회 논문지 2007, Vol. 17, No. 2, pp. 149-153

구간치 퍼지집합상에서 쇼케이적분에 의해 정의된 엔트로피에 관한 연구

A note on entropy defined by Choquet integral on interval-valued fuzzy sets

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요 약

본 논문에서 우리는 Wang와 Li(1998)와 Turksen(1986)에 의해 소개된 구간치 퍼지집합을 생각하고 구간치 퍼지집합상에서 쇼케이적분에 의해 정의된 엔트로피를 조사한다. 더욱이, 이러한 엔트로피와 관련된 성질들을 토의하고 간단한 예들을 알아본다. 이 공식은 구간치 퍼지집합상의 결정이론 및 정보이론과 같은 응용 영역에서 중요한 역할을 한다.

Abstract

In this paper, we consider interval-valued fuzzy sets which were suggested by Wang and Li(1998) and Turksen(1986) and investigate entropy defined by Choquet integral on interval-valued fuzzy sets. Furthermore, we discuss some properties of them and give some examples related this entropy. This tool has drawn much attention due to numerous applications areas, such as decision making and information theory on interval-valued fuzzy sets.

Key words : interval-valued fuzzy sets, entropy, Choquet integrals.

1. 서 론

Sugeno et al. [5,6] have studied some characterizations of Choquet integrals which is a generalized concept of Lebesgue integral, because two definitions of Choquet integral and Lebesgue integral are equal if a fuzzy measure is a classical measure. And also Choquet integral is often used in information nonlinear aggregation tool(see[2,5,6]).

Many researchers, such as Grabisch and Turksen[7], Burillo and Bustince[1], Liu Xuechang[8], and Jang and Kim[4] gave the axiom definitions of distance measure, similarity measure, and entropy on interval-valued fuzzy sets and have been applied to the fields of approximate inference, information theory, and control theory.

We consider interval-valued fuzzy sets which were suggested by Turksen [7]. Based on this, Burillo and Bustine[1], and Wang and Li[9] introduced entropy on interval-valued fuzzy sets. We note that entropy in[1,3,7,10] was defined by Lebesgue integral with respect to a classical measure.

In this paper, by using Choquet integral with respect to a fuzzy measure instead of Lebesgue integral with respect to a classical measure, we define entropy on interval-valued fuzzy sets. In section 2, we list arithmetic operations and some basic characterizations of interval-valued fuzzy sets and interval-valued Choquet integrals. In section 3, we introduce entropy defined by Choquet integral on interval-valued fuzzy sets and discuss their some characterizations.

접수일자 : 2006년 10월 21일
완료일자 : 2007년 1월 13일

2. Preliminaries and Definitions

Throughout this paper, I will denote the unit interval $[0,1]$,

$$[I]=\{\bar{a}=[a^-,a^+]\mid a^-,a^+\in I \text{ and } a^-\le a^+\}.$$

Then, according to Zadeh's extension principle[14], we can popularize these operations such as maximum ($\wedge$), minimum ($\vee$) and complement (c) to $[I]$ defined by

$$\bar{a}\vee\bar{b}=[a^-\vee b^-,a^+\vee b^+],$$
$$\bar{a}\wedge\bar{b}=[a^-\wedge b^-,a^+\wedge b^+], \text{ and}$$
$$\bar{a}^c=[a^{+c},a^{-c}]=[1-a^+,1-a^-],$$

thus $([I],\vee,\wedge,c)$ is a complete lattice with a minimal element $\bar{0}=[0,0]$ and a maximal element $\bar{1}=[1,1]$.

Definition 2.1. Let $\bar{a}=[a^-,a^+]$, $\bar{b}=[b^-,b^+]\in[I]$. Then we define

$\bar{a}=\bar{b}$ if and only if $a^-=b^-$ and $a^+=b^+$,
$\bar{a}\le\bar{b}$ if and only if $a^-\le b^-$ and $a^+\le b^+$,
$\bar{a}<\bar{b}$ if and only if $[a^-,a^+]\le[b^-,b^+]$
but $[a^-,a^+]\ne[b^-,b^+]$.

Let X be the discourse set, $IF(X)$ stands for the set of all interval-valued fuzzy sets in X, $F(X)$ and $\wp(X)$ stand for the set of all fuzzy sets and crisp sets in X, respectively.

Definition 2.2. For every $A\in IF(X)$ and $x\in X$, $A(x)=[A^-(x),A^+(x)]$ is called the degree of membership of an element x to A, then fuzzy sets $A^-:X\to I$ and $A^+:X\to I$ are called a lower fuzzy set of A and an upper fuzzy set of A, respectively.

For simplicity, we denote $A=[A^-,A^+]$. Then, three operations such that such as $\vee,\wedge,c$ can be introduced into $IF(X)$ as follows: for every $A,B\in IF(X)$ and $x\in X$,

$(A\vee B)(x)=A(x)\vee B(x)$,
$(A\wedge B)(x)=A(x)\wedge B(x)$,
$(A^c)(x)=(A(x))^c=[A^{+c}(x),A^{-c}(x)]$
$=[1-A^+(x),1-A^-(x)]$.

Then $(IF(X),\vee,\wedge,c)$ is a complete lattice with minimal element $\bar{0}(x)=[0,0]$ for all $x\in X$ and maximal element $\bar{1}=[1,1]$ for all $x\in X$. If $A,B\in IF(X)$, we define the following operations (see [7,10,11]):

$A\le B$ if and only if for all $x\in X$,
$A^-(x)\le B^-(x)$ and $A^-(x)\le B^-(x)$,
$A=B$ if and only if for all $x\in X$,
$A^-(x)=B^-(x)$ and $A^-(x)=B^-(x)$,
$A<B$ if and only if $A\le B$ and $A\ne B$.

Now, we introduce Choquet integrals and their basic properties which are used in the next section(see[5,6]).

Definition 2.3. (1) A fuzzy measure μ on a measurable space $(X,\mathfrak{I})$ is a nonnegative mapping $\mu:\mathfrak{I}\to[0,1]$ satisfying

(i) $\mu(\emptyset)=0$, $\mu(X)=1$
(ii) $\mu(E_1)\le\mu(E_2)$,
whenever $E_1,E_2\in\mathfrak{I}$, $E_1\subset E_2$.

(2) A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{E_n\}$ of measurable sets, we have $\mu(\bigcup_{n=1}^{\infty}E_n)=\lim_{n\to\infty}\mu(E_n)$.

(3) A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1)<\infty$, we have $\mu(\bigcap_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$.

(4) If μ is both lower semi-continuous and upper semi-continuous, it is said to be continuous.

We note that "$x\in X$ $\mu-a.e.$" stands for "$x\in X$ μ-almost everywhere". The property $P(x)$ holds for $x\in X$ $\mu-a.e.$ means that there is a measurable set A such that $\mu(A)=0$ and the property $P(x)$ holds for all $x\in A^c$, where A^c is the complement of A.

Definition 2.4. The Choquet integral of a measurable profile $f:X\to I$ with respect to a fuzzy measure μ is defined by

$$(C)\int f d\mu=\int_0^1\mu_f(r)\,dr$$

where $\mu_f(r)=\mu(\{x\in X\mid f(x)>r\})$ and the integral on the right-hand side is an ordinary one.

(2) If X is a finite set, that is, $X=\{x_1,\cdots,x_n\}$, then the Choquet integral of f on X is defined by

$$(C)\int f d\mu=\sum_{i=1}^{n}f(x_{(i)})[\mu(A_{(i)})-\mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on $\{1,2,,\cdots,n\}$ such that

$f(x_{(1)})\le\cdots\le f(x_{(n)})$.
Also, $A_{(i)}=\{(i),\cdots,(n)\}$ and $A_{(n+1)}=\emptyset$.

Definition 2.5 Let f,g be measurable nonnegative functions. We say that f and g are comonotonic, in symbol $f\sim g$ if and only if

$$f(x)<f(x')\Rightarrow g(x)\le g(x')\text{ for all }x,x'\in X.$$

Theorem 2.6 Let f,g,h be measurable functions. Then we have

(1) $f\sim f$,
(2) $f\sim g\ \Rightarrow\ g\sim f$,
(3) $f\sim a$ for all $a\in R^+$,
(4) $f\sim g$ and $f\sim h\Rightarrow f\sim(g+h)$.

Theorem 2.7 Let f,g be nonnegative measurable functions.

(1) If $f\le g$, then $(C)\int fd\mu\le(C)\int gd\mu$.
(2) If $f\sim g$ and $a,b\in R^+$, then
$(C)\int(af+bg)d\mu=a(C)\int fd\mu+b(C)\int gd\mu$.
(3) If $f\vee g$, then
$(C)\int f\vee g\,d\mu\ge(C)\int fd\mu\vee(C)\int gd\mu$.
(4) If $f\wedge g$, then
$(C)\int f\wedge g\,d\mu\le(C)\int fd\mu\wedge(C)\int gd\mu$.

3. Entropy defined by Choquet integral on interval-valued fuzzy sets

In this section, we introduce entropy defined by Choquet integral on interval-valued fuzzy sets. We recall that for $A, B \in IF(X)$, $A \equiv B$ if and only if

$\mu(\{x \in X | A(x) \neq B(x)\}) = 0$,

that is, A is equal to B $\mu - a.e.$ on X.

Definition 3.1. A real function $E: IF(X) \to I$ is called an entropy on $IF(X)$ if E satisfies the following properties :

(E_1) $E(A) = 0$ if A is a crisp set;

(E_2) $E(A) = 1$ if and only if

$A^-(x) + A^+(x) = 1$;

(E_3) $E(A) \leq E(B)$ if A is fuzzy less than B, that is, $A^-(x) \leq B^-(x)$ and $A^+(x) \leq B^+(x)$ for $B^-(x) + B^+(x) \leq 1$

or $A^-(x) \geq B^-(x)$ and $A^+(x) \geq B^+(x)$ for $B^-(x) + B^+(x) \geq 1$;

(E_4) $E(A) = E(A^c)$.

We define a real function $E_c: IF(X) \to I$ by

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^{+c}(x))d\mu(x)$$

where $A = [A^-, A^+] \in IF(X)$ and d_H is the Hausdorff metric between $A(x)$ and $B(x)$. Since $A(x) = [A^-(x), A^+(x)]$ and $B(x) = [B^-(x), B^+(x)]$, it is easily to see that

$$d_H(A(x), B(x)) = max\{|A^-(x) - B^-(x)|, |A^+(x) - B^+(x)|\}.$$

Theorem 3.2 A real function $E: IF(X) \to I$ is called an entropy on $IF(X)$, we say that E_c is a Choquet entropy on $IF(X)$.

Proof. (E_1) We note that if A is a csisp set, that is,

$A^+(x) = A^-(x) = 1$ for all $x \in A$

and

$A^+(x) = A^-(x) = 0$ for all $x \not\in A$,

then

$d_H(A^-(x), A^+(c)^c) = 1$ for all $x \in X$.

Thus

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)\, d\mu(x)$$

$$= 1 - (C)\int 1\, d\mu(x)$$

$$= 1 - \mu(X) = 0.$$

(E_2) If $A^-(x) + A^+(x) = 1$, then

$d_H(A^-(x), A^+(x)^c)$

$= |A^-(x) + A^+(x) - 1|$

$= 0.$

Thus

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)\, d\mu(x)$$

$$= 1 - 1 = 0.$$

If $E_c(A) = 1$, then

$$1 - (C)\int d_H(A^-(x), A^+(x)^c)d\mu(x)$$

$$= E_c(A) = 1.$$

Thus $(C)\int d_H(A^-(x), A^+(x)^c)d\mu(x) = 0$ and hence $A^- = (A^+)^c$. That is,

$A^+(x) = A^-(x) = 1$

for almost all $x \in X$. It means

$A^- + (A^+)^c \equiv 0$.

(E_3) If $B^-(x) + B^+(x) \leq 1$ and $A^-(x) \leq B^-(x)$ and $A^+(x) \leq B^+(x)$, then

$A^-(x) + A^+(x) \leq B^-(x) + B^+(x) \leq 1$.

Thus,

$$\begin{aligned} d_H(B^-(x), B^+(x)^c) &= |B^-(x) + B^+(x) - 1| \\ &= 1 - (B^-(x) + B^+(x)) \\ &\leq 1 - (A^-(x) + A^+(x)) \\ &= d_H(A^-(x), A^+(x)^c) \end{aligned}$$

and hence

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)d\mu(x)$$

$$\leq 1 - (C)\int d_H(B^-(x), B^+(x)^c)d\mu(x)$$

$$= E_c(B).$$

We note that entropy on an interval-valued fuzzy set is important topic in fuzzy set theory and describes the fuzziness degree of an interval-valued fuzzy set. Zeng and Li [10] gave the following formulas to calculate entropy of interval-valued fuzzy set:

(1) If $X = \{x_1, \cdots, x_n\}$ is a finite set and $A \in IF(X)$, then

$$E_1(A) = 1 - \sum_{i=1}^{n} |A^-(x_i) + A^+(x_i) - 1|.$$

(2) If X is a set and $A \in IF(X)$, then

$$E_1(A) = 1 - \frac{1}{b-a}\int x|A^-(x) + A^+(x) - 1|\, dx.$$

Theorem 3.3 If m is a counting measure on a finite set $X = \{x_1, \cdots, x_n\}$ and if we put $\mu = \frac{1}{n}m$, then we have

$E_c(A) = E_1(A)$ for all $A \in IF(X)$.

Proof. We let $\{(i) | i = 1, 2, \cdots, n\}$ such that

$|A^-(x_{(i)}) + A^+(x_{(i)}) - 1|$

$\leq |A^-(x_{(i+1)}) + A^+(x_{(i+1)}) - 1|$

for all $i = 1, 2, \cdots, n$. Since $A_{(i)} = \{(i), \cdots, (n)\}$ and

$$\mu(A_{(i)}) - \mu(A_{(i-1)}) = \frac{i}{n} - \frac{i-1}{n} = 1$$

for all $i = 1, 2, \cdots, n$,

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)d\mu(x)$$

$$= 1 - (C)\int | A^-(x_{(i)}) + A^+(x)^c)d\mu(x)$$

$$= 1 - \sum_{i=1}^{n} | A^-(x_{(i)}) + A^+(x_{(i)}) - 1 | \cdot \{\mu(A_{(i)}) - \mu(A_{(i-1)})\}$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} | A^-(x_{(i)}) + A^+(x_{(i)}) - 1 |$$

$$= E_1(A).$$

Theorem 3.4 Let $0 \le a < b$. If m is Lebesgue measure on a set $X = [a, b]$ and if we put $\mu = \frac{1}{b-a}m$, then we have

$E_c(A) = E_2(A)$ for all $A \in IF(X)$.

Proof. We note that μ is Lebesgue measure. Then the Choquet integral is equal to the Lebesgue integral. Thus,

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)d\mu(x)$$

$$= 1 - \frac{1}{b-a}\int_a^b d_H(A^-(x), A^+(x)^c)dm$$

$$= E_2(X).$$

Example 3.5 Let $X = \{a, b\}$ and $\Omega = \wp(X)$ be the class of all subsets of X. If μ is defined by

$$\mu(A) = \begin{cases} 1 & \text{if } b \in A \\ 0.5 & \text{if } A = \{a\} \\ 0 & else \end{cases}$$

Then clearly μ is a fuzzy measure. If we give the following interval-valued fuzzy sets;

$$A(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x = b, \end{cases}$$

$$B(x) = \begin{cases} [0.7, 1] & \text{if } x = a \\ [0.2, 0.5] & \text{if } x = b, \end{cases}$$

$$C(x) = \begin{cases} [0.4, 0.6] & \text{if } x = a \\ [0.1, 0.3] & \text{if } x = b, \end{cases}$$

$$D(x) = \begin{cases} [0.2, 0.4] & \text{if } x = a \\ [0.1, 0.2] & \text{if } x = b, \end{cases}$$

then it is easily to see that A is a crisp set, B is an interval-valued fuzzy set, and D is less fuzzy than C. In fact, we can investigate the properties of the entropy of these fuzzy sets as followings;

Since $A^-(x) = A^+(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x = b, \end{cases}$

$$d_H(A^-(x), A^+(x)^c) = | A^-(x) + A^+(x) - 1 | = \begin{cases} 1 & \text{if } x = a \\ 1 & \text{if } x = b, \end{cases}$$

we can calculate the entropy of A

$$E_c(A) = 1 - (C)\int d_H(A^-(x), A^+(x)^c)\,d\mu$$

$$= 1 - \{1 \times 1\} = 0.$$

Now, we calculate the entropy of three interval-valued fuzzy sets B, C, D .

$$B^-(x) = \begin{cases} 0.5 & \text{if } x = a \\ 0.1 & \text{if } x = b, \end{cases}$$

$$B^+(x) = \begin{cases} 0.9 & \text{if } x = a \\ 0.4 & \text{if } x = b, \end{cases}$$

and hence

$$d_H(B^-(x), B^+(x)^c) = \begin{cases} 0.4 & \text{if } x = a \\ 0.5 & \text{if } x = b. \end{cases}$$

So, the entropy of B is

$$E_c(B) = 1 - (C)\int d_H(B^-(x), B^+(x)^c)\,d\mu$$

$$= 1 - \{0.4 \times 1 + 0.5 \times 0\} = 0.6.$$

Similarly, we obtain

$$E_c(C) = 1 - (0.0 \times 1 + 0.6 \times 0.5) = 0.7$$

and

$$E_c(D) = 1 - (0.4 \times 1 + 0.7 \times 0) = 0.6.$$

Since $D^-(x) \le C^-(x)$, $D^+(x) \le C^-(x)$, and $C^-(x) + C^+(x) \le 1$, we see that the property E_3 of the entropy satisfies. Thus we can see that $E_c(D) \le E_c(C)$. That is, D is less fuzzy than C.

4. References

[1] P.Burillo and H. Bustince, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy sets and systems Vol. 78, pp.305-316, 1996.

[2] G. Choquet, Theory of capacities, Annales de Institut Fourier Vol.5 pp. 131-295, 1953.

[3] Jin-Lum Fan, Yuan-Liang Ma and Wei-Xin Xie, On some properties of distance measures, Fuzzy Sets and Systems Vol.117,pp.355-361, 2001.

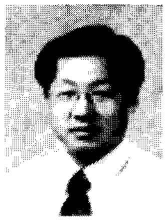
[4] Lee-Chae Jang and Won Joo Kim, Some properties of Choquet distance measures for interval-valued fuzzy numbers, J. of Fuzzy Logic and Intelligent Systems, Vol.15 No.7, pp.789-793, 2005.,

[5] T. Murofushi and M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems Vol. 29 pp. 201-227, 1989.

[6] T. Murofushi and M. Sugeno, A theory of Fuzzy measures: representations, the Choquet integral,

and null sets, J. Math. Anal. and Appl. Vol. 159 pp. 532-549, 1991.
[7] B. Turksen, Interval-valued fuzzy sets on normal forms, Fuzzy Sets and Systems Vol.20, pp.191-210, 1986.
[8] Liu Xuechang, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets and Systems Vol.52, pp.201-227, 1992.
[9] G. Wang and X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems Vol. 98, pp. 331-335, 1998.
[10] W. Zeng and H. Li, Relationship between similarity measure and entropy of interval-valued fuzzy sets, Fuzzy Sets and Systems Vol. 157, pp.1477-1484, 2006.

저 자 소 개



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