The Role of Negative Binomial Sampling In Determining the Distribution of Minimum Chi-Square

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ABSTRACT

The distributions of the minimum correlated F-variable arises in many applied statistical problems including simultaneous analysis of variance (SANOVA), equality of variance, selection and ranking populations, and reliability analysis. In this paper, negative binomial sampling technique is employed to derive the distributions of the minimum of chi-square variables and hence the distributions of the minimum correlated F-variables. The work presented in this paper is divided in two parts. The first part is devoted to develop some combinatorial identities arised from the negative binomial sampling. These identities are constructed and justified to serve important purpose, when we deal with these distributions or their characteristics. Other important results including cumulants and moments of these distributions are also given in somewhat simple forms. Second, the distributions of minimum, chi-square variable and hence the distribution of the minimum correlated F-variables are then derived within the negative binomial sampling framework. Although, multinomial theory applied to order statistics and standard transformation techniques can be used to derive these distributions, the negative binomial sampling approach provides more information regarding the nature of the relationship between the sampling vehicle and the probability distributions of these functions of chi-square variables. We also provide an algorithm to compute the percentage points of the distributions. The computation methods we adopted are exact and no interpolations are involved.

Key words: *Incomplete gamma function, Incomplete beta function, Poisson sum, Binomial sum, Chi-square distribution, Negative binomial distribution*

1. INTRODUCTION

Let X, Y be independent and identically distributed $\chi^2_{(2m)}$ random variables. Denote the minimum of the chi-square random variables by $V = \min(X, Y)$. The distribution of V or some function of it as $W = (n/m)V/X_0$, where X_0 is a random variable distributed as $\chi^2_{(2n)}$ independent of both X and Y, is known as studentized minimum chi-square variable or the minimum of two correlated F-variable. These distributions arised in applied statistical problems when parent distributions are gamma, exponential, pareto, weibull, and rayleigh.

For example in reliability analysis of series systems, statements such as $P(V > C_1)$ for some known constant C_1 is encountered. Testing the equality of the scale parameters of three exponential populations

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Manuscript received Dec.19, 2006 ; accepted Feb. 27, 2007

The authors are thankful to professor E. Elneweihi, the University of Southern Illinois, for the fruitful discussion, during his visit to Kuwait University, which led to fundamental improvement of the results at various places.

$f(t,\theta_i)=\theta_i^{-1}e^{-t/\theta_i}\quad ,\theta_i, i=1,2,3,\ t>0$ gives rise to statements similar to $P(W>C_2)$ for some known constant C_2. Mukhopadhyay and Hamdy (1984a) proposed a two-stage sequential procedure to estimate the difference of location parameters of two exponential populations, their stopping rule requires the values of C_2^{-1} to implement in real life situations. Also, Mukhopadhyay and Hamdy (1984b) proposed a two stage selection procedure to select the best exponential location where C_2^{-1} must be determined in advance to carry out the selection procedures.

The fundamental works of Gupta and Sobel (1962a), (1962b), Hartely (1983), Finney (1941), Nair (1984), Ramachandran (1958), Krishna and Armitag (1964), and Hamdy and Son (1993) focused on the distributions of forms of chi-square. In most of the work done in these papers, the standard transformation techniques, or the inverse probability, were central to the analyses. However, in the present work, we intend to focus on the role of negative binomial sampling techniques to determine the distribution of functions of chi-square variables. In specific, section 2 presents the truncated negative binomial probability mass functions and some related combinatorial identities, which are essential in simplifying the results when dealing with distributions of functions of chi-square random variables. Similar approach to the distribution of maximum correlated F-distribution, is given in Son and Hamdy (2006). In subsequent sections, cumulants moments regarding the sample size required to stop the negative binomial sampling procedure are also given. In sections 3 and 4, we introduce the role of negative binomial sampling technique to determine the probability density functions of V, and their relations with the sampling procedures. Section 5 presents computational algorithms to compute critical values of the distribution of the minimum of two chi-square variables and the minimum of two correlated F-distributions.

2. IDENTITIES GENERATED BY NEGATIVE BINOMIAL SAMPLING

Consider a random experiment where a fair coin is flipped a sequence of independent trials until the head appears m times, where m is a specific positive integer predetermined beforehand. We also assume that the probability of observing a head equals to the probability of observing a tail equals ½, at any trial. Upon observing the m^{th} head, the experiment is then terminated. Let J denotes the number of observed tails preceding the realization of the m^{th} head. It is known that the random variable J follows the negative binomial distribution with the following probability mass function,

$$P(J=j)=\binom{m+j-1}{j}(1/2)^j\,(1/2)^m,\quad j=0,1,2,\ldots \tag{2.1}$$

The following lemma is essential to discuss results regarding the distributions of functions of chi-square and functions of correlated F–variable in subsequent developments.

Lemma.

Consider an experiment leads to the generation of the negative binomial random variable J as defined above then for any positive integer m, the following identities hold

$$(i)\ \sum_{j=0}^{\infty}\binom{m+j-1}{j}(1/2)^j=2^m$$

$$(ii)\ \sum_{j=0}^{m-1}\binom{m+j-1}{j}(1/2)^j=2^{m-1}$$

$$(iii)\ \sum_{j=m}^{\infty}\binom{m+j-1}{j}(1/2)^j=2^{m-1}$$

Proof of the lemma.

To prove (i) of the lemma, we recall the negative binomial probability mass function given in (2.1) and take the summation over $j=0,1,\ldots,\infty$. and equate by 1 then (i) is immediate. It cab be also obtained from the well known expansion

$$\sum_{i=1}^{\infty}\binom{t+i-1}{i}a^i=(1-a)^{-t},\ |a|<1\cdot$$

To prove (ii) and (iii), it is sufficient to prove that $P(J\le m-1)=P(J\ge m)=1/2$. Let J be a random variable represents the number of tails we continue to obtain till we observe the m^{th} head and define the random variable $Y=J+m$ as the total number of trials necessary to generate m heads. Clearly the event that $(Y>n)=(Z\le m-1)$, where Z is a binomial random variable with the binomial probability mass function $Bi(n\ ,1/2\ )$, where n is generic. Now, let $Z\sim Bi(2m-1,1/2)$ then

$$\begin{aligned}P(J\ge m)&=P(J+m\ge 2m)=P(Y\ge 2m)\\&=P(Y>2m-1)=P(Z\le m-1)\\&=P((2m-1)-Z\le m-1)=P(Z\ge m)\\&=1-P(Z\le m-1)\\&=1-P(Y>2m-1)=1-P(J+m>2m-1)\\&=1-P(J>m-1)=1-P(J\ge m)=P(J\le m-1)\end{aligned} \tag{2.2}$$

and the use of part (i) completes proof of (ii) and (iii) of the lemma. It is of interest to elaborate further on our findings in part (ii) and (iii) above. These results can be also justified using induction, however we omit details for briefly. Moreover, (ii) of the lemma is a special case of a riff-shuffle distribution of the form

$$P(J=j)=\binom{m+j-1}{j}(p^m q^j+q^m p^j),\quad j=0,1,2,\ldots,m-1$$

for, $p=q=1/2$ (see Johnson, Kotz and Kemp(1990), page 234, for details). The sampling setup is also related to Banach's matchbox problem (see Rohatgi(1976), page 188, for details). In addition, the summation in part (ii) can be rewritten as $\sum_{j=0}^{m-1}\binom{m+j-1}{j}(1/2)^j((1/2)^m$ which in turns can be rewritten as incomplete beta function of the form $\int_0^{1/2} t^{m-1}(1-t)^{m-1}dt/\beta(m,m)$ where, $\beta(m,m)$ is the known beta function. It is not hard to see that the function $f(t)=t^{m-1}(1-t)^{m-1},\ 0<t<1,$ is symmetric around ½ and attains its maximum at ½. Therefore, ½ is the median of the above beta distribution function and consequently the incomplete beta integral equals ½, which again justify (ii) of the lemma. It can also be shown that (ii) and (iii) are natural consequences considering truncation of (2.1) from right and left, respectively, since $P(J\le m-1)=1/2$ and $P(J\ge m)=1/2$, which result the following right-truncated mass functions of the negative binomial distributions

$$P(J=j)=\binom{m+j-1}{j}(1/2)^j(1/2)^{m-1},\quad j=0,1,2,\ldots m-1. \tag{2.3}$$

and left-truncated probability mass functions of the negative binomial distribution

$$P(J=j)=\binom{m+j-1}{j}(1/2)^j(1/2)^{m-1},\quad j=m,m+1,\ \ldots \tag{2.4}$$

2.1 The r^{th} Cumulants Truncated Negative Binomial Distributions

In this context the r^{th} cumulant of the random variable $m+J$, where J is defined in (2.3), is given by the factorial moments $\mu'_{(r)}=E\left(\frac{(m+J+r-1)!}{(m+J-1)!}\right)$. For the random variable J which is distributed according to the right-truncated negative binomial probability mass function given in (2.3) the r^{th} cumulant has the form

$$\mu'_{(r)}=\frac{(m+r-1)!}{(m-1)!}\left[2^r-\sum_{j=m}^{m+r-1}(1/2)^{m+j-}\binom{m+j+r-1}{j}\right] \tag{2.5}$$

Therefore, the first two factorial moments of the right truncated negative binomial random variable J are

$$\mu'_{(1)}=E(m+J)=2m\left[1-(1/2)^{2m}\binom{2m}{m}\right]$$

$$\mu'_{(2)}=E\{(m+J+1)(m+J)\}=4m(m+1)\left[1-(1/2)^{2m}\binom{2m+1}{m}\right]$$

Where, we have used part (ii) of the lemma to simplify the above results. Thus the variance of the right-truncated negative binomial random variable J can be written as

$$Var(J)=2m\left\{1-(1/2)^{2m}\binom{2m}{m}\left(1+2m(1/2)^{2m}\binom{2m}{m}\right)\right\}$$

Similarly, for the random variable J which is distributed according to the left-truncated negative binomial probability mass function given in (2.4), the r^{th} cumulant has the form

$$\mu'_{(r)}=\frac{(m+r-1)!}{(m-1)!}\left[2^r+\sum_{j=m}^{m+r-1}(1/2)^{m+j}\binom{m+j+r-1}{j}\right]. \tag{2.6}$$

Special cases of $r=1,2$ provide the following first two factorial moments

$$\mu'_{(1)}=E(m+J)=2m\left[1+(1/2)^{2m}\binom{2m}{m}\right]$$

$$\mu'_{(2)}=E\{(m+J+1)(m+J)\}=4m(m+1)\left[1+(1/2)^{2m}\binom{2m+1}{m}\right]$$

Hence, the variance of the left- truncated negative binomial random variable J can be written as

$$Var(J)=2m\left\{1+(1/2)^{2m}\binom{2m}{m}\left(1-2m(1/2)^{2m}\binom{2m}{m}\right)\right\}$$

3. NEGATIVE BINOMIAL SAMPLING AND THE DISTRIBUTION OF V

Consider two urns A and B each contains m balls. An experiment involves sequence of independent trials. First select an urn at random with equal probability of ½ for each urn, then a ball from the selected urn is drawn out. Assume further that, selecting a ball from urn A is associated with taking a realization $x_{1i}, i = 1,2,...,m-1$ on the random variable X_1 while selecting a ball from urn B is associated with taking a realization $y_{1j}, j = 1,2,...,m-1$ on the random variables Y_1. The random variables X_1 and Y_1 are independent and identically distributed each $\chi^2_{(2)}$. Sequence of independent trials are performed till one of the urns is completely exhausted and consequently the minimum is completely identified. Obviously, if the random variables X_1 and Y_1 have been observed s and k times respectively then, $X = \sum_{i=1}^{s} x_{1i}$, $Y = \sum_{j=1}^{k} y_{1j}$ are distributed according to $\chi^2_{(2s)}$ and $\chi^2_{(2k)}$ in that order. We stress that the urn which is exhausted first provides the minimum of two chi-square variables $V = min(X,Y)$, in the sense that the random variable which had been observed completely will be identified as the minimum. Define the random variable J as the number of balls drawn out of one urn when the other urn is exhausted. Hence, the none exhausted urn still has $m - J$ balls. Clearly, the probability mass function of the random variable J is given by (2.3). Moreover, Since $V = min(X,Y)$ is generated by exhausting either urn, its average realization is (½ v) at any given trial. On the other hand the random variable J counts the number of failures at any given trial, with mean (½ v), then the conditional probability mass function of the random variable J given V is given by the following Poisson probability mass function

$$P(J = j \mid V) = \frac{(v\ /2)^j e^{-v/2}}{j!}, \quad j = 0,1,... \tag{3.1}$$

Consequently, the joint probability density function of J and V is given by

$$P(j,v) = f_A(v)P_B(J = j \mid V) + f_B(v)P_A(J = j \mid V)$$

Where, $f_A(v)$ is the probability density function associated with the random variable X if urn A is exhausted first and $f_B(v)$ is the probability density function of the random variable Y if urn B is exhausted first; where $f(\cdot)$ is the $\chi^2_{(2m)}$. While, $P_A(J = j \mid V)$, and $P_B(J = j \mid V)$ are similarly defined as in (2.3) for non-exhausted urn. Therefore,

$$P\ (j,v) = \frac{v^{m+j-1}\ e^{-v}}{2^{m+j-1}\ (m-1)!\ j!}, \quad j = 0,1,2,...,m-1\ ; v > 0. \tag{3.2}$$

The marginal probability mass function of the random variable J is then obtained from (3.2) as

$$P(J = j) = \frac{\int_0^\infty v^{m+j-1} e^{-v} dv}{2^{m+j-1}(m-1)!\ j!} = \binom{m+j-1}{j} (1/2)^j (1/2)^{m-1}, \quad j = 0,1, \ldots, m-1.$$

Also, the conditional probability density function of V given $J = j$ is given by

$$h(v \mid J = j) = \frac{v^{m+j-1} e^{-v}}{\Gamma(m+j)}, \quad v > 0. \tag{3.3}$$

which is gamma probability density function with $m + j$. Finally, the marginal density function of V is given by

$$g(v) = \sum_{j=0}^{m-1} \frac{v^{m+j-1}\ e^{-v}}{2^{m+j-1}(m-1)!\ j!}, v > 0 \tag{3.4}$$

which is a finite sum of weighted gamma probability density functions. Part (ii) of the lemma can be used to check whether $g(v)$ is a proper probability density function. The above results are summarized in the following theorem.

Theorem.

Let X and Y be iid $\chi^2_{(2m)}$ random variables. Further, let J be a random variable generated by a negative binomial sampling process to determine the distribution of the random variable $V = \min(X,Y)$, then the following results hold

1. The random variable J follows a right truncated negative binomial distribution as in (2.3)
2. The conditional probability density function of the random variable V given $J = j$ is gamma as in (3.3)
3. The marginal probability density function of V is weighted gamma as in (3.4)

4. For a real integer $r \geq 1$,
$E(V^r) = E_J E_V(V^r \mid J = j) = E\left[\frac{(m+J+r-1)!}{(m+J-1)!}\right]$ as given in (2.5).

4. NEGATIVE BINOMIAL SAMPLING AND THE DISTRIBUTION OF CORRELATED F-VARIABLES

Let X_0 be $\chi^2_{(2n)}$ random variable independent of both X and Y, defined previously. The random variable $W = (n/m)V/X_0$ is known as studentized minimum or the minimum of two correlated F-variable. The distribution of W arise in many statistical applications including two stage sequential estimation of the difference between two exponential locations, ranking and selection of exponential distributions with smallest location parameter, two stage estimation of the common location and also in reliability estimation of series system. In this section, we proceed to relate the distributions of W to the negative binomial sampling. First, we treat the distribution of the minimum correlated F variables. Recall the joint probability function in (3.2) and write the joint probability function of J, V and X_0 as

$$P(j,v,x_0) = \frac{v^{m+j-1} e^{-v} x_0^{n-1} e^{-x_0/2}}{2^{m+j+n-1}(m-1)!\, j!\, (n-1)!}, \quad j = 0,1,\ldots,m-1,\ v \geq 0,\ x_0 \geq 0. \tag{4.1}$$

Hence, consider the transformation, $W = (n/m)V/X_0$ and integrate (4.1) out of X_0 to obtain

$$P(j,w) = \frac{2\binom{m+j-1}{j}(m/n)^{m+j} w^{m+j-1}}{\beta(m+j,n)(1+(2m/n)w)^{m+n+j}}, \quad j = 0,1,\ldots,m-1,\ w \geq 0. \tag{4.2}$$

which is a negative binomial-beta second kind compound distribution. Therefore, the marginal probability density function of W is then obtained from (4.2) as

$$f(w) = \sum_{j=0}^{m-1}\binom{m+j-1}{j}\frac{2\ (m/n)^{m+j} w^{m+j-1}}{\beta(m+j,n)(1+(2m/n)w)^{m+n+j}}, \quad w \geq 0. \tag{4.3}$$

It is not hard to show that (4.3) is a proper probability density function if we take part (ii) of the lemma into account. Similar result to (4.2) can be achieved considering the conditional probability density function of V given $J = j$ in (3.3). The distribution of X_0 and the probability mass function of J in (2.3). It follows that the conditional probability density function of W given $J = j$ can be written as

$$f(w \mid J = j) = \frac{2^{m+j}\ (m/n)^{m+j} w^{m+j-1}}{\beta(m+j,n)(1+(2m/n)w)^{m+n+j}}, \quad w \geq 0. \tag{4.4}$$

which is beta second kind distribution with $m + j$ and n. The r^{th} moment of W whenever exists, is then obtained from (4.4) as

$$E(W^r) = (n/m)^r \binom{n-1}{r}^{-1}\binom{m+r-1}{r}\left[1 - \sum_{j=m}^{m+r-1}(1/2)^{m+j+r-1}\binom{m+j+r-1}{j}\right]$$

where we have utilized part of The lemma to simplify the above result.

5. COMPUTATIONAL ALGORITHMS AND THE CONSTRUCTION OF TABLES

In this section, we develop algorithms to compute critical values for the random variable V given in section 3 and the random variable W given in section 4. First, recall the probability mass function of the random variable given in (3.4) and let C be the solution of the following integral equation for some given values of m and α such that

$$\alpha = \int_0^C g(v)\, dv = \sum_{j=0}^{m-1}\binom{m+j-1}{j}(1/2)^{m+j-1}\int_0^C \frac{v^{m+j-1} e^{-v}}{\Gamma(m+j)}\, dv \tag{5.1}$$

Meanwhile, for fixed α denote by C_1 and $C2$, the inverse gamma function at m and $2m-1$ respectively. It follows that $C_1 \leq C \leq C_2$ by the monotonicity of the gamma function. Therefore, iterative method of root finding and the bisection method can be used to locate the value of C accurate to 6 decimal places for given values of m and α.

A Fortran program with the aid of the well known IMSL routines are used to generate tables for the minimum chi-square random variable V for $m = 1(1)30$ and $35(5)250$ and values of $\alpha = 0.005, 0.001, 0.025, 0.10, 0.90, 0.975, 0.99$ and 0.995. We report only some selected values of C with only three decimal places, in this study, presented in Table(1) for illustration purpose.

Table 1. Critical Value C for the Minimum of Two Chi-square Variables

	α									
m	0.005	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99	0.995
1	0.005	0.01	0.025	0.051	0.105	2.303	2.996	3.689	4.605	5.298
2	0.145	0.207	0.335	0.488	0.721	4.729	5.689	6.608	7.779	8.641
3	0.527	0.676	0.95	1.244	1.653	7.05	8.203	9.286	10.65	11.63
4	1.105	1.345	1.764	2.189	2.757	9.317	10.63	11.85	13.36	14.45
5	1.828	2.157	2.712	3.258	3.97	11.55	13	14.34	15.99	17.17
6	2.662	3.075	3.758	4.417	5.261	13.76	15.34	16.78	18.55	19.81
7	3.583	4.077	4.879	5.644	6.611	15.95	17.64	19.18	21.06	22.4
8	4.574	5.144	6.061	6.925	8.006	18.13	19.93	21.56	23.54	24.95
9	5.624	6.267	7.292	8.25	9.439	20.3	22.19	23.91	25.99	27.46
10	6.724	7.437	8.565	9.612	10.9	22.46	24.44	26.24	28.41	29.95
11	7.866	8.646	9.873	11.01	12.4	24.61	26.68	28.55	30.81	32.41
12	9.045	9.89	11.21	12.43	13.91	26.75	28.91	30.85	33.2	34.85
13	10.26	11.16	12.58	13.87	15.44	28.89	31.13	33.14	35.56	37.27
14	11.5	12.47	13.97	15.34	17	31.02	33.34	35.42	37.92	39.67
15	12.77	13.79	15.38	16.82	18.56	33.15	35.54	37.68	40.26	42.06
16	14.06	15.14	16.81	18.32	20.15	35.27	37.73	39.94	42.59	44.44
17	15.37	16.51	18.26	19.84	21.74	37.39	39.92	42.19	44.9	46.8
18	16.7	17.89	19.72	21.37	23.35	39.5	42.11	44.43	47.21	49.16
19	18.05	19.29	21.2	22.91	24.97	41.62	44.28	46.67	49.51	51.5
20	19.42	20.71	22.69	24.47	26.6	43.73	46.46	48.89	51.81	53.84
21	20.8	22.14	24.2	26.04	28.23	45.83	48.63	51.12	54.09	56.17
22	22.2	23.59	25.71	27.61	29.88	47.94	50.79	53.33	56.37	58.48
23	23.61	25.05	27.24	29.2	31.53	50.04	52.95	55.55	58.64	60.8
24	25.03	26.52	28.78	30.79	33.2	52.14	55.11	57.76	60.91	63.1
25	26.47	28	30.33	32.4	34.86	54.23	57.27	59.96	63.17	65.4
26	27.91	29.49	31.88	34.01	36.54	56.33	59.42	62.16	65.42	67.69
27	29.37	30.99	33.45	35.63	38.22	58.42	61.57	64.36	67.67	69.98
28	30.83	32.5	35.02	37.26	39.91	60.52	63.71	66.55	69.92	72.26
29	32.31	34.02	36.6	38.89	41.6	62.61	65.86	68.74	72.16	74.54
30	33.79	35.54	38.19	40.53	43.3	64.69	68	70.92	74.4	76.81
35	41.34	43.28	46.22	48.81	51.86	75.12	78.67	81.81	85.53	88.11

Second, consider the probability density function of the random variable W given in (4.3) and write the following integral equation (5.2) to solve for $0 < C < \infty$ for some given values of n, m and α

$$\alpha = \int_0^C f(w)\,dw = \sum_{j=0}^{m-1}\binom{m+j-1}{j}\int_0^C \frac{2\,(m/n)^{m+j}\,w^{m+j-1}}{\beta(m+j,n)(1+(2m/n)w)^{m+n+j}}\,dw \tag{5.2}$$

We then make use of the following transformation to restrict our search for C or any function of it in the interval $(0,1)$. Therefore, if we consider the transformation $Z = (1+2m/nw)^{-1}$ the integral equation in (5.2) becomes

$$\alpha = \int_h^1 f(z)\,dz = \sum_{j=0}^{m-1}\binom{m+j-1}{j}(1/2)^{m+j-1}\int_h^1 \frac{Z^{n-1}\,(1-Z)^{m+j-1}}{\beta(m+j,n)}\,dZ \tag{5.3}$$

It follows that

$$(1-\alpha) = \sum_{j=0}^{m-1}\binom{m+j-1}{j}(1/2)^{m+j-1}\int_0^h \frac{Z^{n-1}\,(1-Z)^{m+j-1}}{\beta(m+j,n)}\,dZ \tag{5.4}$$

where, $C = n(1-h)/2mh$. Part (ii) of the lemma is used to reach the above result in (5.4).
Now, iterative methods of root finding and the bisection method is used to locate h for given values of $\alpha = 0.005, 0.001, 0.025, 0.10, 0.90, 0.975, 0.99$ and 0.995 ; $m = 1(1)30$ and $35(5)250$ and $n = 1(1)30$ and $35(5)250$. We report some critical values of W in the following Table (2).

Table 2. Critical values C for the Upper $\alpha = 0.05$

	n													
M	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026
2	0.097	0.106	0.11	0.113	0.114	0.116	0.116	0.117	0.118	0.118	0.118	0.119	0.119	0.119
3	0.144	0.164	0.175	0.181	0.185	0.188	0.191	0.193	0.194	0.195	0.196	0.197	0.198	0.198
4	0.174	0.204	0.219	0.229	0.236	0.241	0.245	0.248	0.25	0.252	0.254	0.255	0.257	0.258
5	0.194	0.231	0.251	0.264	0.273	0.28	0.285	0.289	0.292	0.295	0.298	0.3	0.301	0.303
6	0.209	0.252	0.275	0.291	0.302	0.31	0.316	0.321	0.325	0.329	0.332	0.334	0.336	0.338
7	0.221	0.268	0.294	0.311	0.324	0.333	0.34	0.346	0.351	0.355	0.359	0.362	0.364	0.367
8	0.23	0.28	0.309	0.328	0.342	0.352	0.36	0.367	0.373	0.377	0.381	0.385	0.388	0.39
9	0.238	0.291	0.321	0.342	0.357	0.368	0.377	0.384	0.39	0.395	0.4	0.404	0.407	0.41
10	0.244	0.299	0.332	0.353	0.369	0.381	0.391	0.399	0.405	0.411	0.416	0.42	0.424	0.427
11	0.249	0.307	0.34	0.363	0.38	0.393	0.403	0.411	0.418	0.424	0.429	0.434	0.438	0.441
12	0.254	0.313	0.348	0.372	0.389	0.402	0.413	0.422	0.429	0.436	0.441	0.446	0.45	0.454
13	0.258	0.319	0.354	0.379	0.397	0.411	0.422	0.432	0.439	0.446	0.452	0.457	0.461	0.465
14	0.261	0.323	0.36	0.385	0.404	0.419	0.43	0.44	0.448	0.455	0.461	0.466	0.471	0.475
15	0.264	0.328	0.365	0.391	0.41	0.425	0.437	0.447	0.456	0.463	0.469	0.475	0.479	0.484
16	0.267	0.331	0.37	0.396	0.416	0.431	0.444	0.454	0.463	0.47	0.477	0.482	0.487	0.492
17	0.27	0.335	0.374	0.401	0.421	0.437	0.45	0.46	0.469	0.477	0.483	0.489	0.494	0.499
18	0.272	0.338	0.378	0.405	0.426	0.442	0.455	0.466	0.475	0.483	0.489	0.495	0.501	0.505
19	0.274	0.341	0.381	0.409	0.43	0.446	0.46	0.471	0.48	0.488	0.495	0.501	0.507	0.512
20	0.276	0.344	0.384	0.413	0.434	0.451	0.464	0.475	0.485	0.493	0.5	0.506	0.512	0.517
21	0.278	0.346	0.387	0.416	0.437	0.454	0.468	0.48	0.489	0.498	0.505	0.511	0.517	0.522
22	0.279	0.348	0.39	0.419	0.441	0.458	0.472	0.484	0.493	0.502	0.509	0.516	0.522	0.527
23	0.281	0.35	0.392	0.422	0.444	0.461	0.475	0.487	0.497	0.506	0.513	0.52	0.526	0.531
24	0.282	0.352	0.395	0.424	0.447	0.464	0.479	0.491	0.501	0.509	0.517	0.524	0.53	0.535
25	0.283	0.354	0.397	0.427	0.449	0.467	0.482	0.494	0.504	0.513	0.521	0.528	0.534	0.539
26	0.285	0.356	0.399	0.429	0.452	0.47	0.484	0.497	0.507	0.516	0.524	0.531	0.537	0.543
27	0.286	0.357	0.401	0.431	0.454	0.472	0.487	0.499	0.51	0.519	0.527	0.534	0.54	0.546
28	0.287	0.359	0.402	0.433	0.456	0.475	0.489	0.502	0.513	0.522	0.53	0.537	0.544	0.549
29	0.288	0.36	0.404	0.435	0.458	0.477	0.492	0.504	0.515	0.525	0.533	0.54	0.546	0.552
30	0.289	0.361	0.406	0.437	0.46	0.479	0.494	0.507	0.518	0.527	0.535	0.543	0.549	0.555
35	0.293	0.367	0.412	0.444	0.468	0.488	0.503	0.517	0.528	0.538	0.546	0.554	0.561	0.567

Applied Case: In testing the equality of scale parameters of three exponential distributions, θ_1, θ_2 and θ_0, we let $X_{i1}, X_{i2}, \ldots, X_{im}$ to be random sequences of independent random variables for $i = 0,1,2$. We also denote the corresponding MLE estimators by $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_0$ respectively. Moreover, we define the set of null hypotheses by $H_{0i} : \theta_i = \theta_0$, $i = 1,2$ and $H_0 = \bigcap_{i=1}^{2} H_{0i}$.

While the set of alternatives is defined as $H_{1i} : \theta_i \succ \theta_0$ and $H_1 = \bigcup_{i=1}^{2} H_{1i}$. Then, the hypotheses H_{01}, H_{02} and H_0 can be tested simultaneously against the respective alternatives H_{11}, H_{12}, and H_1 as follows: We reject the null hypothesis H_0 if $F_i \geq F_{i\alpha}$, where $F_i = \hat{\theta}_i / \hat{\theta}_0$, $i = 1,2.$ and the value of $F_{i\alpha}$ is chosen such that $P(\hat{\theta}_i / \hat{\theta}_0 \geq F_{i\alpha}; i = 1,2 \mid H_o) = P(V \geq C) = \alpha$, where $C = F_{i\alpha}$.

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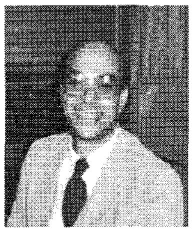

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