Perfect Tracking Control for Linear Systems with State Constraint

Dane Baang, Jin Young Choi*, and Hyungbo Shim

Abstract: This paper presents a new Perfect Tracking Control (PTC) scheme for linear systems with state constraint. The proposed controller increases the number of the steps on-line for perfect tracking to satisfy the given ellipsoid-type state constraint. The unavoidable step delay that we impose is minimized by solving LMI feasibility problems and the possible feedback information loss is avoided. The proposed schemes are easy to develop, theoretically simple and clear, and include the conventional PTC as its special case.

Keywords: Linear systems, perfect tracking control, right inverse, state constraint.

1. INTRODUCTION

In 1997, a new control scheme called Perfect Tracking Control(PTC) has been proposed in [1-3]. In this design, linear systems with the control action N-times faster than sampler, have been considered.

This scheme basically adopted the design of N-step open-loop control scheme to achieve perfect tracking, i.e., exactly deriving the system output to the reference value, after N control-steps where N means the system order. This is why the scheme was called N-delay control in its beginning in [1].

As the author mentioned in [3], this idea is based the conventional dead-beat control [4,5], but differed from it since the considered systems are multi-rate and the tracking performance does not have step-delay, by applying the reference in advance than the actual reference.

This technique has been applied to the motion control for dc-servo motor [2]. In [6], the PTC scheme has been tried to control hard disk drive and visual servoing. Similar scheme has been applied to hard disk drive and servo motor again in [7]. How to generate desired state trajectory from output reference trajectory was studied in [8]. [3] contains most of these results. In [3], several extensions of the main scheme including delayed systems, multi-variable systems, disturbance rejection, inter-sample observer were developed.

The basic condition for the perfect tracking is that the control action speed is exactly N-times faster than the sampling speed. This possibly restrictive condition has been relaxed in [3] into slightly more general case, i.e., control action is multiple-of-N times faster than the sampling. This still restrictive condition has been assumed in most of the researches so far.

In this paper, we extend the existing PTC to state-constrained linear systems. The proposed scheme extends the system class into general $m(\geq N)$ case and achieves perfect tracking with delay, but satisfies the given state-constraint. The proposed scheme contains the conventional PTC as its special case, and minimizes the unavoidable perfect tracking delay.

2. PTC FOR STATE-CONSTRAINED LINEAR SYSTEMS

2.1. Conventional PTC [3]

Consider the single-input single-output (SISO) controllable discrete-time system described by

$$x[k+1] = Ax[k] + bu[k], \quad (1)$$
$$y[k] = cx[k],$$

where k represents the discrete time, $u[k]$ and $y[k]$ are input signal and output signal, respectively. $x[k] \in R^n$ is a state vector and N is the system order. Assume that the full state information is available.

It is also assumed that the measurement of the output y is performed at every N time steps. If the discrete-time model (1) is derived from its continuous-time counterpart, this means that the output is sampled at the frequency of $1/(NT_s)$ while the input is applied at the frequency of $1/T_s$ where

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T_s is the sampling period. This difference between the input and the output frequencies is the key to the perfect tracking control. See [3] for more details.

Now we consider the behavior of this system during N steps. In the external section, the system acts with the time index i. Thus, the input sequence for the next steps is defined as

$$U_N[i] = [u[i], u[i+1], \ldots, u[i+N-1]]^T. \quad (2)$$

Using the time variable i, the system in the external section's point of view is described as

$$\begin{aligned} x[i+N] &= A_N x[i] + B_N U_N[i], \\ y[i] &= cx[i], \\ A_N &= A^N, \\ B_N &= [A^{N-1}b, A^{N-2}b, \ldots, b]. \end{aligned} \quad (3)$$

The control purpose is to derive the future output $y[i+N]$ to exactly match to the desired output trajectory $y_d[i]$ given in advance, i.e., to perform perfect tracking at each N steps. The control sequence $U_N[i]$ is designed as

$$U_N[i] = -B_N^{-1} A_N x[i] + B_N^{-1} x_d[i], \quad (4)$$

where $x_d[i]$ denotes the desired state at the step i. By putting (4) into (3), we see that the control (4) leads to $x[i+N] = x_d[i]$, i.e., perfect tracking is performed at each N steps if each element of $U_N[i]$ is sequentially injected to the plant during the step durations from i to $(i+N-1)$.

2.2. PTC for state-constrained linear systems

The main idea is to give step delay to the perfect tracking performance and, in return for it, to obtain additional design parameters for dealing with state-constraint. We first start with an extended PTC structure. Assume that the sensing of the system state occurs at every mT_s time point where m is any fixed integer such that $m \geq N$. We next show that it is possible to achieve dm-step perfect tracking where d is any positive integer. Consider the following dm-step control strategy for scalar design parameters $\gamma_1, \gamma_2, \ldots, \gamma_{dm-N}$.

$$\begin{aligned} U_{dm}[i] &= Fx[i] + Kx_d[i] \\ &\quad + \gamma_1 n_1 + \gamma_2 n_2 + \ldots \gamma_{dm-N} n_{dm-N}, \end{aligned} \quad (5)$$

where

$$F = -B_{dm}^T (B_{dm} B_{dm}^T)^{-1} A_{dm},$$
$$K = B_{dm}^T (B_{dm} B_{dm}^T)^{-1},$$
$$\begin{aligned} B_{dm} &= [A^{dm-1}b, A^{dm-2}b, \ldots, b] \\ &= [A^{dm-1}b, \ldots, A^N b, B_N], \end{aligned}$$
$$A_{dm} = A^{dm},$$

whereas n_t is a basis vector for the nullspace of B_{dm}, i.e., $B_{dm} n_t = 0$, $t = 1, 2, \ldots, dm-N$. Clearly, the following holds.

$$\begin{aligned} x[i+dm] &= A^{dm} x[i] + [A^{dm-1}b \ldots b] U_{dm}[i] \\ &= A_{dm} x[i] + B_{dm} U_{dm}[i] \\ &= x_d[i]. \end{aligned} \quad (6)$$

Thus, the control (5) performs $x[i+dm] = x_d[i]$. Now consider the state-constraint defined as follows.

Definition 1: Let $x[k]$, $k = 0, 1, 2, \ldots, \infty$ be any state value to the system (1). The ellipsoid type state-constraint is defined as $x[k]^T Q x[k] \leq 1$, $k = 0, 1, 2, \ldots, \infty$, where Q is a pre-given positive definite matrix.

Assumption 1: The initial state $x[i]$ and the desired state $x_d[i]$ satisfy the state constraint given in Definition 1.

We provide LMI conditions such that the state trajectory of the system (1) satisfies the pre-given state constraint. We start with the following two definitions for state-constraint and one assumption.

Since we already developed the general form of dm-step control in (5), the dm-step control that satisfies the state constraint is given as follows.

Theorem 1 (PTC for State-Constrained Linear Systems)**:** Under the assumption 1, and for the system (1), suppose that the system state is constrained by the Definition 1. Then the dm-step control (5) achieves dm-step perfect tracking and satisfies the state constraint if, and only if the following LMI feasibility problem (7) is solvable for a scalar d and a vector $\gamma \in R^{dm-N}$.

$$\begin{aligned} 0 \leq diag\left\{ \begin{bmatrix} 1 & [\zeta_q + \tau_q \bar{N}\gamma]^T \\ \zeta_q + \tau_q \bar{N}\gamma & Q^{-1} \end{bmatrix} \right\} \\ \in R^{(dm-1)(N+1)\times(dm-1)(N+1)}, \; q = 1, 2, .., dm-1, \end{aligned} \quad (7)$$

where

$$\tau_q := [A^{q-1}b, \ldots, b, 0_{N\times dm-q}] \in R^{N\times dm},$$
$$\zeta_q = A^q x[i] + \tau_q [Fx[i] + Kx_d[i]] \in R^N,$$
$$\gamma = [\gamma_1, \ldots, \gamma_{dm-N}]^T \in R^{dm-N},$$
$$\bar{N} = [n_1, \ldots, n_{dm-N}] \in R^{dm\times(dm-N)}.$$

F, K, and n_t are from (5).

Proof: Since the input $U_{dm}[i]$ always performs dm-step perfect tracking, we need to show only that the feasibility of the LMI (7) is equivalent to the satisfaction of the state-constraint in Definition 1.

The state $x[q]$ is expressed as $x[q] = A^q x[i] + \tau_q U_{dm}[i] = \zeta_q + \tau_q \bar{N}\gamma$. Thus, the state-constraint

$$x[q]^T Qx[q] \le 1, \quad q = 0,1,2,\dots,dm-1 \tag{8}$$

can be expressed as

$$[\zeta_q + \tau_q \bar{N}\gamma]^T Q[\zeta_q + \tau_q \bar{N}\gamma] \le 1. \tag{9}$$

Then, (7) is obtained from (9) by using schur complements [9]. □

Fig. 1 represents how we can manipulate the system state by using the proposed scheme in Theorem 1. Until perfect tracking is performed, the state trajectory by using the conventional PTC is unique and may violate the state constraint, but the proposed dm-step control strategy in Theorem 1 provides infinite number of state trajectory candidates, determined by the design parameters d and γ.

Therefore, if at least one candidate lies in the ellipsoid representing the state-constraint, we can select it to satisfy the constraint and achieve perfect tracking. This job is done by the scheme and the LMI solver for the LMI (7). Furthermore, they choose a proper state trajectory that results in the minimal perfect tracking delay.

Since the perfect tracking delay d needs to be minimized for the fastest perfect tracking for constrained linear systems, we fix $d = 1$ initially and solve the feasibility problems for the LMI (7), i.e., finding any vector γ that satisfies (7). If this LMI feasibility problem is not feasible, d is increased by one and the same procedure repeats until getting a feasible solution.

The feasibility of the LMI (7) is the equivalent condition to the success of the proposed control scheme. However, it is difficult to provide a qualititive condition for the feasibility of the LMI, since the constraint can be imposed arbitrary hard and the resulting ellipsoid representing the constraint can be very small. Once the LMI is solved and the resulting dm-step control is performed, then the same prodedure repeats after dm-steps.

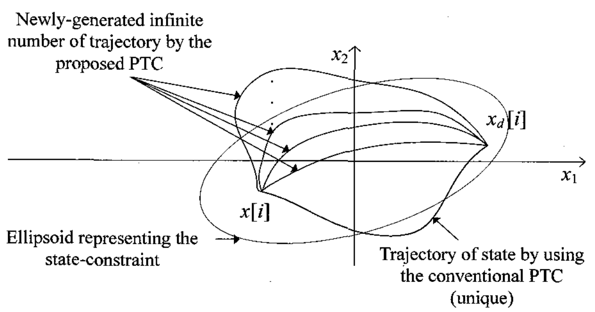


Fig. 1. State constraint and the proposed PTC.

2.3. Usage of feedback information

Note that the property of the control in (5) is that, during the steps $i \sim i+dm-1$, it does not use the feedback information of the state even though the feedback information of the state is available at every m steps. The feedback information in most control designs is important in a sense that there exist uncertainties in any real plant, and the feedback itself gives robustness to the uncertainties. Thus, the proposed scheme in (5) is reformulated as follows to avoid this possible information loss. Suppose that $\gamma_1, \gamma_2, \dots, \gamma_{dm-N}$ are obtained from (7) and let

$$\gamma N_{p\sim q} := \gamma_1 n_{1,p\sim q} + \gamma_2 n_{2,p\sim q} + \dots \gamma_{dm-N} n_{dm-N,p\sim q},$$

where $n_{t,p\sim q}$ is $p \sim q$-th elements of $n_t, t = 1,2,\dots, dm-N$. Since the feedback information at step i is used for step $i \sim i+m-1$, the control inputs for these steps are not changed, i.e.,

$$U^{i\sim i+m-1} = F_{1\sim m} x[i] + K_{1\sim m} x_d[i] + \gamma N_{1\sim m}, \tag{10}$$

where $F_{i\sim j}$ and $K_{i\sim j}$ denote the submatrices of F and K, consisting of the i-th row to the j-th row of the matrices F and K, respectively. For the steps $i+m \sim i+2m-1$, the control in (5) is reformulated to use the feedback information obtained at the step $i+m$, i.e.,

$$\begin{aligned}
&U^{i+m\sim i+2m-1} \\
&= (Fx[i] + Kx_d[i])_{(m+1)\sim 2m} + \gamma N_{(m+1)\sim 2m} \\
&= F_{(m+1)\sim 2m} x[i] + K_{(m+1)\sim 2m} x_d[i] + \gamma N_{(m+1)\sim 2m} \\
&= [-B_{dm}^T (B_{dm} B_{dm}^T)^{-1}]_{(m+1)\sim 2m} A^{dm} x[i] \\
&\quad + [-B_{dm}^T (B_{dm} B_{dm}^T)^{-1}]_{(m+1)\sim 2m} x_d[i] \\
&\quad + \gamma N_{(m+1)\sim 2m} \\
&= -K_{(m+1)\sim 2m} (A^{dm} x[i] - x_d[i]) + \gamma N_{(m+1)\sim 2m} \\
&= -K_{(m+1)\sim 2m} (A^{dm-m} A^m x[i] - x_d[i]) \\
&\quad + \gamma N_{(m+1)\sim 2m} \\
&= -K_{(m+1)\sim 2m} (A^{dm-m} (x[i+m] \\
&\quad - [A^{m-1}b \ \dots \ b][u[i] \ \dots \ u[i+m-1]]^T) \\
&\quad - x_d[i]) + \gamma N_{(m+1)\sim 2m}.
\end{aligned} \tag{11}$$

Similarly, for the steps $i+nm \sim i+(n+1)m-1$, $n = 2,3,\dots,d-1$, we obtain

$$U^{i+nm\sim i+(n+1)m-1}$$
$$= -K_{(nm+1)i+nm\sim i+(n+1)m-1,(n+1)m}(A^{dm-nm}(x[i+nm]$$
$$-[A^{nm-1}b \ldots b][u[i] \ \ldots \ u[i+nm-1]]^T) - x_d[i])$$
$$+\gamma N_{(nm+1)\sim(n+1)m}. \tag{12}$$

Note that the control strategy (10)~(12) is equivalent to the control (4) if the model (1) is exact and there is no uncertainty.

3. EXAMPLE

Simulations have been performed for 6-axis industrial robot model described as a fourth order SISO plant.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{J_1} & -\frac{c_1}{J_1} & \frac{k_1}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{J_2} & 0 & -\frac{k_1+k_2}{J_2} & -\frac{c_2}{J_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x,$$

$$J_1 = 0.01063, \quad J_2 = 0.01063,$$
$$k_1 = 2.76, \quad k_2 = 2.76,$$
$$c_1 = 0.027, \quad c_2 = 0.0022.$$

This plant has been discretized with the sampling time $T_s = 0.01$ seconds by using zero-order holder. The initial states are at origin and the reference signal for the output is sine wave with the amplitude equal to 1.

Fig. 2 shows the performance of conventional PTC by (4). It is shown that perfect tracking is performed since the error between y and the reference r (sine wave), sampled at every m-steps, is almost zero. In Fig. 2, one can see that the 2-norm of the unconstrained state goes up to 100 when the conventional PTC is used. The system state is now constrained by Definition 1 with $Q = 0.01 \times I$, i.e., the 2-norm of the state should be equal to or less than 10. Fig. 3 shows that the state satisfies the given constraint and the perfect tracking is performed. The perfect tracking delay d increases when the reference varies rapidly.

To test the effect of the usage of feedback information, variations were added to the system matrices A and b, of the continuous-counterpart of the discrete plant (1). Fig. 4 shows the comparison results between two controls for three different cases of variations.

The left-side pictures represent the result when dm-step control sequence (5) is directly used, and the right-side pictures represent the result when the reformulated control (10)~(12) is used. In left-side pictures, available feedback information can be lost and the response oscillated relatively big, while the right ones had small oscillation or better performance since the scheme updated the control at each m-steps, by using all available feedback information.

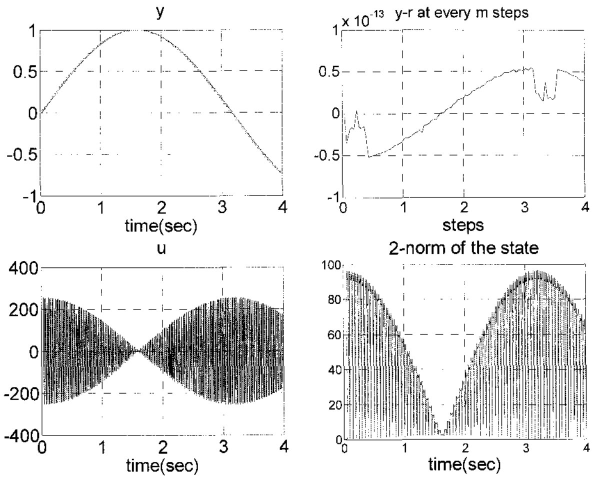


Fig. 2. Conventional PTC (unconstrained).

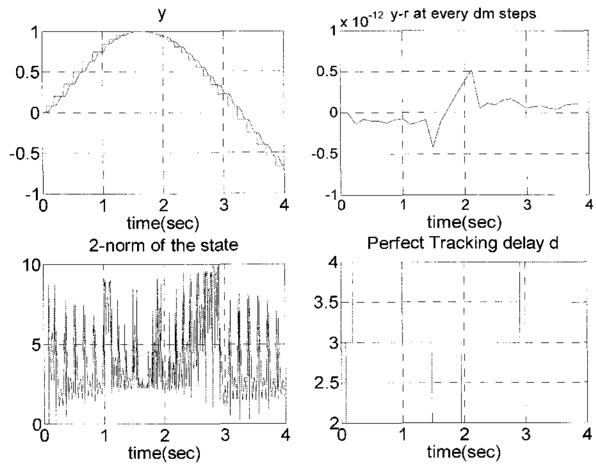
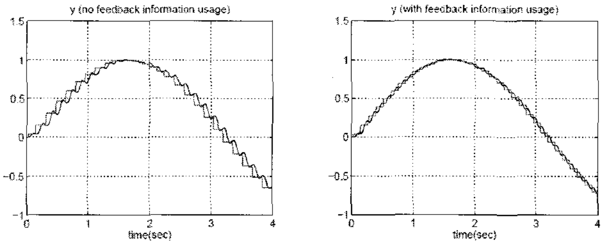


Fig. 3. State-constrained case by using Theorem 1.

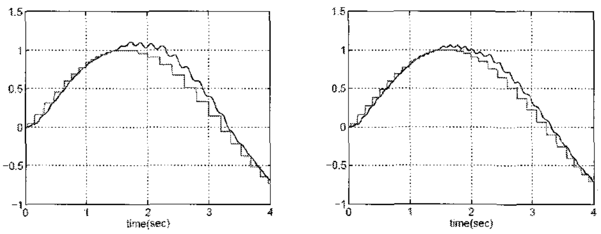
For the cases where $m > N$, the results was similar, they were not always better than $m = N$ case. This is because the sampling time T_s also varies according to the increment of m and the corresponding discretization should be performed.

4. CONCLUSIONS

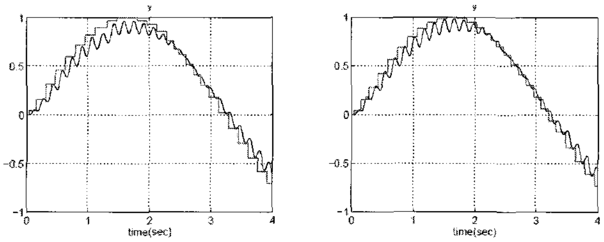
A new Perfect Tracking Control (PTC) scheme for linear systems with state constraints is proposed. The proposed scheme extends PTC structure to more general cases, and increases the the number of steps for perfect tracking to satisfy the given state-constraint. The proposed scheme includes the conventional PTC as its special case and the



(a) $A \to 1.1A,\ b \to 1.1b.$



(b) $A \to 0.9A,\ b \to b.$



(c) $A \to 1.1A,\ b \to b.$

Fig. 4. Effectiveness of feedback information usage.

unavoidable perfect tracking delay is minimized by solving LMI feasibility problems. Future researches would include robustness approach, extension to MIMO linear systems.

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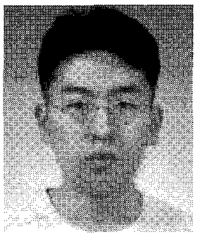

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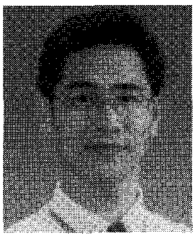

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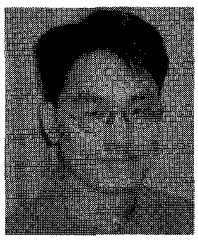

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