A Note on Discrete Interval System Reduction via Retention of Dominant Poles

Younseok Choo

Abstract: In a recently proposed method of model reduction for discrete interval systems, the denominator polynomial of a reduced model is computed by applying interval arithmetic to dominant poles of the original system. However, the denominator polynomial obtained via interval arithmetic usually has poles with larger intervals than desired ones. Hence an unstable polynomial can be derived from the stable polynomial. In this paper a simple technique is presented to partially overcome such a stability problem by accurately preserving desired real dominant poles.

Keywords: Interval poles, interval system, model reduction, root locus.

1. INTRODUCTION

In many engineering problems, it is often desirable to approximate a high-order system by a low-order model since it facilitates simulation and implementation of the system. Over the decades the model reduction problem has been an ample area of research and a variety of techniques have been proposed for the order reduction of fixed-coefficients systems.

Many systems have coefficients that are constants but uncertain within a finite range. Those systems can be modeled as interval systems. Recently, the model reduction problem for interval systems has been considered in the literature for discrete [1] and continuous [2,3] systems. In [1], the denominator polynomial of a reduced model was obtained so that dominant poles of the original system are preserved in the reduced model. The numerator polynomial was then determined by matching some initial time-moments. In [2,3], a technique was suggested to construct the interval Routh array from which we can obtain a stable denominator polynomial for the reduced model. The method of [1] can be used for continuous systems. However, the approach taken in [2,3] cannot be applied to discrete systems.

This paper discusses the method of [1] in the viewpoint of stability of the reduced model. In [1], the interval poles of the original system were obtained using the results of [4]. Then the denominator polynomial of a reduced model was computed by applying interval arithmetic [5] to dominant poles. Among others, the primary goal of retaining dominant poles is to guarantee the stability of the reduced model. In case of fixed-coefficients systems, the reduced model is obviously stable if its denominator polynomial is computed from dominant poles of the original stable system. However, the situation is different for interval systems with interval poles. The denominator polynomial obtained by interval arithmetic as in [1] usually has poles with larger intervals than dominant poles. Consequently, an unstable polynomial can be derived from the stable polynomial. For example, assume that the following two interval poles are to be retained in the second-order reduced model

$$\lambda_1 = [-0.82, -0.8],\ \lambda_2 = [-0.92, -0.9].$$

Then, using the interval arithmetic, we obtain

$$(z-\lambda_1)(z-\lambda_2) = z^2 + [1.7, 1.74]z + [0.72, 0.7544]. \quad (1)$$

Clearly the interval polynomial in (1) is unstable since $z^2 + 1.74z + 0.72$ has a root at $z = -1.0621$.

In this paper a simple method is given to obtain the denominator polynomial of the reduced model that accurately preserves desired real dominant poles, which can be used to partially overcome the stability problem in [1]. To this end it is first shown that two opposite edge polynomials of a polynomial with distinct real interval poles can be represented via the minima and maxima of its poles. Then the result is applied to the model reduction problem.

The paper is organized as follows. In Section 2, some definitions are given and the results of [4] are briefly reviewed. The main results of this paper are

Manuscript received March 9, 2005; revised January 2, 2006; accepted October 30, 2006. Recommended by Editorial Board member Guang-Ren Duan under the direction of past Editor-in-Chief Myung Jin Chung.

Younseok Choo is with the Department of Electronic and Electrical Engineering, Hongik University, San 34, Sinan-Ri Jochiwon-Eup, Yeongi-Gun, Chungnam 339-701, Korea (e-mail: yschoo@wow.hongik.ac.kr).

contained in Section 3 with an example, and the paper is concluded in Section 4.

2. PRELIMINARIES

2.1. Definitions

2.1.1 Interval arithmetic

The addition, subtraction and multiplication of two intervals $[a,b]$ and $[c,d]$ are respectively defined as follows [5].

Addition: $[a,b]+[c,d]=[a+b,c+d]$

Subtraction: $[a,b]-[c,d]=[a-b,b-c]$

Multiplication: $[a,b][c,d]=[\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$

2.1.2 Interval polynomial

The family of (monic) polynomials $C(z)$ defined by $C(z)=z^n+[c_{n-1}^-,c_{n-1}^+]z^{n-1}+\cdots+[c_1^-,c_1^+]z+[c_0^-,c_0^+]$ with $c_i^-\le c_i^+$ is called an (monic) interval polynomial. $C(z)$ is said to be stable if every polynomial with fixed coefficients in $C(z)$ is stable.

2.1.3 Reciprocal eigenvector

For an $n\times n$ matrix A with distinct eigenvalues λ_i, $1\le i\le n$, let $\mathbf{x}_k$ be the eigenvector associated with λ_k. Define

$$U=\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{bmatrix},$$

$$V=[U^*]^{-1}=\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_n\end{bmatrix}.$$

Then $\mathbf{y}_k$ is called the reciprocal eigenvector associated with λ_k.

2.2. Interval poles

Consider an nth-order stable discrete interval system described by the transfer function $G(z)$ with the interval denominator polynomial

$$C(z)=z^n+[c_{n-1}^-,c_{n-1}^+]z^{n-1}+\cdots+[c_1^-,c_1^+]z+[c_0^-,c_0^+], \quad (2)$$

where $c_i^-\le c_i^+$ for each i. Suppose $C(z)$ has only distinct real interval poles. Let

$$A^I=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1\\ -[c_0^-,c_0^+] & -[c_1^-,c_1^+] & -[c_2^-,c_2^+] & \cdots & -[c_{n-1}^-,c_{n-1}^+]\end{bmatrix}. \quad (3)$$

Then interval poles of (2) can be computed from the interval eigenvalues of A^I. Denote A^I by

$$A^I=[A_c-\Delta A, A_c+\Delta A], \quad (4)$$

where A_c is the center matrix of A^I and ΔA is the perturbation matrix given by

$$\Delta a_{ni}=\frac{c_{i-1}^+-c_{i-1}^-}{2},\quad i=1,2,\cdots,n \quad (5)$$

with all other elements equal to 0. Assume that the center matrix A_c has n distinct real eigenvalues λ_i, $1\le i\le n$, and let $\mathbf{x}_k$ and $\mathbf{y}_k$ respectively be the real eigenvector and reciprocal eigenvector associated with the eigenvalue λ_k. Consider the signature matrix

$$S_k=[\operatorname{sgn}(y_{k,i}x_{k,j})]_{i,j=1,2,\cdots,n}. \quad (6)$$

The matrix S_k can be computed for any $A\in A^I$ in the same way. If S_k is the same for all $A\in A^I$ up to the multiplication of -1, then the exact range of λ_k over A^I is given by [4].

$$\lambda_k^I(A^I)=[\lambda_k(A_c-\Delta A\circ S_k),\lambda_k(A_c+\Delta A\circ S_k)], \quad (7)$$

where $\circ$ denotes the component wise multiplica-tion, and $\lambda_k(B)$ is the kth eigenvalue of B.

3. MAIN RESULTS

3.1. Representation of polynomial via interval poles

For the nth-order denominator polynomial given in (2), assume its real interval poles $\lambda_i^I=[\lambda_i^-,\lambda_i^+]$, $1\le i\le n$, have been exactly computed by (7). Without loss of generality, let

$$\lambda_1^I<\lambda_2^I<\cdots<\lambda_n^I$$

i.e., $\lambda_i^+<\lambda_{i+1}^-$. It is now shown that two opposite edge polynomials of $C(z)$ can be represented by the minima and maxima of its interval poles. We deal with several cases separately.

Case 1: λ_i^I s are all negative and n is odd.

In this case each element of S_k in (6) is +1 or −1. Then it follows from (7) that λ_k^- and λ_k^+ are evaluated at opposite edge matrices of A^I. Alternatively, λ_k^- and λ_k^+ respectively are roots of two opposite edge polynomials of $C(z)$. On the other hand, consider the real eigenvector $\mathbf{x}_k$ and real reciprocal eigenvector $\mathbf{y}_k$ corresponding to the eigenvalue λ_k of A_c. From

$$A_c\mathbf{x}_k = \lambda_k \mathbf{x}_k \tag{8}$$

we can assume that

$$\mathbf{x}_k = \begin{bmatrix} 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{n-1} \end{bmatrix}^T$$

for each k. Then $\mathbf{x}_k$ s have the same sign patterns. Since the last row of S_k in (6) is determined by $\mathbf{x}_k$ and the last element of $\mathbf{y}_k$, the last row of each S_k is given by $[1\ -1\ 1\ -1\cdots]$ up to the multiplication of -1. Hence λ_k^- s and λ_k^+ s are roots of the following two edge polynomials of $C(z)$

$$p(z) = z^n + c_{n-1}^- z^{n-1} + c_{n-2}^+ z^{n-2} + c_{n-3}^- z^{n-3} + \cdots + c_0^-, \tag{9}$$
$$q(z) = z^n + c_{n-1}^+ z^{n-1} + c_{n-2}^- z^{n-2} + c_{n-3}^+ z^{n-3} + \cdots + c_0^+. \tag{10}$$

As noted above, if λ_k^- is a root of $p(z)$, then λ_k^+ is a root of $q(z)$, and vice versa.

Let $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_n$ be the roots of $p(z)$, where $\tilde{\lambda}_i \in \{\lambda_i^-, \lambda_i^+\}$. Replace c_0^- in (9) by $c_0^- + k$ and consider the root loci for k>0. As is well known [6], the real intervals $(-\infty, \tilde{\lambda}_1]$ and $[\tilde{\lambda}_{2i}, \tilde{\lambda}_{2i+1}]$, $1 \le i \le (n-1)/2$, are parts of the root loci for k>0. Then $\tilde{\lambda}_1, \tilde{\lambda}_3, \cdots, \tilde{\lambda}_n$ decrease, but $\tilde{\lambda}_2, \tilde{\lambda}_4, \cdots, \tilde{\lambda}_{n-1}$ increases as k starts to increase from zero. Hence we have

$$\tilde{\lambda}_{2i-1} = \lambda_{2i-1}^+, \ 1 \le i \le \frac{n+1}{2},$$
$$\tilde{\lambda}_{2i} = \lambda_{2i}^-, \ \ 1 \le i \le \frac{n-1}{2},$$

i.e.,

$$p(z) = (z - \lambda_1^+)(z - \lambda_2^-)(z - \lambda_3^+)\cdots(z - \lambda_n^+), \tag{11}$$
$$q(z) = (z - \lambda_1^-)(z - \lambda_2^+)(z - \lambda_3^-)\cdots(z - \lambda_n^-). \tag{12}$$

Case 2: λ_i^I s are all negative and n is even.
For this case, let

$$p(z) = z^n + c_{n-1}^+ z^{n-1} + c_{n-2}^- z^{n-2} + c_{n-3}^+ z^{n-3} + \cdots + c_0^-, \tag{13}$$
$$q(z) = z^n + c_{n-1}^- z^{n-1} + c_{n-2}^+ z^{n-2} + c_{n-3}^- z^{n-3} + \cdots + c_0^+. \tag{14}$$

Then, using the same arguments, we can show that

$$p(z) = (z - \lambda_1^-)(z - \lambda_2^+)(z - \lambda_3^-)\cdots(z - \lambda_n^+), \tag{15}$$
$$q(z) = (z - \lambda_1^+)(z - \lambda_2^-)(z - \lambda_3^+)\cdots(z - \lambda_n^-). \tag{16}$$

Case 3: λ_i^I s are all positive.

In this case λ_k^- s and λ_k^+ s are roots of the following two edge polynomials of $C(z)$

$$p(z) = z^n + c_{n-1}^- z^{n-1} + c_{n-2}^- z^{n-2} + \cdots + c_0^-, \tag{17}$$
$$q(z) = z^n + c_{n-1}^+ z^{n-1} + c_{n-2}^+ z^{n-2} + \cdots + c_0^+. \tag{18}$$

Again, using the root locus theory, we get the same results, i.e., (11) and (12) hold for n odd, and (15) and (16) hold for n even.

Hence, for $C(z)$ with real interval poles $\lambda_1^I < \lambda_2^I < \cdots < \lambda_n^I$, if λ_i^I s are either all negative or all positive, then two opposite edge polynomials of $C(z)$ can be determined from the minima and maxima of its interval poles. The results can be used for the case where the denominator polynomial has positive and negative interval poles simultaneously. Suppose $\lambda_1^I < \lambda_2^I < \cdots < \lambda_m^I < 0$ and $0 < \lambda_{m+1}^I < \lambda_{m+2}^I < \cdots < \lambda_n^I$. Then the coefficients of the interval polynomial can be obtained by solving linear algebraic equations. For example, let n=2 and assume $\lambda_1^I < 0$, $0 < \lambda_2^I$. If λ_1^I and λ_2^I have been exactly computed by (7), then we have

$$\lambda_1^{-2} + c_1^+ \lambda_1^- + c_0^- = 0, \tag{19}$$
$$\lambda_1^{+2} + c_1^- \lambda_1^+ + c_0^+ = 0, \tag{20}$$
$$\lambda_2^{+2} + c_1^- \lambda_2^+ + c_0^- = 0, \tag{21}$$
$$\lambda_2^{-2} + c_1^+ \lambda_1^- + c_0^+ = 0, \tag{22}$$

where the first two equations follow from (13)-(16), and the last two equations are derived from (15)-(18). Solving (19)-(22), we obtain the coefficients of the second-order interval polynomial.

3.2. Model reduction

The results obtained above can be directly applied to the model reduction problem. The denominator polynomial of a kth-order reduced model can be easily determined by computing two edge polynomials from k real dominant poles, $\lambda_1^I < \lambda_2^I < \cdots < \lambda_k^I$, to be retained in the reduced model. If λ_i^I s are either all negative or all positive, then two opposite edge polynomials of the kth-order denominator polynomial $C_k(z)$ can be computed as in (11), (12) (k odd) or (15), (16) (k even). Otherwise, equations of the form (19)-(22) can be used to obtain the coefficients of $C_k(z)$.

The polynomial with those interval poles may or may not exist. If such a polynomial exists, it can be exactly derived. Otherwise we obtain a polynomial that closely approximates dominant poles.

Example: Consider a third-order interval system with the denominator polynomial

$$C(z) = z^3 + [1.82, 1.821]z^2 + [0.908, 0.91]z + [0.0736, 0.0738]. \quad (23)$$

Each fixed-coefficients polynomial of $C(z)$ has real, negative, distinct roots [7]. Hence interval poles of (23) can be exactly computed by (7) as

$$\lambda_1^I = [-0.92988, -0.896738],$$
$$\lambda_2^I = [-0.82362, -0.79076],$$
$$\lambda_3^I = [-0.10037, -0.09965].$$

It is easily seen that two opposite edge polynomials of $C(z)$ are represented by

$$(z-\lambda_1^+)(z-\lambda_2^-)(z-\lambda_3^+) = z^3 + 1.82z^2 + 0.91z + 0.0736, \quad (24)$$

$$(z-\lambda_1^-)(z-\lambda_2^+)(z-\lambda_3^-) = z^3 + 1.821z^2 + 0.908s + 0.0738. \quad (25)$$

The second-order model retaining two dominant poles, λ_1^I and λ_2^I, is to be found. Applying the interval arithmetic as in [1] leads to

$$C_2(z) = z^2 + [1.6874, 1.7535]z + [0.7091, 0.7658]. (26)$$

The interval polynomial in (26) is not stable. For example, $z^2 + 1.7535z + 0.7091$ has an unstable pole at $z = -1.1209$. Conversely, we have from $(\lambda_1^-, \lambda_2^+)$ and $(\lambda_1^+, \lambda_2^-)$

$$(z-\lambda_1^-)(z-\lambda_2^+) = z^2 + 1.7206z + 0.7353, \quad (27)$$
$$(z-\lambda_1^+)(z-\lambda_2^-) = z^2 + 1.7203z + 0.7386. \quad (28)$$

Then we obtain the second-order interval polynomial

$$\hat{C}_2(z) = z^2 + [1.7203, 1.7206]z + [0.7353, 0.7386] (29)$$

with the interval poles λ_1^I and λ_2^I.

4. CONCLUSIONS

In this paper it was pointed out that a recently proposed method of model reduction for discrete interval systems suffers a stability problem. To partially overcome such a stability problem, it was shown that the denominator polynomial with real interval poles can be represented from its exactly computed poles. Based on this result, a simple technique was presented to obtain the stable reduced model from the stable interval system by accurately preserving desired dominant real interval poles. Only interval systems with real interval poles were considered in the paper. Further research is required to deal with more general interval systems.

REFERENCES

[1] O. Ismail, B. Bandyopadhyay, and R. Gorez, "Discrete interval system reduction using Pade approximation to allow retention of dominant poles," *IEEE. Trans. on Circuits and Systems I,* vol. 44, no. 11, pp. 1075-1078, 1997.

[2] Y. Dolgin and E. Zeheb "On Routh-Pade model reduction of interval systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 9, pp. 1610-1612, 2003.

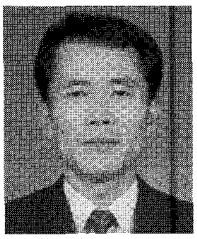
[3] S. F. Yang, "Comments on 'On Routh-Pade model reduction of interval systems'," *IEEE Trans. on Automatic Control*, vol. 50, no. 2, pp. 273-274, 2005.

[4] A. Deif, "The interval eigenvalue problem," *Z. Angew. Math. Mech.*, vol. 71, pp. 61-64, 1991.

[5] G. Alefeld and J. Herzberger, *Introduction to Interval Computations,* Academic Press, 1983.

[6] B. C. Kuo, *Automatic Control Systems,* Prentice Hall, 1995.

[7] C. B. Soh and C. S. Berger, "Strict aperiodic property of polynomials with perturbed coefficients," *IEEE Trans. on Automatic Control*, vol. 34, no. 5, pp. 546-548, 1989.



Younseok Choo was born in Daejeon, Korea, on Oct. 20, 1957. He received the B.S. degree in Electrical Engineering from Seoul National University in 1980, and the M.S. and Ph.D. degrees from the University of Texas, Austin, U.S.A., in 1991 and 1994, respectively. From Dec., 1979 to Aug., 1989, he worked for ADD as a Researcher. He held the Postdoctoral position at ETRI from Sep., 1994, to Feb., 1995. Since March 1995, he has been with the Department of Electronic and Electrical Engineering, Hongik University, Korea, where he is currently a Professor. His research interests are in the areas of stochastic control, adaptive control and model reduction.