Robust Stabilization of Decentralized Dynamic Surface Control for a Class of Interconnected Nonlinear Systems

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Abstract: The analysis and design method for achieving robust stabilization of Decentralized Dynamic Surface Control (DDSC) is presented for a class of interconnected nonlinear systems. While a centralized design approach of DSC was developed in [1], the decentralized approach to deal with large-scale interconnected systems is proposed under the assumption that interconnected functions among subsystems are unknown but bounded. To provide a closed-loop form with provable stability properties, augmented error dynamics for N nonlinear subsystems with DDSC are derived. Then, the reachable set for errors of the closed-loop systems will be approximated numerically in the form of an ellipsoid in the framework of convex optimization. Finally, a numerical algorithm to calculate the L_2 gain of the augmented error dynamics is presented.

Keywords: Decentralized control, dynamic surface control, interconnected systems, reachable set, L_2 gain.

1. INTRODUCTION

Due to dramatic advances in the field of electronics and mechatronics, complexity of control systems increases rapidly. Moreover, large-scale distributed systems have been paid much attention recently with applications to intelligent transportation systems, semiconductor manufacturing systems, and intelligent building and power plants. A decentralized control approach for the large-scale distributed systems was motivated in 1970s to design a set of local controllers to reduce the burden of both computation of a controller and communication of information among subsystems (e.g., see in [2,3] and references therein).

More specifically, while the decentralized adaptive control was developed for the linear system subject to nonlinear interactions among sub-systems [4-6], the decentralized adaptive output feedback design was proposed in the literature for a system of which all states are not measured [7,8]. Although significant progress in the area of decentralized control has been made for last three decades, in recent the decentralized approaches for a class of nonlinear systems were proposed in 1990s. For instance, decentralized adaptive control techniques for the interconnected nonlinear subsystems have been found in the literature [9,10]. However, most of the work is based on an integrator backstepping technique whose complexity due to an 'explosion of terms' may result in difficulties for control synthesis [11].

Among many Lyapunov-based nonlinear control techniques in the literature, a nonlinear control method called Dynamic Surface Control (DSC) [11] was developed to reduce the complexity of integrator backstepping control [12] and mathematical difficulties for analysis of the sliding mode control due to discontinuous functions [13]. The systematic analysis and design method was developed by Song *et al.* [1] in the framework of convex optimization. Furthermore, for the case that all states of the system are not measured, a separation principle for the observer-based DSC was proposed in the form of diagonal norm-bound Linear Differential Inclusions (LDIs) of the augmented error dynamics [14].

The main contribution of this paper is to extend a centralized DSC design methodology to Decentralized Dynamic Surface Control (DDSC) design for a class of interconnected nonlinear systems within the framework of convex optimization. While the idea of DDSC was originally proposed in [15], more detailed mathematical derivations of DDSC are described and its stability and performance are discussed in terms of L_2 gains as well as reachable sets of the error dynamics. The remainder of the paper is divided as follows; While the problem statement is given in Section 2, Section 3 will present a preliminary design

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procedure of DDSC for N subsystems and the augmented closed-loop error dynamics are derived in forms of diagonal norm-bound LDIs [16]. Then, a Linear Matrix Inequality (LMI)-based design methodology to estimate the stability and performance of the DDSC systems will be proposed in Section 3. Finally, an illustrative example will be presented in Section 4.

2. PROBLEM STATEMENT

The interconnected system composed of N *strict-feedback* single-input nonlinear subsystems is considered as follows; for $i = 1,2,\dots,N$,

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{U}_i\mathbf{x}_i + \mathbf{B}_i u_i + \mathbf{f}_i(\mathbf{x}_i) + \sum_{j=1, j\neq i}^{N} \mathbf{g}_{ij}(\mathbf{x}_j) \\ &= \mathbf{U}_i\mathbf{x}_i + \mathbf{B}_i u_i + \mathbf{f}_i(\mathbf{x}_i) + \mathbf{h}_i(\dots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\dots),\end{aligned} \tag{1a}$$

$$y_i = x_{i1} = \mathbf{C}_i\mathbf{x}_i, \tag{1b}$$

where $\mathbf{x}_i = [x_{i1},\dots,x_{in_i}]^T \in \mathbb{R}^{n_i}$ and $\mathbf{x}_j \in \mathbb{R}^{n_j}$ are the state of the ith and jth nonlinear subsystem respectively, and $u_i, y_i \in \mathbb{R}$ are the inputs and outputs of the subsystem respectively. Moreover, the nonlinear function vector $\mathbf{g}_{ij}$ represents the interconnection among subsystems, $\mathbf{h}_i = \sum_{j=1,j\neq i}^{N} \mathbf{g}_{ij}(\mathbf{x}_j) \in \mathbb{R}^{n_i}$, and the matrices are following

$$\mathbf{U}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_i \times n_i},$$

$$\mathbf{B}_i = [0 \ \cdots \ 0 \ 1]^T \in \mathbb{R}^{n_i \times 1},$$

$$\mathbf{C}_i = [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times n_i},$$

where the matrix $\mathbf{U}_i$ is a square matrix whose first super-diagonal elements are one and elsewhere zero, i.e., $\mathbf{U}_i = diag([1,\dots,1],1) \in \mathbb{R}^{n_i \times n_i}$. It is noted that $diag(x,i)$ [or $diag(x,-i)$] denotes a square matrix of size ($n+i$) with the vector x forming the ith super-diagonal [or sub-diagonal] of the matrix, and $diag(x,0) = diag(x)$ stands for a diagonal matrix with the vector x forming the diagonal.

Furthermore, (1a) can be rewritten component-wise as follows: for $1 \le k \le n_i$,

$$\begin{aligned}\dot{x}_{ik} &= x_{i(k+1)} + f_{ik}(x_{i1},\dots,x_{ik}) \\ &\quad + h_{ik}(\mathbf{x}_1,\dots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\dots,\mathbf{x}_N),\end{aligned} \tag{2}$$

where $x_{i(n_i+1)} := u_i$ and h_{ik} is the kth function of $\mathbf{h}_i$. The 'strict-feedback' is said in the sense that the nonlinearities f_{ik}, the k-th element of the nonlinear function vector $\mathbf{f}_i$, depend only upon $x_{i1},\dots,x_{ik}$ [12].

The additional assumptions for the system are as follows:

A-i: Each nonlinear function $f_{ik}: \mathbb{R}^k \to \mathbb{R}$, is a C^∞ function, Lipschitz, and $f_{ik}(0,\dots,0) = 0$.

A-ii: The interconnected nonlinear function $\mathbf{g}_{ij}: \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$ is unknown but bounded such that

$$\mathbf{h}_i^T\mathbf{h}_i \le c, \tag{3}$$

where c is a known positive constant.

A-iii: $\mathbf{x}_j$ is known if $j = i$ in the ith subsystem, but not if $j \neq i$.

The constraint for unknown interconnected functions shown in the *A-ii* is necessary to prove the quadratic stability of the decentralized robust control approach proposed later. The additional constraint for the unknown interconnected function is generally considered for the decentralized robust control approach (e.g., see in [6]), while the system parameters for a linear combination of the known interconnected functions are assumed to be unknown in the decentralized adaptive control approach (e.g., see in [8,9]). However, if the constraint in the *A-ii* is too strict to apply for a class of systems, it can be relaxed in the local domain with guaranteed local stability. Furthermore, it is noted that a class of large-scale nonlinear systems can be transformed to (1a) via a global diffeomorphism [12].

3. STABILIZATION OF DDSC

The analysis and design methodology of DDSC is proposed in this section. After the preliminary design procedure is described, the augmented error dynamics of the N closed-loop nonlinear systems are derived, thus enabling us to provide a systematic method for analyzing the closed loop system. In consequence, algorithms for approximating reachable sets of error trajectory in the form of an ellipsoid and for calculating an L_2 gain will be developed in the framework of LMI.

3.1. Design procedure of DDSC

Suppose the objective is $x_{i1} \to x_{i1,d} = 0$ for the stabilization problem. First, define the ith error surface as $S_{ik} = x_{ik} - x_{ik,d}$ for $1 \le k \le n_i - 1$. After differentiating the error surface and using (2), we get

$$\dot{S}_{ik} = \dot{x}_{ik} - \dot{x}_{ik,d} = x_{i(k+1)} + f_{ik} + h_{ik} - \dot{x}_{ik,d}.$$

If $x_{i(k+1)}$ is considered as the synthetic input to force S_{ik} to converge within an arbitrarily small bound, then the ultimate boundedness is achieved if $x_{i(k+1)} = \bar{x}_{i(k+1)}$ as follows:

$$\bar{x}_{i(k+1)} := -f_{ik} - h_{ik} + \dot{x}_{ik,d} - \lambda_{ik} S_{ik},$$

where $\lambda_{ik} > 0$ is the controller gain chosen later to guarantee the quadratic stability and boundedness [1,11]. However, since it is assumed that h_{ik} is unknown, the synthetic input is

$$\bar{x}_{i(k+1)} := -f_{ik} + \dot{x}_{ik,d} - \lambda_{ik} S_{ik}. \tag{4}$$

It will be shown in Section 3.3 that the synthetic input defined in (4) makes the closed-loop subsystem quadratically stable.

Next, in order to force $x_{i(k+1)} \to \bar{x}_{i(k+1)}$, define $S_{i(k+1)} := x_{i(k+1)} - x_{i(k+1),d}$ where $x_{i(k+1),d}$ equals $\bar{x}_{i(k+1)}$ passed through a first order low-pass filter, i.e.,

$$\tau_{i(k+1)} \dot{x}_{i(k+1),d} + x_{i(k+1),d} = \bar{x}_{i(k+1)}, \tag{5}$$

where $x_{i(k+1),d}(0) := \bar{x}_{i(k+1)}(0)$. After continuing this procedure up to $k = n_i - 1$, define $S_{in_i} := x_{in_i} - x_{in_i,d}$. Finally, the desired control input is chosen as

$$u_i = -f_{in_i} + \dot{x}_{in_i,d} - \lambda_{in_i} S_{in_i}, \tag{6}$$

where $\dot{x}_{in_i,d}$ is calculated as $\dot{x}_{in_i,d} = (\bar{x}_{in_i} - x_{in_i,d}) / \tau_{in_i}$ using (5).

Once the proposed controllers are implemented in the system, the overall layout of the closed-loop system can be described as shown in Fig. 1. It is noted that the complexity of the proposed controller is reduced dramatically compared with integrator backstepping control (see (6)), while the dimension of the closed-loop subsystem with a local controller becomes higher due to the inclusion of the low-pass filter dynamics (see (5)), i.e., the dimension becomes $2n_i - 1$.

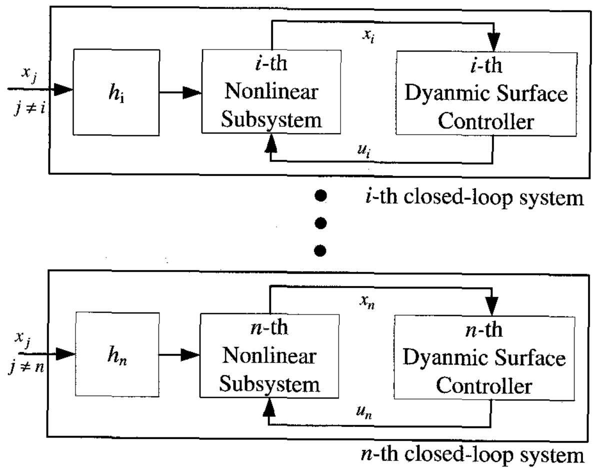

Fig. 1. Schematic diagram of interconnected systems with DDSC.

3.2. Augmented error dynamics

Similar with the DSC design methodology developed by Song *et al.* [1], a set of augmented error dynamics can be derived for the closed-loop interconnected systems. First, after subtracting and adding $x_{(i+1)d}$ and $\bar{x}_{i+1}$ in (2), (2) is written as

$$\begin{aligned}\dot{x}_{ik} = [x_{i(k+1)} - x_{i(k+1),d}] + [x_{i(k+1),d} - \bar{x}_{i(k+1)}] \\ + \bar{x}_{i(k+1)} + f_{ik} + h_{ik}.\end{aligned} \tag{7}$$

Then, using $\bar{x}_{i+1}$ in (4), the definitions of S_i, and the filter error defined as $\xi_{i(k+1)} := x_{i(k+1),d} - \bar{x}_{i(k+1)}$, (7) is rewritten as follows,

$$\dot{x}_{ik} = S_{i(k+1)} + \xi_{i(k+1)} + \dot{x}_{ik,d} - \lambda_{ik} S_{ik} + h_{ik}.$$

After iterating up to $k = n_i$, it is rewritten as for $1 \le k \le n_i - 1$,

$$\dot{S}_{ik} = S_{i(k+1)} + \xi_{i(k+1)} - \lambda_{ik} S_{ik} + h_{ik}, \tag{8a}$$

for $k = n_i$,

$$\dot{S}_{ik} = -\lambda_{ik} S_{ik} + h_{ik}. \tag{8b}$$

Next, since the low-pass filters are added for DDSC, the filter error dynamics should also be included. Differentiating the filter error, $\xi_i \in \mathbb{R}^{n_i - 1}$, and using (4), the filter error dynamics are for $1 \le k \le n_i - 1$,

$$\begin{aligned}\dot{\xi}_{i(k+1)} &= \frac{d}{dt}(x_{i(k+1),d} - \bar{x}_{i(k+1)}) \\ &= -\frac{1}{\tau_{i(k+1)}} \xi_{i(k+1)} + \dot{f}_{ik} - \ddot{x}_{ik,d} + \lambda_{ik} \dot{S}_{ik}.\end{aligned} \tag{9}$$

Since $\ddot{x}_{i1,d} = 0$ for the stabilization problem, (9) can be rewritten as follows;
for $k = 1$,

$$-\lambda_{ik} \dot{S}_{ik} + \dot{\xi}_{i(k+1)} = -\frac{1}{\tau_{i(k+1)}} \xi_{i(k+1)} + \dot{f}_{ik}, \tag{10a}$$

for $2 \le k \le n_i - 1$,

$$-\lambda_{ik}\dot{S}_{ik} + \dot{\xi}_{i(k+1)} - \frac{1}{\tau_{ik}}\dot{\xi}_{ik} = -\frac{1}{\tau_{i(k+1)}}\xi_{i(k+1)} + \dot{f}_{ik}. \quad (10b)$$

Finally, combining (10) with (8), we have the augmented error dynamics in the form of block matrices as follows:

$$\begin{bmatrix} \mathbf{I}_{n_i} & 0 \\ -\mathbf{T}_{S_i} & \mathbf{T}_{\xi_i} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{S}}_i \\ \dot{\xi}_i \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{I}_{n_i-1} \\ 0 & \mathbf{A}_{i2} \end{bmatrix}\begin{bmatrix} \mathbf{S}_i \\ \xi_i \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{I}_{n_i-1} \end{bmatrix}\mathbf{p}_i + \begin{bmatrix} \mathbf{I}_{n_i} \\ 0 \end{bmatrix}\mathbf{h}_i, \quad (11)$$

where

$$\mathbf{S}_i := [S_{i1}, \ldots, S_{in_i}]^T \in \mathbb{R}^{n_i},$$
$$\xi_i := [\xi_{i2}, \ldots, \xi_{in_i}]^T \in \mathbb{R}^{n_i-1},$$
$$\mathbf{p}_i := [\dot{f}_1, \ldots, \dot{f}_{n_i-1}]^T \in \mathbb{R}^{n_i-1},$$

and the submatrices are the following:

$$\mathbf{A}_{i1} := diag([1,\ldots,1],1) - diag(\lambda_{i1},\ldots,\lambda_{in_i}) = U_i - diag(\lambda_{i1},\ldots,\lambda_{in_i}),$$
$$\mathbf{A}_{i2} := -diag(1/\tau_{i2},\ldots,1/\tau_{in_i}),$$
$$\mathbf{T}_{S_i} := \left[diag(\lambda_{i1},\ldots,\lambda_{i(n_i-1)})0_{n_i-1} \right] \in \mathbb{R}^{(n_i-1)\ n_i},$$
$$\mathbf{T}_{\xi_i} := \mathbf{I}_{n_i-1} + \Gamma_i, \Gamma_i = diag\left(\left[-\frac{1}{\tau_{i2}}, \ldots, -\frac{1}{\tau_{i(n_i-1)}} \right], -1 \right).$$

Since the matrix on the left hand side of (11) is invertible with an inverse given by (refer to [1]):

$$\begin{bmatrix} \mathbf{I}_{n_i} & 0 \\ -\mathbf{T}_{S_i} & \mathbf{T}_{\xi_i} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_{n_i} & 0 \\ \mathbf{T}_{\xi_i}^{-1}\mathbf{T}_{S_i} & \mathbf{T}_{\xi_i}^{-1} \end{bmatrix},$$

the augmented closed loop error dynamics of N nonlinear subsystems with DDSC can be reformulated as

$$\dot{\mathbf{z}}_i = \mathbf{A}_i\mathbf{z}_i + \mathbf{B}_{pi}\mathbf{p}_i + \mathbf{B}_{hi}\mathbf{h}_i, \quad (12a)$$
$$y_i = [\mathbf{C}_i 0]\mathbf{z}_i = \tilde{\mathbf{C}}_i\mathbf{z}_i, \quad (12b)$$

where $\mathbf{z}_i := [\mathbf{S}_i^T \xi_i^T]^T \in \mathbb{R}^{2n_i-1}$, $y_i = x_{i1} = S_{i1}$ for the stabilization problem, and the corresponding matrices are

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{I} \\ T_{\xi_i}^{-1}T_{S_i}\mathbf{A}_{i1} & T_{\xi_i}^{-1}(T_{S_i}\mathbf{I} - \mathbf{A}_{i2}) \end{bmatrix},$$

$$\mathbf{B}_{pi} = \begin{bmatrix} 0 \\ T_{\xi}^{-1} \end{bmatrix}, \text{ and } \mathbf{B}_{hi} = \begin{bmatrix} \mathbf{I}_{n_i} \\ T_{\xi_i}^{-1}T_{S_i} \end{bmatrix}.$$

Remark 1: While the error dynamics of centralized DSC for the stabilization problem are derived in the form of $\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{B}_p\mathbf{p}$ (refer to [1]), the closed-loop error dynamics in (12a) are extended with an additional term, $\mathbf{h}_i$. Furthermore, following terminologies used in [16, § 4], it can be regarded as a diagonal norm-bound Linear Differential Inclusion (LDI), if there exists a matrix C_{z_i} such that $|\mathbf{p}_i| \le |C_{z_i}\mathbf{z}_i|$.

Regarding the assumption (A-i) that $\mathbf{f}_i$ is sufficiently smooth and motivating to use linear mapping theory [17], one approach to find the linear mapping is to obtain a linear upper bound of $\mathbf{p}_i$ through linear approximation. That is, there exists a constant row matrix L_i such that for $i = 1, \ldots, n-1$

$$\begin{aligned} |\mathbf{p}_i| &\le |L_i\mathbf{x}_i| \\ &\le |\nu_i(S_{i1},\ldots,S_{in_i},\xi_{i2},\ldots,\xi_{in_i},\lambda_{i1},\ldots,\lambda_{i(n_i-1)}, \tau_{i2},\ldots,\tau_{i(n_i-1)})| \\ &\le |C_{zi}(\Lambda_i,\tau_i)\mathbf{z}_i|, \end{aligned}$$

where ν_i is used to denote a function at the i-th step of the induction. The last inequality above comes from linear approximation of the function ν_i. For more detail, the reader is referred to [1,11].

Remark 2: The overall augmented error dynamics of N nonlinear subsystems with DDSC are written as follows:

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \vdots \\ \dot{\mathbf{z}}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{A}_N \end{bmatrix}\begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_N \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{p1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{B}_{pN} \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_N \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{h1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{B}_{hN} \end{bmatrix}\begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_N \end{bmatrix}.$$

If the centralized DSC proposed in [1] is used, the calculation of a positive definite matrix $P \in \mathbb{R}^{M \times M}$ where $M := \sum_{i=1}^{N} n_i$ for showing the existence of a quadratic Lyapunov function candidate is necessary to guarantee the quadratic stabilization. This will be a heavier burden for appropriate assignment of controller gains and filter time constants compared with the decentralized approach (i.e., the dimension of P for DDSC is n_i). This is a main motivation to develop DDSC.

3.3. Reachable sets and L_2 gain

While the quadratic stability for the DSC was proposed to guarantee the stability of DSC systems and tested numerically in [1], reachable sets for DDSC systems will be estimated to predict the upper bound of errors of DDSC systems which result from the unknown interconnected functions in (1a). First, we need to define the reachable set as follows:

Definition 1 (also see in [16]): Suppose $\Lambda_i = \{\lambda_{i1},\ldots,\lambda_{in_i}\}$ and $\tau_i = \{\tau_{i2},\ldots,\tau_{in_i}\}$ are given. A set, R_{up} described in (13) below is called *reachable sets* of the augmented error dynamics given by (12a)

$$R_{up} = \left\{\mathbf{z}_i(T)\middle|\mathbf{z}_i,\mathbf{p}_i,\mathbf{h}_i \text{ satisfy (12a)}, \mathbf{z}_i(0)=0, T\geq 0\right\}. \tag{13}$$

Definition 2 (also see in [16]): Suppose that there exists a quadratic function $V(\mathbf{z}_i)=\mathbf{z}_i^T P\mathbf{z}_i$ with $P\succ 0$, and $dV(\mathbf{z}_i)/dt\leq 0$ for all $\mathbf{z}_i,\mathbf{p}_i,\mathbf{h}_i$ satisfying (12a), $\mathbf{h}_i^T\mathbf{h}_i\leq 1$ and $V(\mathbf{z}_i)\geq 1$. Then, the ellipsoid $E=\{\mathbf{z}_i \in \mathbb{R}^{n_i} \mid \mathbf{z}_i^T P\mathbf{z}_i\leq 1\}$ contains the reachable set R_{up}.

Following the assumption *A-ii* in (3), the augmented error dynamics shown in (12a) can be rewritten as

$$\dot{\mathbf{z}}_i = \mathbf{A}_i\mathbf{z}_i + \mathbf{B}_{pi}\mathbf{p}_i + \tilde{\mathbf{B}}_{hi}\mathbf{u}_i, \tag{14}$$

where $\mathbf{u}_i$ is a unit-peak function such as $\mathbf{u}_i^T\mathbf{u}_i\leq 1$, i.e., $\mathbf{u}_i := \mathbf{h}_i/\sqrt{c}$, and $\tilde{\mathbf{B}}_{hi} := \sqrt{c}\mathbf{B}_{hi}$. Once the diagonal norm-bound condition $|\mathbf{p}_i|\leq|C_{zi}\mathbf{z}_i|$ are obtained and the assumption *A-ii* is satisfied, the closed-loop error dynamics in (14) can be regarded as a set of diagonal norm-bound LDIs (refer to [16, § 4]). Then, the ellipsoidal bound of the reachable sets is obtained as follows;

Theorem 1: Suppose $\mathbf{h}_i$ is bounded with a known constant c in (3) and the matrices $\mathbf{A}_i$, $\mathbf{B}_{pi}$ and $\tilde{\mathbf{B}}_{hi}$ in (14) are given. The ellipsoid, $E=\{\mathbf{z}_i\in\mathbb{R}^{n_i} \mid \mathbf{z}_i^T P\mathbf{z}_i\leq 1\}$, contains the reachable sets of the augmented error dynamics in (14) if there exist a set of controller gains $\Lambda_i := \{\lambda_{i1},\ldots,\lambda_{in_i}\}$ and filter time constants $\tau_i := \{\tau_{i2},\ldots,\tau_{in_i}\}$ satisfying

i. there exists $\mathbf{C}_{zi}(\Lambda_i,\tau_i)$ such that $|\mathbf{p}_i|\leq|\mathbf{C}_{zi}\cdot\mathbf{z}_i|$,

ii. there exist P, Σ, and α such that

$P\succ 0$, $\Sigma\succeq 0$ and diagonal, $\alpha\geq 0$

$$\begin{bmatrix} \mathbf{A}_i^T P + P\mathbf{A}_i + \alpha P + \mathbf{C}_{zi}^T\Sigma\mathbf{C}_{zi} & P\mathbf{B}_{pi} & P\tilde{\mathbf{B}}_{hi} \\ \mathbf{B}_{pi}^T P & -\Sigma & 0 \\ \tilde{\mathbf{B}}_{hi}^T P & 0 & -\alpha I \end{bmatrix} \preceq 0. \tag{15}$$

It is noted that the proof of the theorem is derived from the quadratic tracking idea of the diagonal norm-bound linear differential inclusion in [1], and the interested readers can refer to [16, § 6] for a detailed proof. Also it is remarked that inequality (15) is not an LMI in P, Σ, and α. However, it is an LMI in P and Σ for fixed α.

Remark 3: While it is assumed that the interconnection function $\mathbf{h}_i$ satisfies a unit-peak condition, other assumptions can be used instead. For instance, if $\mathbf{h}_i$ is a unit-energy such that $\int_0^T \mathbf{h}_i^T\mathbf{h}_i dt\leq 1$, the matrix inequality condition like the second condition in Theorem 1 can be derived [16]. Furthermore, to obtain the smallest outer approximation of the reachable set, we need to minimize a maximum diameter of the ellipsoid. That is, to minimize the objective function $\lambda_{max}(P^{-1})$ [1].

Definition 3 (also see in [16]): When the L_2 gain of the augmented error dynamics in (12a) is defined as

$$\sup_{\|\mathbf{h}_i\|_2\neq 0}\frac{\|y_i\|_2}{\|\mathbf{h}_i\|_2},$$

where the L_2 norm of x is $\|x\|_2^2=\int_0^\infty x^T x dt$, and supremum is taken over all nonzero trajectories of the augmented error dynamics, starting from $\mathbf{z}_i(0)=0$, the augmented error dynamics are said to be *nonexpansive* if its L_2 gain is less than one.

Suppose there exists a quadratic function $V(\zeta_i) = \zeta_i^T P\zeta_i$, $P\succ 0$, and $\gamma\geq 0$ such that for all t,

$$\frac{d}{dt}V(\mathbf{z}_i) + y_i^T y_i - \gamma^2\mathbf{h}_i^T\mathbf{h}_i\leq 0$$

for all $\mathbf{z}_i$ and $\mathbf{h}_i$ satisfying (12a). Then the L_2 gain of the augmented error dynamics is less than γ [16]. The above condition is equivalent to

$$\begin{aligned} &\zeta_i^T(\mathbf{A}_i^T P + P\mathbf{A}_i + \tilde{\mathbf{C}}_i^T\tilde{\mathbf{C}}_i)\zeta_i \\ &+2\zeta_i^T P(\mathbf{B}_{pi}\pi_i + \mathbf{B}_{hi}\mathbf{h}_i) - \gamma^2\mathbf{h}_i^T\mathbf{h}_i\leq 0 \end{aligned} \tag{16}$$

for all ζ_i and π_i satisfying

$$\pi_{i,j}^T \pi_{i,j} \le (\mathbf{C}_{zi,j}\zeta_{i,j})^T (\mathbf{C}_{zi,j}\zeta_{i,j}),\ \ j = 1,\dots,n_{pi}.$$

Furthermore, the condition (16) is equivalent to

$$P \succ 0,\ \Sigma \succeq 0 \text{ and diagonal,}$$

$$\begin{bmatrix} \mathbf{A}_i^T P + P\mathbf{A}_i + \tilde{\mathbf{C}}_i^T \tilde{\mathbf{C}}_i + \mathbf{C}_{zi}^T \Sigma \mathbf{C}_{zi} & P\mathbf{B}_{pi} & P\mathbf{B}_{hi} \\ \mathbf{B}_{pi}^T P & -\Sigma & 0 \\ \mathbf{B}_{hi}^T P & 0 & -\gamma^2 I \end{bmatrix} \preceq 0. \tag{17}$$

Theorem 2: Suppose the matrices $\mathbf{A}_i$, $\mathbf{B}_{pi}$, $\mathbf{B}_{hi}$, and $\tilde{\mathbf{C}}_i$ in (12) are determined for a given set of Λ_i and τ_i. The augmented error dynamics in (12) is nonexpansive

i. there exists $\mathbf{C}_{zi}(\Lambda_i, \tau_i)$ such that $|\mathbf{p}_i| \le |\mathbf{C}_{zi}\cdot \mathbf{z}_i|$ and,

ii. if there exist $|\gamma|<1$, $P \succ 0$, and $\Sigma \succeq 0$ satisfying the condition (17).

Therefore, we compute the smallest upper bound on the L_2 gain of the augmented error dynamics provable via quadratic functions by minimizing γ over the variables P, Σ, and γ satisfying the condition (17).

4. ILLUSTRATIVE EXAMPLE

The stabilization problem of coupled inverted double pendulums shown in Fig. 2 [5,6] is considered to illustrate the proposed design technique. It can be modeled as a fourth-order differential equation of motion using Newton's law as follows:

$$\ddot{\theta}_1 = \frac{g}{l}\sin\theta_1 + u_1 + \frac{ka^2}{m_1 l^2}(\sin\theta_2\cos\theta_2 - \sin\theta_1\cos\theta_1),$$

$$\ddot{\theta}_2 = \frac{g}{l}\sin\theta_2 + u_2 + \frac{ka^2}{m_2 l^2}(\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2).$$

The model can be written in the form of (1) as follows:

$$\dot{\mathbf{x}}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x}_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ \frac{g}{l}\sin x_{i1} - \frac{ka^2}{m_i l^2}\sin x_{i1}\cos x_{i1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{ka^2}{m_i l^2}\sin x_{j1}\cos x_{j1} \end{bmatrix}_{j\ne i}$$

$$= U_i\mathbf{x}_i + B_i u_i + \mathbf{f}_i + \sum_{j=1, j\ne i}^{2} \mathbf{g}_{ij}(\mathbf{x}_j),$$

$$y_i = [1\ \ 0]\mathbf{x}_i = \mathbf{C}_i\mathbf{x}_i,$$

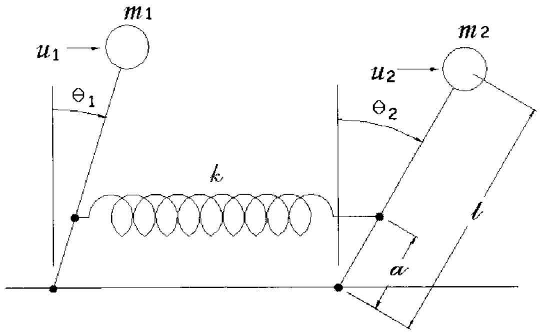

Fig. 2. Schematic of interconnected inverted pendulums.

where $\mathbf{x}_i = [\theta_i\ \dot{\theta}_i]^T$ and the assumptions *A-ii* are satisfied as follows:

$$h_i^T h_i = \left(\frac{ka^2}{m_i l^2}\right)^2 \sin^2 x_{j1}\cos^2 x_{j1} \le \left(\frac{1}{2}\frac{ka^2}{m_i l^2}\right)^2 = c. \tag{18}$$

Suppose that $a = \frac{g}{l} = 1$ and $b_i = \frac{ka^2}{m_i l^2} = 0.25$ for simulation, thus $c = 1/64$.

4.1. Design of DDSC

Define the first error surface as $S_{i1} := x_{i1}$, and the first low-pass filter as follows.

$$\dot{S}_{i1} = \dot{x}_{i1} = x_{i2} \Rightarrow \bar{x}_{i2} = -\lambda_{i1} S_{i1},$$
$$\tau_{i2}\dot{x}_{i2,d} + x_{i2,d} = \bar{x}_{i2}, x_{i2,d}(0) := \bar{x}_{i2}(0).$$

Next, defining the second surface as $S_{i2} := x_{i2} - x_{i2,d}$ and differentiating it, we get

$$\dot{S}_{i2} = u_i + \frac{g}{l}\sin x_{i1} - \frac{ka^2}{m_i l^2}\sin x_{i1}\cos x_{i1} - \dot{x}_{i2,d} + \frac{ka^2}{m_i l^2}\sin x_{j1}\cos x_{j1},$$

$$u_i = -\frac{g}{l}\sin x_{i1} + \frac{ka^2}{m_i l^2}\sin x_{i1}\cos x_{i1} + \dot{x}_{i2,d} - \lambda_{i2} S_{i2}.$$

Then, using $f_{i1} = 0$ in the given example, we can derive the simpler augmented closed loop error dynamics than these presented in (12) as follows:

$$\dot{z}_i = \mathbf{A}_i z_i + \mathbf{B}_{hi} h_i, \tag{19a}$$
$$y_i = \tilde{\mathbf{C}}_i z_i, \tag{19b}$$

where

$$z_i = [S_{i1}\ S_{i2}\ \xi_{i2}]^T \in \mathbb{R}^3, h_i = 2\sin x_{j1}\cos x_{j1} = \sin 2x_{j1},$$

and the submatrices are as follows:

$$\mathbf{A}_i = \begin{bmatrix} -\lambda_{i1} & 1 & 1 \\ 0 & -\lambda_{i2} & 0 \\ -\lambda_{i1}^2 & \lambda_{i1} & \lambda_{i1}-1/\tau_{i2} \end{bmatrix}, \ \mathbf{B}_{hi} = \begin{bmatrix} 0 \\ \dfrac{ka^2}{2m_i l^2} \\ 0 \end{bmatrix},$$

$$\tilde{\mathbf{C}}_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

It is noted that the constant c in (18) is reflected in $\mathbf{B}_{hi}$, so $h_i^T h_i = \sin^2 2x_{j1} \le 1$.

4.2. Estimation of reachable sets

When the DDSC gain set, $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}$, is given, a solution for the problem minimizing the largest diameter of ellipsoids subject to (15) is obtained by solving the following algorithm.

For a fixed $\alpha > 0$,

$$\begin{aligned} &\max \ t \\ &\text{subject to} \ \ P \succeq tI, \\ &\begin{bmatrix} \mathbf{A}_i^T P + P\mathbf{A}_i + \alpha P & P\mathbf{B}_{hi} \\ \mathbf{B}_{hi}^T P & -\alpha I \end{bmatrix} \preceq 0. \end{aligned} \tag{20}$$

It is remarked that the above LMI is simpler than LMI (15) because $p_i = 0$, thus the error dynamics in (19) are used.

When a set of the DDSC gains are given as $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=1} = \{1, 4, 0.02\}$ and $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=2} = \{2, 8, 0.02\}$, the ellipsoidal bound is calculated using both LMITOOL for a MATLAB-based graphical user interface [18] and SeDuMi for semidefinite programming [19]. When α is varied from 0.01 to 1.5, Fig. 3 shows the corresponding maximum diameter of the ellipsoidal error bound which is defined as $2\sqrt{\lambda_{\min}(P)}$. Since the maximum diameter increases as α does, the smallest ellipsoidal error bound is obtained for $\alpha = 0.01$ as follows;

$$E = \{z_i \in \mathbb{R}^3 \mid z_i^T P z_i \le 1\},$$

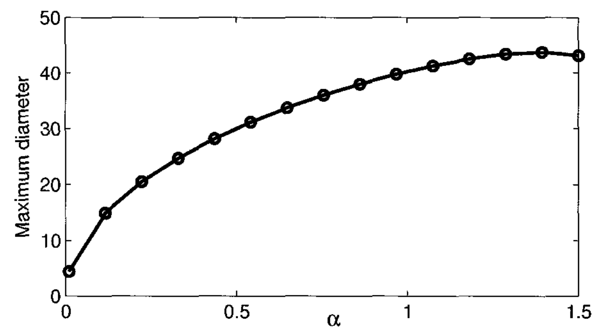

Fig. 3. Smallest ellipsoidal bound by minimizing the largest diameter through the line search of α.

where

$$P = \begin{bmatrix} 31.5121 & -6.5853 & 11.6885 \\ -6.5853 & 6.6376 & -12.2990 \\ 11.6885 & -12.2990 & 661.5193 \end{bmatrix}. \tag{21}$$

4.3. L_2 gain of DDSC

When the DDSC gain set, $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}$, is given, a solution for the problem minimizing the L_2 gain of augmented error dynamics subject to (17) is obtained by solving the following algorithm

$$\begin{aligned} &\min \ \gamma \\ &\text{subject to} \ \ P \succeq 0, \\ &\begin{bmatrix} \mathbf{A}_i^T P + P\mathbf{A}_i + \tilde{\mathbf{C}}_i^T \tilde{\mathbf{C}}_i & P\mathbf{B}_{hi} \\ \mathbf{B}_{hi}^T P & -\gamma^2 I \end{bmatrix} \preceq 0. \end{aligned} \tag{22}$$

For the given set of DDSC gain used above, the L_2 gain of the augmented error dynamics is 0.0312. Therefore, the given augmented error dynamics are nonexpansive.

4.4. Time response of error

For the initial condition given by $\theta_1 = 1.0$, $\theta_2 = -0.8$, and $\dot{\theta}_i = 0$, Fig. 4 shows time responses of θ_i, which converge to zero via DDSC. Furthermore, in the presence of a parametric uncertainty, its results are compared with those of the nonlinear control, $u_i = -(1.25 + |2.9256\theta_i + 2.8251\dot{\theta}_i|)(2.9256\theta_i + 2.8251\dot{\theta}_i)$, proposed by Gong *et al.* [6]. When $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=1} = \{1, 4, 0.02\}$, it is shown in Fig. 4 that performance of both controllers is robust and quite similar in terms of trajectories of θ_1. However, when $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=2} =$

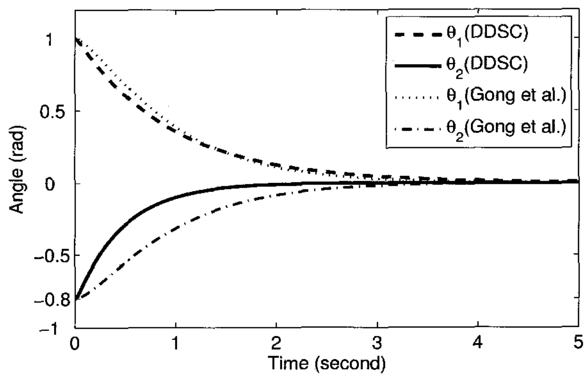

Fig. 4. Time responses of θ_i in the presence of a parametric uncertainty $b_i(t) = 0.25[1 + \sin(50t)]$ for the given $\{\theta_1(0), \theta_2(0)\} = \{1.0, -0.8\}$.

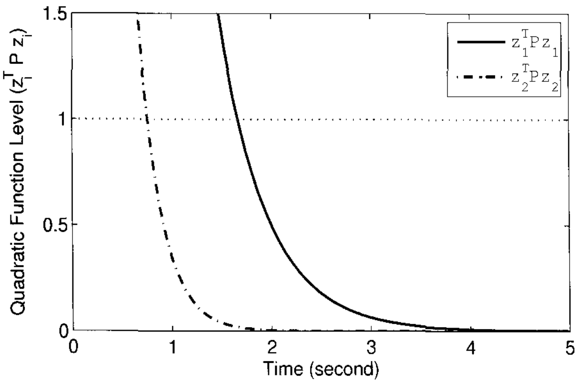

Fig. 5. Time responses of a quadratic function level, $\mathbf{z}_i^T P \mathbf{z}_i$.

$\{2,8,0.02\}$, the stabilization response of θ_2 is faster due to increase of the gain set of DDSC (see Fig. 4).

More precisely speaking, when the eigenvalues of A_i for two different gain sets are calculated as follows:

for $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=1} = \{1,4,0.02\}$,

$$\text{eig}(\mathbf{A}_1) = \{-1.0208, -48.9792, -4.0\},$$

for $\{\lambda_{i1}, \lambda_{i2}, \tau_{i2}\}_{i=2} = \{2,8,0.02\}$,

$$\text{eig}(\mathbf{A}_2) = \{-2.0871, -47.9129, -8.0\},$$

all eigenvalues are placed on the negative real axis in s-plane. If the largest eigenvalues for two cases are compared, the former case has a negative real eigenvalue (i.e., $\lambda_{max}(A_i) = -1.0208$) which is closer to $j\omega$ axis in S-plane, thus causing the slower time responses (compare θ_1 with θ_2 in Fig. 4). Finally, it is shown in Fig. 5 that trajectories of $\mathbf{z}_i$ are contained in the ellipsoidal bound which is the approximation of reachable sets. That is, the quadratic function level $\mathbf{z}_i^T P \mathbf{z}_i$, where P is obtained in (21), becomes less than one after a certain period of time.

5. CONCLUSIONS

This paper developed the analysis and design method of DDSC for a class of large-scale nonlinear systems. By using the fact that the augmented error dynamics for each nonlinear subsystem can be expressed as a diagonal norm-bound LDI, it was shown that the overall closed-loop system is presented as N augmented error dynamics in forms of polytopic diagonal norm-bound LDIs. Using the concepts of reachable sets and L_2 gain, numerical algorithms to estimate the reachable sets of the error dynamics and to calculate the L_2 gain were proposed in the framework of convex optimization. This approach enables us to predict the performance as well as input-output stability of the DDSC with respect to the given controller gains and filter constants before conducting simulation or experiments.

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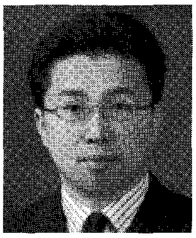

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