International Journal of Fuzzy Logic and Intelligent Systems, vol. 7, no. 1, March 2007 pp. 66-70

Some characterizations of a mapping defined by interval-valued Choquet integrals

Lee-Chae Jang*Hyun-mee Kim

Dept. of Mathematics and Computer Science, Konkuk University
***Dept. of Mathematics, Kyunghee University**

Abstract

Note that Choquet integral is a generalized concept of Lebesgue integral, because two definitions of Choquet integral and Lebesgue integral are equal if a fuzzy measure is a classical measure. In this paper, we consider interval-valued Choquet integrals with respect to fuzzy measures (see [4,5,6,7]). Using these Choquet integrals, we define a mappings on the classes of Choquet integrable functions and give an example of a mapping defined by interval-valued Choquet integrals. And we will investigate some relations between m-convex mappings ϕ on the class of Choquet integrable functions and m-convex mappings T_ϕ defined by the class of closed set-valued Choquet integrals with respect to fuzzy measures.

Key words : fuzzy measure, m-convex, m-concave, interval-Choquet integral.

1. Introduction

Sugeno et al. [8,9] have studied some characterizations of Choquet integrals which is a generalized concept of Lebesgue integral, because two definitions of Choquet integral and Lebesgue integral are equal if a fuzzy measure is a classical measure. And also Choquet integral is often used in information nonlinear aggregation tool(see[2,8,9]).

In this paper, we define fuzzy mappings on the classes of Choquet integrable functions and give an example of fuzzy mapping defined by closed set-valued Choquet integrals. In Section 2, we list various definitions and notations which are used in the proof of our results. In Section 3, using these definitions and properties, we investigate some relations between m-convex mappings ϕ on the class of Choquet integrable functions and m-convex mappings T_ϕ defined by the class of closed set-valued Choquet integrals with respect to fuzzy measures.

2. Preliminaries and definitions

Throughout this paper, we assume that X is a locally compact Hausdorff space, K is the class of continuous functions on X with compact support, Ω is the class of Borel sets, C is the class of compact sets, and O is the class open set. The class of measurable functions is denoted by M and the class of non-negative measurable functions is denoted by M^+. Let (X,Ω,μ) be a fuzzy complete measure space.

A fuzzy measure μ on a measurable space is a set function $\mu:\Omega\to[0,\infty]$ satisfying

(1) $\mu(\varnothing)=0$,

(2) $\mu(A)\le\mu(B)$,

whenever $A,B\in\Omega, A\subset B$.

A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets, we have

$$\mu(\cup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n).$$

A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1)<\infty$, we have

$$\mu(\cap_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n).$$

If μ is both lower and upper semi-continuous, it is said to be continuous(see [4,5,6,7]).

Recall that a function $f: X\to[0,\infty]$ is said to be measurable if $\{x|\, f(x)>\alpha\}\in\Omega$ for all $\alpha\in(-\infty,\infty)$.

Definition 2.1 ([4,5,6,7]) (1) The Choquet integral of measurable function f on A with respect to a fuzzy measure μ is defined by

$$(C)\int_A f\,d\mu=\int_0^\infty \mu(\{x|\, f(x)>r\}\cap A)\,dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet of f can be defined and its value is finite.

Manuscript received Mar. 8. 2007; revised Mar. 15. 2007

Instead of $(C)\int_X f\,d\mu$, we will write $(C)\int f\,d\mu$. Throughout this paper, R^+ will denote the interval $[0,\infty)$. We say $f: X\to R^+$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C)\int f\,d\mu < \infty$.

Theorem 2.2 ([4,5,6,7]). Let f, g be measurable functions.

(1) If $f\le g$, then $(C)\int f\,d\mu \le (C)\int g\,d\mu$.

(2) If $A\subset B$ and $A, B\in\Omega$, then

$$(C)\int_A f\,d\mu \le (C)\int_B f\,d\mu.$$

(3) If $(f\vee g)(x) = max\{f(x), g(x)\}$ and $(f\wedge g)(x) = min\{f(x), g(x)\}$ for all $x\in X$, then

$$(C)\int (f\vee g)\,d\mu \ge (C)\int f\,d\mu \vee (C)\int g\,d\mu$$

and

$$(C)\int (f\wedge g)\,d\mu \le (C)\int f\,d\mu \wedge (C)\int g\,d\mu.$$

Definition 2.3 ([4,5,6,7]) Let f, g be measurable non-negative functions. Then we say that f is comonotonic to g, in symbol $f\sim g$ if and only if

$f(x) < f(x') \to g(x)\le g(x')$ for all $x, x'\in X$.

Let M^+ be the set of continuous non-negative functions $f: X\to R^+$ with compact support and let

$$\mathbb{F} = \{f\in M^+ \mid (C)\int f\,d\mu < \infty\}$$

and for each $g\in\mathbb{F}$,

$\mathbb{F}_g = \{f\in\mathbb{F} \mid f\sim g\}$.

Definition 2.4 Let $m\in[0,1]$ arbitrary. A subset $\mathbb{F}_0$ of $\mathbb{F}$ is said to be m-convex if $tf + m(1-t)g\in\mathbb{F}_0$ whenever $f, g\in\mathbb{F}_0$, $t\in[0,1]$.

It is easily to see that for each $g\in\mathbb{F}$, $\mathbb{F}_g$ is m-convex.

Definition 2.5 ([10]) Let $m\in[0,1]$ arbitrary. A mapping $\phi:\mathbb{F}_0\to R^+$ defined on m-convex set $\mathbb{F}_0$ is said to be m-convex, if

$$\phi(tf + m(1-t)g) \le t\phi(f) + m(1-t)\phi(g)$$

for all $f, g\in\mathbb{F}_0$ and $t\in[0,1]$; and strictly m-convex if strict inequality holds for $f\ne g$ and $t\in(0,1)$. $\phi:\mathbb{F}_0\to R^+$ is said to be m-concave, if

$$\phi(tf + m(1-t)g) \ge t\phi(f) + m(1-t)\phi(g)$$

for all $f, g\in\mathbb{F}_0$ and $t\in[0,1]$; and strictly m-concave if strict inequality holds for $f\ne g$ and $t\in(0,1)$.

Throughout this paper, we denote

$$I(R^+) = \{[a,b] \mid a, b\in R^+ \text{and } a\le b\}.$$

Then an element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a,b], [c,d]\in I(R^+)$ and $k\in R^+$,

$[a,b] + [c,d] = [a+c, b+d]$,

$[a,b]\cdot[c,d] = [a\cdot c, b\cdot d]$,

$k[a,b] = [ka, kb]$,

$[a,b]\le[c,d]$ if and only if $a\le c$ and $b\le d$,

$\max\{[a,b], [c,d]\}\le[a\vee c, b\vee d]$,

$\min\{[a,b], [c,d]\}\le[a\wedge c, b\wedge d]$.

Then $(I(R^+), d_H)$ is a metric space, where of the Hausdorff metric defined by

$$d_H(A, B) = max\{sup_{x\in A} inf_{y\in B}|x-y|,\ sup_{y\in B} inf_{x\in A}|x-y|\}$$

for all $A, B\in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.6 ([7]) For each pair $[a,b]$, $[c,d]\in I(R^+)$,

$d_H([a,b],[c,d]) = max\{|a-c|, |b-d|\}$.

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F: X\to C(R^+)\setminus\{\varnothing\}$ and an interval number-valued function $F: X\to I(R^+)\setminus\{\varnothing\}$.

We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if

$$\lim_{n\to\infty} d_H(A_n, A) = 0,$$

where $A\in I(R^+)$ and $\{A_n\}\subset I(R^+)$.

Definition 2.7 ([4,5,6,7]) A closed set-valued function F is said to be measurable if for each open set $O\subset R^+$,

$F^{-1}(O) = \{x \mid F(x)\cap O\ne\varnothing\}\in\Omega$.

Definition 2.8 ([4,5,6,7]) Let F be a closed set-valued function. A measurable function $f: X\to R^+$ satisfying $f(x)\in F(x), \forall x\in X$ is called a measurable selection of F.

We note that "$x\in X$ $\mu-a.e.$" stand for "$x\in X$ $\mu-$almost everywhere". The property $P(x)$ holds for $x\in X$ $\mu-a.e.$ means that there is a measurable set A such that $\mu(A)=0$ and the property $P(x)$ holds for all $x\in A^c$, where A^c is the complement of A.

Definition 2.9 ([4,5,6,7]) (1) Let F be a closed set-valued function and $A \in \Omega$. The set-valued Choquet integral of F on A is defined by

$$(C)\int_A F d\mu = \{(C)\int_A f d\mu \mid f \in S(F)\},$$

where $S(F)$ is the family of $\mu-a.e.$ measurable selections of F, that is,

$$S(F) = \{f \in L_c^1(\mu) \mid f(x) \in F(x),\ x \in X,\ \mu - a.e\}.$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int F\, d\mu$ exists and dose not include ∞.

(3) A set-valued function F is said to be Choquet integrably bounded if there is a function $g \in M^+$ such that $\| F(x) \| = \sup_{r \in F(x)} |r| \le g(x),\ \forall x \in X.$

Instead of $(C)\int_X F\, d\mu$, we write $(C)\int F\, d\mu$. Obviously, $(C)\int F\, d\mu$ may be empty.

Theorem 2.10 ([4,5]). (1) If a closed set-valued function F is a Choquet integrable, then $A \subset B$ and $A, B \in \Omega \rightarrow$

$$(C)\int_A F\, d\mu \subset (C)\int_B F\, d\mu.$$

(2) If a closed set-valued function F is a Choquet integrable, then for all $a \in R^+$, $(C)\int aF\, d\mu = a(C)\int F\, d\mu$.

Theorem 2.11 ([4]). A closed set-valued function F is measurable if and only if there exists a sequence of measurable selections $\{f_n\}$ of F such that

$$F(x) = cl\{f_n(x)\} \text{ for all } x \in X.$$

Theorem 2.12 ([4,5,6,7]). If F is a closed set-valued function and Choquet integrably bounded and if we define $f^+(x) = sup\{r \mid r \in F(x)\}$ and $f^-(x) = inf\{r \mid r \in F(x)\}$ for all $x \in X$, then f^+ and f^- are Choquet integrable selections of F.

Theorem 2.13 ([4,5,6,7]). Let μ be continuous fuzzy measure and F a measurable and Choquet integrably bounded set-valued function. If F is interval number-valued, i.e., $F(x) = [f^-(x), f^+(x)],\ x \in X$, then

$$(C)\int F\, d\mu = [\,(C)\int f^-\, d\mu,\ (C)\int f^+ d\mu\,].$$

3. Main results

Let M^+ be the set of continuous non-negative functions $f: X \rightarrow R^+$ with compact support. We denote the following classes;

$\mathbb{T} = \{F \mid F: X \rightarrow I(R^+)$ is measurable and Choquet integrable bounded$\}$ and for each $G \in \mathbb{T}$,

$$\mathbb{T}_G = \{F \in \mathbb{T} \mid F \sim G, S(F) \subset M^+\}.$$

Definition 3.1 ([4,5,6,7]) Let $F, G \in \mathbb{T}$. We say that F and G are comonotonic, in symbol $F \sim G$ if and only if

(1) $f^+(x) < f^+(x') \rightarrow g^+(x) \le g^+(x')$ for all $x, x' \in X$ and

(2) $f^-(x) < f^-(x') \rightarrow g^-(x) \le g^-(x')$ for all $x, x' \in X$,

where $F(x) = [f^-(x), f^+(x)]$ and $G(x) = [g^-(x), g^+(x)]$ for all $x, x' \in X$.

Definition 3.2 Let $m \in [0,1]$ arbitrary. A subset $\mathbb{T}_0$ of $\mathbb{T}$ is said to be m-convex if $tF + m(1-t)G \in \mathbb{T}_0$ whenever $F, G \in \mathbb{T}_0$, $t \in [0,1]$.

It is easily to see that for each $G \in \mathbb{T}$, $\mathbb{T}_G$ is m-convex.

Definition 3.3 (1) Let $m \in [0,1]$ arbitrary. A mapping $\Gamma: \mathbb{T}_0 \rightarrow I(R^+)$ defined on m-convex set $\mathbb{T}_0$ is said to be m-convex, if

$\Gamma(tF + m(1-t)G) \le t\Gamma(F) + m(1-t)\Gamma(G)$ for all $F, G \in \mathbb{T}_0$ and $t \in [0,1]$; and strictly m-convex if strict inequality holds for $F \ne G$ and $t \in (0,1)$. If $m = 1$, then we say that Γ is convex.

(2) $\Gamma: \mathbb{T}_0 \rightarrow I(R^+)$ is said to be m-concave, if

$\Gamma(tF + m(1-t)G) \ge t\Gamma(F) + m(1-t)\Gamma(G)$ for all $F, G \in \mathbb{T}_0$ and $t \in [0,1]$; and strictly m-concave if strict inequality holds for $F \ne G$ and $t \in (0,1)$. If $m = 1$, then we say that Γ is concave.

Definition 3.4 (1) Let $m \in [0,1]$ arbitrary. A mapping $\Gamma: \mathbb{T}_0 \rightarrow I(R^+)$ defined on m-convex set $\mathbb{T}_0$ is said to be quasi-m-convex with respect to max($\vee$), if $\Gamma(tF + m(1-t)G) \le \vee\{\Gamma(F), \Gamma(G)\}$ for all $F, G \in \mathbb{T}_0$ and $t \in [0,1]$.

(2) $\Gamma: \mathbb{T}_0 \rightarrow I(R^+)$ is said to be quasi-m-concave with respect to min($\wedge$), if $\Gamma(tF + m(1-t)G) \ge \wedge\{\Gamma(F), \Gamma(G)\}$ for all $F, G \in \mathbb{T}_0$ and $t \in [0,1]$.

Definition 3.5 Let $\phi: \mathbb{F}_0 \rightarrow R^+$ be a real-valued functional.

(1) For any $f \in \mathbb{F}_0$, $\phi(f)$ is defined by $\phi(f) = (C)\int f\, d\mu$.

(2) Let a mapping $T_\phi: \mathbb{T}_0 \rightarrow I(R^+)$ be an interval-valued

fuzzy mapping induced by ϕ; that is, for any $F \in \mathbb{T}_0$, T_ϕ is defined by $T_\phi(F) = (C)\int F\,d\mu$.

We remark that if F is an interval-valued Choquet integrably bounded and μ is continuous fuzzy measure, then we have

$$T_\phi(F) = (C)\int F\,d\mu$$

$$= [(C)\int f^-\,d\mu,\ (C)\int f^+\,d\mu]$$

$$= [\phi(f^-), \phi(f^+)],$$

where $f^+(x) = sup\{r \mid r \in F(x)\}$ and $f^-(x) = inf\{r\ r \mid r \in F(x)\}$.

Theorem 3.6 Let $m \in [0,1]$ and $\mathbb{F}_0$ be m-convex. Then a mapping $\phi : \mathbb{F}_0 \to R^+$ satisfies the following property: for all $f, g \in \mathbb{F}_0$, and $t \in [0,1]$,

$$\phi(tf + m(1-t)g) = t\phi(f) + m(1-t)\phi(g)$$

Proof. Let $m \in [0,1]$. Then clearly, we have

$$\phi(tf + m(1-t)g)$$

$$= (C)\int\leq tf + m(1-t)g\,d\mu$$

$$= (C)\int tf\,d\mu + (C)\int m(1-t)g\,d\mu$$

$$= t(C)\int f\,d\mu + m(1-t)\,(C)\int g\,d\mu$$

$$= t\phi(f) + m(1-t)\phi(g).$$

We note that Theorem 3.6 implies ϕ is m-convex on $\mathbb{F}_0$, but if $\mathbb{F}_0$ is m-convex of $\mathbb{F}$, then ϕ is not m-convex.

Theorem 3.7 Let $m \in [0,1]$, $\mathbb{F}_0$ and $\mathbb{T}_0$ be m-convex on $\mathbb{F}$ and $\mathbb{T}$, respectively. If a mapping $\phi : \mathbb{F}_0 \to R^+$ is m-convex, then $T_\phi : \mathbb{T}_0 \to I(R^+)$ is m-convex.

Proof. Let $F, G \in \mathbb{T}_0$ and

$$F = [f^-, f^+],\ G = [g^-, g^+].$$

By Theorem 2.12, we have

$$f^-, f^+, g^-, g^+ \in \mathbb{F}_0.$$

Then, there exist $h^-, h^+ \in \mathbb{F}_0$ such that

$$h^- = tf^- + m(1-t)g^-$$

and

$$h^+ = tf^+ + m(1-t)g^+,$$

for all $m \in [0,1]$, $t \in [0,1]$. So, $H = [h^-, h^+] \in \mathbb{T}_0$. Since ϕ is m-convex, then we have

$$T_\phi(tF + m(1-t)G)$$

$$= T_\phi(t[f^-, f^+] + m(1-t)[g^-, g^+])$$

$$= T_\phi([tf^- + m(1-t)g^-,\ tf^+ + m(1-t)g^+])$$

$$= T_\phi([h^-, h^+]) = T_\phi(H)$$

$$= [(C)\int h^-\,d\mu,\ (C)\int h^+\,d\mu]$$

$$= [\phi(h^-), \phi(h^+)]$$

$$= [\phi(tf^- + m(1-t)g^-), \phi(tf^+ + m(1-t)g^+)]$$

$$\leq [t\phi(f^-) + m(1-t)\phi(g^-),$$
$$t\phi(f^+) + m(1-t)\phi(g^+)]$$

$$= [t\phi(f^-),\ t\phi(f^+)] +$$
$$[m(1-t)\phi(g^-), m(1-t)\phi(g^+)]$$

$$= t[\phi(f^-),\ \phi(f^+)] + m(1-t)\,[\phi(g^-), \phi(g^-)]$$

$$= t[(C)\int f^-\,d\mu,\ (C)\int f'\,d\mu] + m(1-t)$$

$$[(C)\int g^-\,d\mu,\ (C)\int g^+ d\mu]$$

$$= t(C)\int F\,d\mu + m(1-t)(C)\int G\,d\mu$$

$$= tT_\phi(F) + m(1-t)T_\phi(G).$$

By similar method, we obtain the following theorems.

Theorem 3.8 Let $m \in [0,1]$, $\mathbb{F}_0$ and $\mathbb{T}_0$ be m-concave on $\mathbb{F}$ and $\mathbb{T}$, respectively. If a mapping $\phi : \mathbb{F}_0 \to R^+$ is m-concave, then $T_\phi : \mathbb{T}_0 \to I(R^+)$ is m-concave.

4. References

[1] P.Burillo and H. Bustince, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy sets and systems Vol. 78, pp.305-316, 1996.

[2] G. Choquet, Theory of capacities, Annales de Institut Fourier Vol.5 pp. 131-295, 1953.

[3] Jin-Lum Fan, Yuan-Liang Ma and Wei-Xin Xie, On some properties of distance measures, Fuzzy Sets and Systems Vol.117,pp.355-361, 2001.

[4] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, On comonotonically additive interval-valued Choquet integrals (Ⅱ), J. of Fuzzy Logic and Intelligent Systems, Vol.14, No. 1, pp.33-38, 2004.

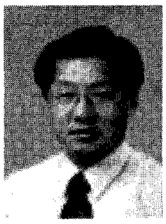
[5] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, "On set-valued Choquet integrals and convergence theorem (Ⅱ), Bull. Korean Math. Soc., Vol. 20, No. 1, pp.139-147, 2003.

[6] Lee-Chae Jang, Interval-valued Choquet integrals and their applications, Journal of Applied Mathematics and Computing, Vol. 16, No.1-2, pp.429-443, 2004.

[7] Lee-Chae Jang and Won Joo Kim, Some properties of Choquet distance measures for interval-valued fuzzy

numbers, J. of Fuzzy Logic and Intelligent Systems, Vol.15 No.7, pp.789-793, 2005.,
[8] T. Murofushi and M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems Vol. 29 pp. 201-227, 1989.
[9] T. Murofushi and M. Sugeno, A theory of Fuzzy measures: representations, the Choquet integral, and null sets, J. Math. Anal. and Appl. Vol. 159 pp. 532-549, 1991.
[10] Yu-Ru Syau, " On convex and concave fuzzy mappings, Fuzzy sets and systems, Fuzzy Sets and Systems Vol.103, pp. 163-168, 1999.
[11] B. Turksen, Interval-valued fuzzy sets on normal forms, Fuzzy Sets and Systems Vol.20, pp.191-210, 1986.
[12] G. Wang and X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems Vol. 98, pp. 331-335, 1998.
[13] W. Zeng and H. Li, Relationship between similarity measure and entropy of interval-valued fuzzy sets, Fuzzy Sets and Systems Vol. 157, pp.1477-1484, 2006.

저 자 소 개



Lee-Chae Jang received his MS and Ph.D at Kyungpook University under the direction of Han-Soo Kim. Since 1987 he has been a professor at Konkuk University. His research interest is analysis, fuzzy measure and Choquet integral, information theory, and p-adic analysis.

Department of Mathematics and Computer Science, Konkuk University, Chungju, 380-701, Korea.
E-mail: leechae.jang@kku.ac.kr

Hyun-Mee Kim received her Ph.D at Kyunghee University under the direction of Jong-Duck Jeon. Since 1991 he has been a parttime instructor at Kyunghee University and Konkuk University, etc. His research interest is fuzzy theory and functional analysis.
E-mail: love4g@smu.ac.kr