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A NEW APPROACH TO FUZZY CONGRUENCES

Kul Hur[1], Su Youn Jang[2] and Keon Chang Lee[3]

[1] **Division of Mathematics and Informational Statistics, Wonkwang University,Iksan, Chonbuk, Korea 570-749**

[2] **Division of Mathematics and Informational Statistics, Wonkwang University,Iksan, Chonbuk, Korea 570-749**

[3] **Department of Computer Science, Dongshin Universuty, Naju, Cheonnam, Korea, 520-714**

Abstract

First, we investigate fuzzy equivalence relations on a set X in the sense of Youssef and Dib. Second, we discuss fuzzy congruences generated by a given fuzzy relation on a fuzzy groupoid. In particular, we obtain the characterizations of $\rho \circ \sigma \in \mathrm{FC}(S)$ for any two fuzzy congruences ρ and σ on a fuzzy groupoid $(S, \odot)$. Finally, we study the lattice of fuzzy equivalence relations (congruences) on a fuzzy semigroup and give certain lattice theoretic properties.

Key words : fuzzy function, fuzzy (equivalence) relation, fuzzy binary generation, fuzzy groupoid, fuzzy semigroup, fuzzy congruence, (modular) lattice.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh[11] in 1965. Since its inception, the theory of fuzzy sets has developed in many directions and found applications in a wide variety of fields. In particular, many researchers [2,4,5,8,9] considered fuzzy relations on a set as fuzzy sets in $X \times X$ and studied them. However, in 1992, Youssef and Dib[3] established a new approach to fuzzy relations. Crisp congruences on structures, such as groupoids, semigroups, groups, rings and lattices, are very well-known. In this paper, we attempts a study of a new approach to fuzzy congruences on fuzzy groupoids and fuzzy semigroups. In Section 1, we list some definitions and some results needed in the later sections. In Section 2, we define a fuzzy equivalence relation on a set X generated by a given fuzzy relation, give a description for it and form a lattice of fuzzy eguivalence relations. In Section 3, we define a fuzzy congruence on a fuzzy groupoid S and obtain some results for it. In Section 4, we give a description for the fuzzy congruence generated by a given fuzzy relation on a fuzzy semigroup S. In Section 5, we form the lattice of fuzzy equivalence relations (congruences) on a fuzzy semigroup S and prove some properties of the lattice of fuzzy congruence on a fuzzy semigroup S.

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2. Preliminaries

Throughout this paper, the following notation will be used:

I : the complete and completely distributive lattice [0,1] with the usual order of real numbers.

$I \wedge I$: the vector lattice $I \times I$ with the partial order defined as follows:

(i) $(r_1, r_2) \leq (s_1, s_2)$ if and only if $r_1 \leq s_1, r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$.

(ii) $(0,0) = (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$.

$I^* = I - \{0\}, (I \wedge I)^* = I \wedge I - (0,0)$.

L : an arbitrary complete and completely distributive lattice with least and greatest elements denoted respectively by 0 and 1.

L' : an arbitrary sublattice of L containing both 0 and 1.

A is called an *L-fuzzy set* in a set X if $A : X \to L$ is a mopping. The notation $\{(x, A(x)) : x \in X\}$ or simply $\{(x, r)\}$, where $r = A(x)$, will be used to denote a L-fuzzy set A in X. Similarly, an $I \wedge I$-fuzzy set in $X \times Y$ will be denoted by $\{(x, y), (r, s)\}$. A fuzzy point of X with support $x \in X$ and value $r \in I^*$ may be denoted by $[x, r]$ or x_r and, analogously, an $I \wedge I$-fuzzy point of $X \times Y$ may be denoted by $[(x, y), (r, s)]$ or $(x, y)_{(r,s)}$, where $(r, s) \in (I \wedge I)^*$.

Throughout this paper, the notation $(x, r) \in A$, where $A \in I^X$, will mean that $A(x) = r$ and the notation $((x, y), (r, s)) \in B$, where $B \in (I \wedge I)^{X \times Y}$, will mean that $B(x, y) = (r, s)$.

We set now some fundamental concepts from [3] which are of great importance in this work. The *fuzzy Cartesian product* of two ordinary sets X and Y, simbolically $X \underline{\times} Y$, is the collection of all $I \wedge I$-fuzzy sets in $X \times Y$. Hence $X \underline{\times} Y = (I \wedge I)^{X \times Y}$. The *fuzzy Cartesian product* of a fuzzy set $A = \{(x, r)\}$ in X and a fuzzy set $B = \{(y, s)\}$ in Y is the $I \wedge I$-fuzzy set $A \underline{\times} B$ in $X \times Y$ defined as follows:

$$\begin{aligned} A \underline{\times} B &= \{((x, y), A(x), b(y)) : x \in X, y \in Y\} \\ &= \{((x, y), (r, s))\}. \end{aligned}$$

It is clear that $A \underline{\times} B \in X \underline{\times} Y$ for each $A \in I^X$ and $B \in I^Y$.

Definition 2.1[3]. Let X and Y be nonempty sets. A fuzzy function from X to Y is a function $\mathbf{F}$ from I^X to I^Y characterized by the ordered pair $(F, \{f_x\}_{x \in X})$, where $F : X \to Y$ is a function and $\{f_x\}_{x \in X}$ is a family of functions $f_x : I \to I$ satisfying the conditions:

(i) f_x is nondecreasing,

(ii) $f_x(0)$ and $f_x(1) = 1$,

such that the *image of any fuzzy set A in X under* $\mathbf{F}$ is the fuzzy set $\mathbf{F}(A)$ in Y defined as follows : For each $y \in Y$,

$$\mathbf{F}(A)(y) = \begin{cases} \bigvee_{x \in F^{-1}(y)} f_x(A(x)) & \text{if } \quad F^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \quad F^{-1}(y) = \emptyset. \end{cases}$$

We write $\mathbf{F} = (F, f_x) : X \to Y$ to denote a fuzzy function from X to Y and we call the function $f_x, x \in X$, the *comembership function* to $\mathbf{F}$.

A fuzzy function $\mathbf{F} = (F, f_x)$ is said to be *uniform* if the comembership functions f_x are identical for all $x \in X$. Two fuzzy functions $\mathbf{F} = (F, f_x)$ and $\mathbf{G} = (G, g_x)$ from X to Y are said to be *equal*, denoted by $\mathbf{F} = \mathbf{G}$, if $F = G$ and $f_x = g_x$ for each $x \in X$ (See [3]).

The above definitions can be generalized in an obvious way by replacing the unit closed interval I by an arbitrary complete and completely distributive lattice L.

3. The lattice of fuzzy equivalence relations

Definition 3.1[3]. ρ is called a *fuzzy relation from a set X to a set Y* if $\rho \subset X \underline{\times} Y$. In particular, ρ is called a *fuzzy relation in X* if $\rho \subset X \underline{\times} X$.

It is clear that $X \underline{\times} Y$ is itself a fuzzy relation from X to Y. Any collection of $A \underline{\times} B$, where $A \in I^X$ and $B \in I^Y$, is a fuzzy relation from X to Y.

The fuzzy cartesian product $X \underline{\times} X$ is called the *universal fuzzy relation in X*. The fuzzy relation $\emptyset \underline{\times} \emptyset = \emptyset$ is called the *empty fuzzy relation.* Between these two extreme cases, lies the *identity fuzzy relation*, denoted by Δ_X, where Δ_X is the fuzzy relation in X whose members are the $I \star I$-fuzzy sets $\{((x, x), (r, r)) : x \in X \text{ and } r \in I\}$.

Definition 3.2[3]. Let $\rho_1, \rho_2 \subset X \underline{\times} Y$.

(1) We say that ρ_1 *is contained in* ρ_2 if whenever $((x, y), (r_1, r_2)) \in A \in \rho_1$, there exists $B \in \rho_2$ such that $((x, y), (r_1, r_2)) \in B$.
In this case, we write $\rho_1 \subset \rho_2$.

(2) We say that ρ_1 and ρ_2 *are equal* if $\rho_1 \subset \rho_2$ and $\rho_2 \subset \rho_1$.
In this case, we write $\rho_1 = \rho_2$.

To each $I \Lambda I$-fuzzy set $C = \{((x, y), (r, s))\}$ in $X \times Y$ we associate a $I \Lambda I$-fuzzy set C^{-1} in $Y \times X$ defined by $C^{-1} = \{((y, x), (s, r))\}$.

Definition 3.3[3]. Let ρ be a fuzzy relation from X to Y. Then the *inverse* of ρ, denoted ρ^{-1}, is the fuzzy relation from Y to X defined by $\rho^{-1} = \{C^{-1} : C \in \rho\}$.

Definition 3.4[3]. Let ρ be a fuzzy relation from X to Y and let σ be a fuzzy relation from Y to Z. Then the *composition* of ρ and σ, denoted $\sigma \circ \rho$, is the fuzzy relation from X to Z whose constituting of $I \star I$-fuzzy sets $C \in X \underline{\times} Z$ are defined as follows:

$((x, z), (r_1, r_3)) \in C$ if and only if there exists $(y, r_2) \in Y \times I$ such that $((x, y), (r_1, r_2)) \in A$ and $((y, z), (r_2, r_3)) \in B$ for some $A \in \rho$ and $B \in \sigma$.
Hence $\sigma \circ \rho = \{C \in X \underline{\times} Z : C \text{ is as defined above}\}$.

It is clear that if ρ is a fuzzy relation on X, then $\Delta_X \circ \rho \subset \rho$ and $\rho \circ \Delta_X \subset \rho$.

Result 3.A[3, Proposition in p.303]. Let $\rho, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2$ be any fuzzy relations defined on the appropriate sets. Then we have

(1) $(\rho_1 \circ \rho_2) \circ \rho_3 = \rho_1 \circ (\rho_2 \circ \rho_3)$.

(2) $\rho_1 \subset \rho_2$ and $\sigma_1 \subset \sigma_2 \ \Rightarrow \ \rho_1 \circ \sigma_1 \subset \rho_2 \circ \sigma_2$.

(3) $\rho_1 \circ (\rho_2 \cup \rho_3) = (\rho_1 \circ \rho_2) \cup (\rho_1 \circ \rho_3)$.

(4) $\rho_1 \circ (\rho_2 \cap \rho_3) \subset (\rho_1 \circ \rho_2) \cap (\rho_1 \circ \rho_3)$.

(5) $\rho_1 \subset \rho_2 \ \Rightarrow \ \rho_1^{-1} \subset \rho_2^{-1}$.

(6) $(\rho^{-1})^{-1} = \rho$ and $(\rho_1 \circ \rho_2)^{-1} = {\rho_2}^{-1} \circ {\rho_1}^{-1}$.

(7) $(\rho_1 \cup \rho_2)^{-1} = {\rho_1}^{-1} \cup {\rho_2}^{-1}$.

(8) $(\rho_1 \cap \rho_2)^{-1} = {\rho_1}^{-1} \cap {\rho_2}^{-1}$.

Definition 3.5[3]. Let ρ be a fuzzy relation on X. Then ρ is said to be:

(1) *reflexive* in X if for each $x \in X$ and $r \in I$, there exists $A \in \rho$ such that $((x,x),(r,r)) \in A$, i.e., $\Delta_X \subset \rho$.

(2) *symmetric* in X if whenever $((x,y),(r,s)) \in A \in \rho$, there exists $B \in \rho$ such that $((y,x),(s,r)) \in B$, i.e., $\rho^{-1} = \rho$.

(3) *transitive* in X if whenever $((x,y),(r,s)) \in A \in \rho$ and $((y,z),(s,t)) \in B \in \rho$, there exists $C \in \rho$ such that $((x,z),(r,t)) \in C$, i.e., $\rho \circ \rho \subset \rho$.

(4) a *fuzzy equivalence relation* on X if it is reflexive, symmetric and transitive.

We will denote the set of all fuzzy equivalence relations in X as $\mathrm{F}Rel_E(X)$. It is clear that $X \underline{\times} X$, $\Delta_X \in \mathrm{F}Rel_E(X)$.

Result 3.B[3, Proposition in p.303]. Let ρ and σ be fuzzy relations in a nonempty set X. Then:

(1) If ρ is reflexive [resp. symmetric and transitive], then ρ^{-1} is reflexive [resp. symmetric and transitive].

(2) If ρ is reflexive [resp. symmetric and transitive], then $\rho \circ \rho$ is reflexive [resp. symmetric and transitive].

(3) If ρ is reflexive, then $\rho \circ \rho \supset \rho$.

(4) If ρ is symmetric, then $\rho \cup \rho^{-1}$, $\rho \cap \rho^{-1}$ are symmetric and $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho$.

(5) If ρ and σ are reflexive [resp. symmetric and transitive], then $\rho \cap \sigma$ is reflexive [resp. symmetric and transitive].

(6) If ρ and σ are symmetric, then $\rho \cup \sigma$ is symmetric.

From (1), (2) and (5), it is clear that if $\rho, \sigma \in \mathrm{F}Rel_E(X)$, then ρ^{-1}, $\rho \circ \rho$, $\rho \cap \sigma \in \mathrm{F}Rel_E(X)$.

Result 3.C[7, Proposition 2.2]. If $\rho \in \mathrm{F}Rel_E(X)$, then $\rho \circ \rho = \rho$.

Result 3.D[7, Proposition 2.5]. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. Then $\rho \circ \sigma \in \mathrm{F}Rel_E(X)$ if and only if $\rho \circ \sigma = \sigma \circ \rho$.

Result 3.E[7, Proposition 2.7]. Let $\{\rho_\alpha\}_{\alpha \in \Gamma}$ be an indexed family of fuzzy equivalence relations on X. Then $\bigcap_{\alpha \in \Gamma} \rho_\alpha \in \mathrm{F}Rel_E(X)$.

Definition 3.6. Let ρ be a fuzzy relation on a set X and let $\{\rho_\alpha : \rho \subset \rho_\alpha\}_{\alpha \in \Gamma}$ be the indexed family of all the fuzzy equivalence relation on X containing ρ. Then the fuzzy equivalence relation *generated by* ρ, denoted by ρ^e, is defined by

$$\rho^e = \bigcap_{\rho \subset \rho_\alpha, \alpha \in \Gamma} \rho_\alpha.$$

From Result 3.E, it is clear that ρ^e is the smallest fuzzy equivalence relation on X containing ρ and for any two fuzzy relations ρ on σ on X, $(\rho \cup \sigma)^e = \rho \vee \sigma$, where $\rho \vee \sigma$ denotes the least upper bound for $\{\rho, \sigma\}$ with respect to the inclusion " $\subset$ ".

Definition 3.7. Let ρ be a fuzzy relation on a set X. Then the transitive closure of ρ, denoted by ρ^∞, is defined as follows:

$$\rho^\infty = \bigcup_{n=1}^{\infty} \rho^n, \text{ where } \rho^n = \rho \circ \rho \circ \cdots \circ \rho \ (n \text{ factors}).$$

The following is the immediate result of Definitions 3.5 and 3.7 and Result 3.A(3).

Proposition 3.8. Let ρ be a fuzzy relation on a set X. Then ρ^∞ is the smallest fuzzy transitive relation on X containing ρ.

The following is the immediate result of Result 3.A(2) and Definition 3.7.

Proposition 3.9. Let ρ, σ and η be fuzzy relations on a set X. Then

(1) If $\rho \subset \sigma$, then $\rho \circ \eta \subset \sigma \circ \eta$.

(2) If $\rho \subset \sigma$, then $\rho^\infty \subset \sigma^\infty$.

Proposition 3.10. If ρ is fuzzy symmetric on a set X, then so is ρ^∞.

Proof. For any $n \geq 1$, let $((x,y),(r,s)) \in A \in \rho^n$. Then there exist $(z_i, t_i) \in X \times I$ and $A_i \in \rho$ $(i = 1, 2, \cdots, n-1)$ such that $((x,z_1),(r,t_1)) \in A_1, ((z_1,z_2),(t_1,t_2)) \in A_2, \cdots$, and $((z_{n-1},y),(t_{n-1},z)) \in A_{n-1}$. Since ρ is fuzzy symmetric, there exist $B_i \in \rho$ $(i = 1, 2, \cdots, n-1)$ such that $((y,z_{n-1}),(s,t_{n-1})) \in B_{n-1}, \cdots$, and $((z_1,x),(t_1,r)) \in B_1$. Thus there exist $B \in \rho^n$ such that $((y,x),(s,r)) \in B$. So ρ^n is fuzzy symmetric for any $n \geq 1$. Hence ρ^∞ is fuzzy symmetric. □

The following is the immediate result of Proposition 3.10.

Corollary 3.10. Let ρ be a fuzzy relation on a set X, If, for some k, $\rho^{k+1} = \rho^k$. Then $\rho^\infty = \rho \cup \rho^2 \cup \cdots \cup \rho^k$.

Proposition 3.11. Let ρ and σ be fuzzy equivalence relations on a set X. If $\rho \circ \sigma = \sigma \circ \rho$, then $(\rho \circ \sigma)^\infty = \rho \circ \sigma$

Proof. For any $n \geq 1$,

$$\begin{aligned}(\rho\circ\sigma)^n &= (\rho\circ\sigma)\circ(\rho\circ\sigma)\circ\cdots\circ(\rho\circ\sigma)\ (n \text{ factors})\\ &= (\rho\circ\rho\circ\cdots\circ\rho)\circ(\sigma\circ\sigma\circ\cdots\circ\sigma)\\ &\qquad \text{(By Result 3.A(1) and the hypothesis)}\\ &= \rho\circ\sigma. \text{ (By Result 3.C and the hypothesis)}\end{aligned}$$

Thus $(\rho\circ\sigma)^n = \rho\circ\sigma$ for any $n \geq 1$. Hence $(\rho\circ\sigma)^\infty = \rho\circ\sigma$.

Result 3.F[6]. If R is a relation on a set X. Then $R^e = [R \cup R^{-1} \cup 1_X]^\infty$.

The following is the fuzzy analogue for Result 3.F.

Theorem 3.12. If ρ is a fuzzy relation on a set X, then
$\rho^e = [\rho \cup \rho^{-1} \cup \triangle_X]^\infty$.

Proof. Let $\sigma = [\rho \cup \rho^{-1} \cup \triangle_X]^\infty$. Then, by Proposition 3.8, σ is fuzzy transitive and $\rho \subset \sigma$. Moreover, $\triangle_X \subset \rho \cup \rho^{-1} \cup \triangle_X \subset \sigma$. Thus σ is fuzzy reflexive. On the other hand, it is clear that $\rho \cup \rho^{-1} \cup \triangle_X$ is fuzzy symmetric. Thus, by Proposition 3.10, σ is fuzzy symmetric. So $\sigma \in \mathrm{F}Rel_E(X)$ and $\rho \subset \sigma$. Now suppose $\eta \in \mathrm{F}Rel_E(X)$ with $\rho \subset \eta$. Then clearly $\triangle_X \subset \eta$ and $\rho^{-1} \subset \eta^{-1} = \eta$ by Result 3.A(5). Thus $\rho \cup \rho^{-1} \cup \triangle_X \subset \eta$. So $\sigma \subset \eta$. Hence $\rho^e = \sigma = [\rho \cup \rho^{-1} \cup \triangle_X]^\infty$. □

The following gives another description for $\rho \vee \sigma$ of two fuzzy equivalence relations ρ and σ.

Proposition 3.13. Let X be a set and let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. Then $(\rho \cup \sigma)^\infty \in \mathrm{F}Rel_E(X)$. In fact, $(\rho \cup \sigma)^\infty = \rho \vee \sigma$.

Proof. From Proposition 3.8, it is clear that $(\rho\cup\sigma)^\infty$ is the smallest fuzzy transitive relation on X containing $\rho\cup\sigma$. Since ρ and σ are fuzzy symmetric, by Result 3.B(6), $\rho\cup\sigma$ is fuzzy transitive. Then, by Proposition 3.10, $(\rho,\sigma)^\infty$ is fuzzy symmetric. Since ρ and σ are fuzzy reflexive, $\triangle_X \subset \rho$ and $\triangle_X \subset \sigma$. Then, by Result 3.A(2), $\triangle_X \subset \rho\cup\sigma$. Thus $\triangle_X \subset (\rho\cup\sigma)^\infty$. So $(\rho\cup\sigma)^\infty$ is fuzzy reflexive. Hence $(\rho\cup\sigma)^\infty \in \mathrm{F}Rel_E(X)$ containing $\rho\cup\sigma$.

Now let $\eta \in \mathrm{F}Rel_E(X)$ containing ρ and σ. Then η is a fuzzy transitive relation on X containing ρ and σ. Since $(\rho\cup\sigma)^\infty$ is the smallest fuzzy transitive relation on X containing ρ and σ, $(\rho\cup\sigma)^\infty \subset \eta$. Therefore $(\rho\cup\sigma)^\infty = \rho\vee\sigma$. □

Theorem 3.14. Let X be a set. If $\rho, \sigma \in \mathrm{F}Rel_E(X)$, Then $\rho\vee\sigma = (\rho\circ\sigma)^\infty$.

Proof. Suppose $\rho, \sigma \in \mathrm{F}Rel_E(X)$. Then, by Theorem 3.12, $(\rho\cup\sigma)^e = [(\rho\cup\sigma)\cup(\rho\cup\sigma)^{-1}\cup\triangle_X]^\infty$. Since ρ and σ are fuzzy symmetric, by Result 3.B(6), $(\rho\cup\sigma)\cup(\rho\cup\sigma)^{-1}\cup\triangle_X = \rho\cup\sigma$. Since $\rho \subset \rho\cup\sigma$ and $\sigma \subset \rho\cup\sigma$, by (2) and (3) of Result 3.A, $\rho\circ\sigma \subset (\rho\cup\sigma)\circ(\rho\cup\sigma) = \rho\cup\sigma$. Thus, by Corollary 3.A(2), $(\rho\cup\sigma)^\infty \subset (\rho\cup\sigma)^\infty$. On the other hand, since $\rho, \sigma \in \mathrm{F}Rel_E(X))$, $\rho \subset \rho\circ\sigma$ and $\sigma \subset \rho\circ\sigma$. Then $\rho\cup\sigma \subset \rho\circ\sigma$. Thus, by Proposition 3.9(2), $(\rho\cup\sigma)^\infty \subset (\rho\circ\sigma)^\infty$. So $(\rho\circ\sigma)^\infty = (\rho\cup\sigma)^\infty$. Hence, by Proposition 3.13, $\rho\vee\sigma = (\rho\cup\sigma)^\infty$. □

Theorem 3.15. Let X be a set and let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. If $\rho\circ\sigma \in \mathrm{F}Rel_E(X)$, then $\rho\circ\sigma = \rho\vee\sigma$.

Proof. Let $((x,y),(r,s)) \in A \in \rho$. Since σ is reflexive, $\triangle_X \subset \sigma$. Then there exists $B \in \sigma$ such that $((y,y),(s,s)) \in \sigma$. Thus there exists $C \in \rho\circ\sigma$ such that $((x,y),(r,s)) \in \rho\circ\sigma$. So $\rho \subset \rho\circ\sigma$. By the similar arguments, $\sigma \subset \rho\circ\sigma$. Hence $\rho\circ\sigma$ is an upper bound for $\{\rho,\sigma\}$ with respect to " $\subset$ ".

Now let $\eta \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \eta$ and $\sigma \subset \eta$. Then, by Results 3.A(2) and 3.C, $\rho\circ\sigma \subset \eta\circ\eta \subset \eta$. Thus $\rho\circ\sigma$ is the least upper bound for $\{\rho,\sigma\}$ with respect to " $\subset$ ". Therefore $\rho\circ\sigma = \rho\vee\sigma$. □

The following is the immediate result Theorem 3.15 and Result 3.D.

Corollary 3.15. Let X be a set and let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ such that $\rho\circ\sigma = \sigma\circ\rho$. Then $\rho\vee\sigma = \rho\circ\sigma$.

For a set X, it is clear that $\mathrm{F}Rel_E(X)$ is a partially ordered set with respect to the inclusion " $\subset$ ". Moreover, for any $\rho, \sigma \in \mathrm{F}Rel_E(X)$, $\rho\cap\sigma$ is the greatest lower bound for $\{\rho,\sigma\}$ in $(\mathrm{F}Rel_E(X), \subset)$. Now, we define two binary operation $\vee$ and $\wedge$ on $\mathrm{F}Rel_E(X)$ as follows : For any $\rho, \sigma \in \mathrm{F}Rel_E(X)$,
$\rho\wedge\sigma = \rho\cap\sigma$ and $\rho\vee\sigma = (\rho\cup\sigma)^e$.

Then we obtain the following result.

Theorem 3.16. Let X be a set. Then $(\mathrm{F}Rel_E(X), \vee, \wedge)$ is a complete lattice with the least element $\triangle_X$ and the greatest element $X\underline{\times}X$.

4. Fuzzy congruence on a fuzzy groupoid

Definition 4.1[3]. Let X, Y and Z be arbitrary (ordinary) sets. A fuzzy function from $X \times Y$ to Z is a function $\mathbf{F}$ from the fuzzy Cartesian product $X\underline{\times}Y = (I\Lambda I)^{X\times Y}$ of X and Y to the set I^Z of fuzzy sets in Z, characterized by the ordered pair $(F, \{f_{xy}\}_{(x,y)\in X\times Y})$, where $F : X\times Y \to Z$ is a function and $\{f_{xy}\}_{(x,y)\in X\times Y}$ is a family of functions $f_{xy} : I\Lambda I \to I$ satisfying the conditions:

(i) f_{xy} is nondecreasing on $I\Lambda I$,

(ii) $f_{xy}(0,0) = 0$ and $f_{xy}(1,1) = 1$.

such that the *image of any $I\Lambda I$-fuzzy set C in $X \times Y$ under* **F** is the fuzzy set **F**(C) in Z defined as follows: For each $z \in Z$,

$$\mathbf{F}(C)(z) = \begin{cases} \bigvee_{(x,y)\in F^{-1}(z)} f_{xy}(C(x,y)) & \text{if } F^{-1}(z) \neq \emptyset, \\ 0 & \text{if } F^{-1}(z) = \emptyset. \end{cases}$$

We write $\mathbf{F} = (F, f_{xy}) : X \times Y \to Z$ to denote a fuzzy function from $X \times Y$ to Z and we call the functions $f_{xy}, (x,y) \in X \times Y$, *the comembership functions* to **F**.

From the conditions of f_{xy} and the definition of the partial order on $I\Lambda I$, it is clear that $f_{xy}(r,0) = f_{xy}(0,r) = 0$ for each $r \in I$.

It should be noticed that I in this definition can be replaced by an arbitrary complete and completely distributive lattice L.

By using the above definition, in [3], they defined the concept of a fuzzy binary operation, analogous to the ordinary case, as follows.

Definition 4.2[3]. A *fuzzy binary operation* on a set X is a fuzzy function $\odot = (\cdot, \cdot_{xy})$from $X \times X$ to X. A nonempty set X together with a binary operation $\odot$ on X is said to be a *fuzzy groupoid* and is denoted by $(X, \odot)$.

In the definition of $\odot = (\cdot, \cdot_{xy})$, when I is replaced by an arbitrary complete and completely distributive lattice L, then $\odot$ is called an *L-fuzzy binary operation* on X. If $\odot$ is an L-fuzzy binary operation on X, then $(X, \odot)$ is is called an *L-fuzzy groupoid* and is denoted by $(X, L, \odot)$.

A fuzzy binary operation $\odot = (\cdot, \cdot_{xy})$ on X is said to be *uniform* if the comembership functions $\cdot_{xy}$ are identical for all $x, y \in X$. A fuzzy groupoid $(X, \odot)$ is said to be *uniform* if $\odot$ is uniform.

Example 4.2. Let $S = \{a, b, c\}$. We define the function $\cdot : S \times S \to S$ and the functions $\cdot_{xy} : I\Lambda I \to I, (x,y) \in S \times S$, respectively as follows: For each $(r,s) \in I\Lambda I$,

$\cdot$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

$$\cdot_{aa}(r,s) = \cdot_{ab}(r,s) = \cdot_{ac}(r,s) = \begin{cases} \frac{1}{2}(r+s) & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0, \end{cases}$$

$$\cdot_{ba}(r,s) = \cdot_{bb}(r,s) = \cdot_{bc}(r,s) = \begin{cases} \log_2(r+s) & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0, \end{cases}$$

$$\cdot_{ca}(r,s) = \cdot_{cb}(r,s) = \cdot_{cc}(r,s) = \begin{cases} \frac{1}{2}(2^r+2^s)-1 & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0. \end{cases}$$

Then we can easily see that $\odot = (\cdot, \cdot_{xy}) : S\times S \to S$ is a fuzzy binary operation on S. Hence $(S, \odot)$ is a fuzzy gropoid. Moreover, let us the comembership functions $\cdot_{xy} : I\Lambda I \to I, (x,y) \in S \times S$, be defined as follows: For each $(x,y) \in S \times S$ and $(r,s) \in I \star I$,

$$\cdot_{xy}(r,s) = \begin{cases} \frac{1}{2}(r+s) & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0, \end{cases}$$

or

$$\cdot_{xy}(r,s) = \begin{cases} \log_2(r+s) & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0, \end{cases}$$

or

$$\cdot_{xy}(r,s) = \begin{cases} \frac{1}{2}(2^r+2^s)-1 & \text{if } r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if } r = 0 \text{ or } s = 0. \end{cases}$$

Then clearly $(S, \odot)$ is a uniform fuzzy groupoid. □

Remark 4.2. (a) If $\odot = (\cdot, \cdot_{xy})$ is a fuzzy binary operation on a set X, then F defines an (ordinary) binary operation on X and for each $(x,y) \in X \times X$, f_{xy} defines an (ordinary) binary operation on I.

(b) If $I = \{0,1\}$, then the concept of fuzzy binary operation reduced to the concept of (ordinary) binary operation.

Let $\odot = (\cdot, \cdot_{xy})$ be a fuzzy binary operation on X. For any fuzzy sets A and B in X, $A\underline{\times}B$ is a $I\Lambda I$-fuzzy set in $X \times X$. Thus, by Definition 4.1,

$$(A\odot B)(z) = \begin{cases} \bigvee_{(x,y)\in F^{-1}(z)} \cdot_{xy}(A(x), B(y)) & \text{if } \cdot^{-1}(z) \neq \emptyset, \\ 0 & \text{if } \cdot^{-1}(z) \neq \emptyset, \end{cases} \tag{4.1}$$

for each $z \in X$, where $(A \odot B)$ denote $\odot(A\underline{\times}B)$.

Since any $I\Lambda I$-fuzzy point $[(x,y)(r,s)]$ in $X \times X$ can be written as $[x,r]\underline{\times}[y,s]$

$$\odot[(x,y),(r,s)] = \odot([x,r]\underline{\times}[y,s]) = [\cdot(x,y), \cdot_{xy}(r,s)].$$

If we write $\cdot(x,y) = x \cdot y$ and $\cdot_{xy}(r,s) = r \cdot_{xy} s$ (analogous to the notation used in ordinary algebra), the above equation takes the following form:

$$[x,r] \odot [y,s] = [x \cdot y, r \cdot_{xy} s]. \tag{4.2}$$

Let $[X] = \{[x,r] : x \in X, r \in I^*\}$ be the set of all fuzzy point in a set X. Then, by (4.2), $\odot$ induces an ordinary operation on $[X]$, denoted again $\odot$. Therefore to each fuzzy groupoid $(X, \odot)$ there is associated an ordinary groupoid $([X], \odot)$.

A fuzzy groupoid $(X, \odot)$ is said to be *commutative* [resp. *associative* or *cancellative*] if $\odot$ is commutative [resp. associative or cancellative] on the fuzzy sets in X. A fuzzy set E in X is called an *identity* of $(X, \odot)$ or an $\odot$*-identity* if $A \odot E = A = E \odot A$ for each $A \in I^X$. A fuzzy set A in X is said to be $\odot$*-invertible* if there exists a fuzzy set A' in X such that $A \odot A' = E = A' \odot A$. In this case, A' is called an $\odot$-inverse of A. A fuzzy set A in X is called an $\odot$*-idempotent* if $A \odot A = A$. It is clear that if an $\odot$-identity exists, then it is unique and that if $(X, \odot)$ is associative and identity admitting, then an $\odot$-inverse, if exists, is unique.

Result 4.A[3, Theorem 4]. Let $(X, \odot)$ be a fuzzy groupoid, with $\odot = (\cdot, \cdot_{xy})$, such that $(r \cdot_{xy}, s) = 0$ only if $r = 0$ or $s = 0$. Then, the identity of $(X, \odot)$, if it exists, is a fuzzy point $[e, \varepsilon] \in [X]$, where e is the identity of $(X, \cdot)$ and $\varepsilon \cdot_{ex} r = r = r \cdot_{xe} \varepsilon$ for each $(x, \varepsilon) \in X \times I$.

Result 4.B[3, Theorem 5]. Let $(X, \odot)$ be a fuzzy groupoid with $\odot = (\cdot, \cdot_{xy})$, where $\cdot$ is associative and $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. If $[e, \varepsilon] \in [X]$ is the identity of $(X, \odot)$, then the invertible elements of $(X, \odot)$ must necessarily be fuzzy points of X. Moreover, if $[x, r] \in [X]$ is $\odot$-invertible, then the $\odot$-inverse of $[x, r]$ is the fuzzy point $[x^{-1}, r^{-1}] \in [X]$, where x^{-1} is the $\cdot$ -inverse of x and $r \cdot_{xx^{-1}} r^{-1} = \varepsilon = r^{-1} \cdot_{x^{-1}x} r$.

Let S be a groupoid. A relation R on the set S is called *left compatible* if $(a, b) \in R$ implies $(xa, xb) \in R$ for all $a, b, c \in S$, and is called *right compatible* if $(a, b) \in R$ implies $(ax, bx) \in R$ for all $a, b, c \in S$. It is called *compatible* if $(a, b) \in R$ and $(c, d) \in R$ implies $(ac, bd) \in R$ for all $a, b, c, d \in S$. A left(right) compatible compatible equivalence relation on S is called a *left(right) congruence* on S. As is well-known [6, Proposition 5.1], a relation R on a groupoid S is a congruence if and only if it is both a left and a right congruence on S.

Definition 4.3. Let $(S, \odot)$ be a fuzzy groupoid with $\odot = (\cdot, \cdot_{xy})$, let $A \in S \underline{\times} S$ and let $[x, \lambda] \in [S]$. Then we define $[x, \lambda] \odot A$ and $A \odot [x, \lambda]$ as follows:

(i) $[x, \lambda] \odot A = \{((x \cdot a, x \cdot b), (\lambda \cdot_{xa} r, \lambda \cdot_{xb} s)) : ((a, b), (r, s)) \in A\}$,

(ii) $A \odot [x, \lambda] = \{((a \cdot x, b \cdot x), (r \cdot_{ax} \lambda, s \cdot_{bx} \lambda)) : ((a, b), (r, s)) \in A\}$.

We can see that either $[x, \lambda] \odot A$, $A \odot [x, \lambda] \in S \underline{\times} S$ or $(x, \lambda) \odot A$, $A \odot [x, \lambda] \notin S \underline{\times} S$ according to types of a fuzzy groupoid $(S, \odot)$.

Example 4.3. (1) Let $(S, \odot)$ be the fuzzy groupoid given in Example 4.2 and let A be the $I \star I$-fuzzy set in S defined as follows:

A	a	b	c
a	(λ_{11}, μ_{11})	(λ_{12}, μ_{12})	(λ_{13}, μ_{13})
b	(λ_{21}, μ_{21})	(λ_{22}, μ_{22})	(λ_{23}, μ_{23})
c	(λ_{31}, μ_{31})	(λ_{32}, μ_{32})	(λ_{33}, μ_{33}) ,

where $(\lambda_{ij}, \mu_{ij}) \in I \star I$ $(1 \le i, j \le 3)$. Then, for each $[a, t] \in [S]$,

$[a,t] \odot A$	a
a	$(t \cdot_{aa} \lambda_{11}, t \cdot_{aa} \mu_{11})$
b	$(t \cdot_{ab} \lambda_{21}, t \cdot_{aa} \mu_{21})$
c	$(t \cdot_{ac} \lambda_{31}, t \cdot_{aa} \mu_{31})$

$[a,t] \odot A$	b
a	$(t \cdot_{aa} \lambda_{12}, t \cdot_{ab} \mu_{12})$
b	$(t \cdot_{ab} \lambda_{22}, t \cdot_{ab} \mu_{22})$
c	$(t \cdot_{ac} \lambda_{32}, t \cdot_{ab} \mu_{32})$

$[a,t] \odot A$	c
a	$(t \cdot_{aa} \lambda_{13}, t \cdot_{ac} \mu_{13})$
b	$(t \cdot_{ab} \lambda_{23}, t \cdot_{ac} \mu_{23})$
c	$(t \cdot_{ac} \lambda_{33}, t \cdot_{ac} \mu_{33})$.

Thus $[a, t] \odot A \in S \underline{\times} S$ but $A \odot [a, t] \notin S \underline{\times} S$.

(2) Let $S = \{a, b, c\}$ and $(S, \odot), \odot = (\cdot, \cdot_{xy})$, be the fuzzy groupoid defined as follows: For each $(r, s) \in I \Lambda I$,

$\cdot$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c ,

$$r \cdot_{aa} s = r \cdot_{ab} s = r \cdot_{ac} s = \begin{cases} \frac{1}{2}(r+s) & \text{if} \quad r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if} \quad r = 0 \text{ or } s = 0, \end{cases}$$

$$r \cdot_{ba} s = r \cdot_{bb} s = r \cdot_{bc} s = \begin{cases} \log_2(r+s) & \text{if} \quad r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if} \quad r = 0 \text{ or } s = 0, \end{cases}$$

$$r \cdot_{ca} s = r \cdot_{cb} s = r \cdot_{cc} s = \begin{cases} \frac{1}{2}(2^r + 2^s) - 1 & \text{if} \quad r \neq 0 \text{ and } s \neq 0, \\ 0 & \text{if} \quad r = 0 \text{ or } s = 0. \end{cases}$$

Let A be the $I \Lambda I$-fuzzy set in S defined as follows:

A	a	b	c
a	(λ_{11}, μ_{11})	(λ_{12}, μ_{12})	(λ_{13}, μ_{13})
b	(λ_{21}, μ_{21})	(λ_{22}, μ_{22})	(λ_{23}, μ_{23})
c	(λ_{31}, μ_{31})	(λ_{32}, μ_{32})	(λ_{33}, μ_{33}) ,

where $(\lambda_{ij}, \mu_{ij}) \in I\Lambda I$ $(1 \leq i, j \leq 3)$. Then, for each $[a, t] \in [S]$,

$A \odot [a, t]$	a
a	$(\lambda_{11} \cdot_{aa} t, \mu_{11} \cdot_{aa} t)$
b	$(\lambda_{21} \cdot_{ba} t, \mu_{21} \cdot_{aa} t)$
c	$(\lambda_{31} \cdot_{ca} t, \mu_{31} \cdot_{aa} t)$
$A \odot [a, t]$	b
a	$(\lambda_{12} \cdot_{aa} t, \mu_{12} \cdot_{ba} t)$
b	$(\lambda_{22} \cdot_{ba} t, \mu_{22} \cdot_{ba} t)$
c	$(\lambda_{32} \cdot_{ca} t, \mu_{32} \cdot_{ba} t)$
$A \odot [a, t]$	c
a	$(\lambda_{13} \cdot_{aa} t, \mu_{13} \cdot_{ca} t)$
b	$(\lambda_{23} \cdot_{ba} t, \mu_{23} \cdot_{ca} t)$
c	$(\lambda_{33} \cdot_{ca} t, \mu_{33} \cdot_{ca} t)$.

Thus $A \odot [a, t] \in S \underline{\times} S$ but $[a, t] \odot A \notin S \underline{\times} S$.

Now we shall introduce the notion of a fuzzy compatible relation on a fuzzy groupoid.

Definition 4.4. Let $(S, \odot)$ be a fuzzy groupoid with $\odot = (\cdot, \cdot_{xy})$ and let $\rho \subset S \underline{\times} S$. Then ρ is said to be:

(1) *fuzzy left compatible* if for any $[a, r], [b, s], [x, t] \in [S]$, $[a, r] \underline{\times} [b, s] \in \rho$ implies
$$([x, t] \odot [a, r]) \underline{\times} ([x, t] \odot [b, s]) \in \rho,$$
equivalently,
$((a, b), (r, s)) \in A \in \rho$ and $[x, t] \in [S]$ implies that there exists $B \in \rho$ such that
$$((x \cdot a, x \cdot b), (t \cdot_{xa} r, t \cdot_{xb} s)) \in B.$$

(2) *fuzzy right compatible* if for any $[a, r], [b, s], [x, t] \in [S]$, $[a, r] \underline{\times} [b, s] \in \rho$ implies
$$([a, r] \odot [x, t]) \underline{\times} ([b, s] \odot [x, t]) \in \rho,$$
equivalently,
$((a, b), (r, s)) \in A \in \rho$ and $[x, t] \in [S]$ implies that there exists $B \in \rho$ such that
$$((a \cdot x, b \cdot x), (r \cdot_{ax} t, s \cdot_{bx} t)) \in B.$$

(3) *fuzzy compatible* if for any $[a, r], [b, s], [x, t], [y, p] \in [S]$, $[a, r] \underline{\times} [b, s] \in \rho$ and $[x, t] \underline{\times} [y, p] \in \rho$ implies
$$([a, r] \cdot [x, t]) \underline{\times} ([b, s] \cdot [y, p]) \in \rho,$$
$$\text{and } ([x, t] \odot [a, r]) \underline{\times} ([y, p] \odot [b, s]) \in \rho$$
equivalently,
$((a, b), (r, s)) \in A \in \rho$ and $((x, y), (t, p)) \in B \in \rho$ implies that there exists $C \in \rho$ such that
$$((a \cdot x, b \cdot y), (r \cdot_{ax} t, s \cdot_{by} p)) \in C.$$

We can easily see that if $A \in \rho$, $[x, t] \odot A \in \rho$ [resp, $A \odot [x, t] \in \rho$] for each $[x, t] \in [S]$, then ρ is fuzzy left [resp, right] compatible $\odot$.

Example 4.4. (1) Let $(S, \odot), \odot = (\cdot, \cdot_{xy})$, be the fuzzy groupoid given in Example 4.2. and let $\rho = A \cup \{[x, t] \odot A;\, [x, t] \in [S]\}$ be a collection of $I\Lambda I$-fuzzy sets in S, where A is the $I\Lambda I$-fuzzy set in S given in Example 4.3(1). Then we can easily see that ρ is a fuzzy left compatible relation on S.

(2) Let $S = \{a, b, c\}$ and $(S, \odot), \odot = (\cdot, \cdot_{xy})$, be the fuzzy groupoid and let A be the $I \star I$-fuzzy set in S given in Example 4.3.(2). Let $\rho = \{A\} \cup \{A \odot [x, t] : [x, t] \in [S]\}$ be a $I\Lambda I$-fuzzy sets in S. Then we can easily see that ρ is a fuzzy right compatible relation on S.

Lemma 4.5. Let ρ and σ be any fuzzy compatible relations on a fuzzy groupoid $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$. Then $\rho \circ \sigma$ is also a fuzzy compatible relation on S.

Proof. Let $((a, b), (r, s)) \in A \in \rho \circ \sigma$. Then there exists $(c, p) \in S \times I, B \in \sigma$ and $C \in \rho$ such that $((a, c), (r, p)) \in B$ and $((c, b), (p, s)) \in C$. Let $[z, t] \in [S]$. Since ρ and σ are fuzzy compatible, there exist $B' \in \sigma$ and $C' \in \rho$ such that
$$((x \cdot a, x \cdot c), (t \cdot_{xa} r, t \cdot_{xc} p)) \in B'$$
and
$$((x \cdot c, x \cdot b), (t \cdot_{xc} p, t \cdot_{xb} s)) \in C'.$$
Thus there exists $D \in \rho \circ \sigma$ such that $((x \cdot a, x \cdot b), (t \cdot_{xa} r, t \cdot_{xb} s)) \in D$. So $\rho \circ \sigma$ is fuzzy left compatible. By the similar arguments, we can easily see that $\rho \circ \sigma$ is fuzzy right compatible. Hence $\rho \circ \sigma$ is fuzzy compatible. □

Definition 4.6. Let $(S, \odot), \odot = (\cdot, \cdot_{xy})$, be a fuzzy groupoid and let $\rho \in \mathrm{F}Rel_E(S)$. Then ρ is called a:

(1) *fuzzy left congruence* (in short, FLC) if it is fuzzy left compatible.

(2) *fuzzy right congruence* (in short, FRC) if it is fuzzy right compatible.

(3) *fuzzy congruence* (in short, FC) if it is fuzzy compatible.

We will denote the set of all $\mathrm{F}C_S$ [resp. $\mathrm{F}LC_S$ and $\mathrm{F}RC_S$] and a fuzzy groupoid $(S, \odot)$ as FC(S) [resp. FLC(S) and FRC(S)]. It is clear that $\triangle_X$, $S \underline{\times} S \in$ FC(S).

Theorem 4.7. Let ρ be a fuzzy equivalence relation on fuzzy groupoid $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$. Then $\rho \in \mathrm{F}C(S)$ if and only if $\rho \in \mathrm{L}FC(S) \cap \mathrm{F}RC(S)$.

Proof. ($\Rightarrow$): Suppose $\rho \in \mathrm{F}C(S)$. Let $(x, t) \in S \times I$ and let $((a, b), (r, s)) \in A \in \rho$. Since ρ is fuzzy reflexive, there $B \in \rho$ such that $((x, x), (t, t)) \in B$. Then, by the hypothesis, there exists $C \in \rho$ such that $(x \cdot a, x \cdot b), (t \cdot_{xa} r, t \cdot_{xb} s) \in C$. Thus ρ is fuzzy left compatible. So $\rho \in FLC(S)$. By the similar arguments, we can see that $\rho \in FRC(S)$. Hence $\rho \in FLC(S) \cap FRC(S)$.

($\Leftarrow$): Suppose $\rho \in FLC(S) \cap FRC(S)$. Let $((a,b),(r,s)) \in B \in \rho$ and $((c,d),(t,p)) \in C \in \rho$. Since $\rho \in FRC(S)$ and $(c,t) \in S \times I$, there exists $D \in \rho$ such that $((a \cdot c, b \cdot c),(r \cdot_{ac} t, s \cdot_{bc} t) \in D$. Since $\rho \in FLC(S)$ and $(b,s) \in S \times I$, there exists $E \in \rho$ such that $((b \cdot c, b \cdot d),(s \cdot_{bc} t, s \cdot_{bd} p)) \in E$. Since ρ is fuzzy transitive, there exists $F \in \rho$ such that $((a \cdot c, b \cdot d),(r \cdot_{ac} t, s \cdot_{bd} p)) \in F$. Hence $\rho \in FC(S)$. This completes the proof. □

Theorem 4.8. Let ρ and σ be fuzzy congruences on a fuzzy groupoid $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$. Then the following conditions are equivalent:

(1) $\rho \circ \sigma \in FC(S)$.

(2) $\rho \circ \sigma \in FRel_E(S)$.

(3) $\rho \circ \sigma$ is fuzzy symmetric.

(4) $\rho \circ \sigma = \sigma \circ \rho$.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We shall show that (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): Suppose $\rho \circ \sigma$ is fuzzy symmetric. Let $((a,b),(r,s)) \in A \in \rho \circ \sigma$. Then there exist $(c,t) \in S \times I, B \in \sigma$ and $C \in \rho$ such that $((a,c),(r,t)) \in B$ and $((c,b),(t,s)) \in C$. Since ρ and σ are fuzzy symmetric, there exists $C' \in \rho$ and $B' \in \rho$ such that $((b,c),(s,t)) \in C'$ and $((c,a),(t,r)) \in B'$. Thus there exists $D \in \sigma \circ \rho$ such that $((b,a),(s,r)) \in D$. So $\rho \circ \sigma \subset \sigma \circ \rho$. By the similar arguments, we can see that $\sigma \circ \rho \subset \rho \circ \sigma$. Hence $\rho \circ \sigma = \sigma \circ \rho$.

(4) $\Rightarrow$ (1): Suppose $\rho \circ \sigma = \sigma \circ \rho$. Then, by Result 3.D, $\rho \circ \sigma \in FRel_E(S)$. Since ρ and σ are fuzzy compatible, by Lemma 4.5, $\rho \circ \sigma$ is fuzzy compatible. Hence $\rho \circ \sigma \in FC(S)$. This completes the proof. □

Remark 4.9. For a nonfuzzy case for Theorem 4.8, See Rosenfeld[10] Proposition 2.

5. Fuzzy congruences on fuzzy semigroups

Let ρ be fuzzy relation on a fuzzy semigroup $(S, \odot)$ with $\odot = (\cdot, \cdot_{xy})$ and let $\{\rho_\alpha : \rho \subset \rho_\alpha\}_{\alpha \in \Gamma}$ be the indexed family of all fuzzy congruence on S containing ρ. Let $\widehat{\rho}$ be the fuzzy relation on S defined as follows:

$$\widehat{\rho} = \bigcap_{\alpha \in \Gamma_{\rho \subset \rho_\alpha}} \rho_\alpha .$$

Then we can easily see that $\widehat{\rho}$ is the fuzzy congruence on S containing ρ. In this case, $\widehat{\rho}$ is called the fuzzy congruence on S *generated by* ρ.

Let $(S, \odot)$ be a fuzzy semigroup, where $\odot = (\cdot, \cdot_{xy})$ and $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. We shall consistently use notation $S^{[e,\varepsilon]}$ with the following meaning:

$$S^{[e,\varepsilon]} = \begin{cases} S \text{ if } S \text{ has the fuzzy identity } [e,\varepsilon], \\ S \cup \{[e,\varepsilon]\} \text{ otherewise.} \end{cases}$$

Definition 5.1. Let ρ be a fuzzy relation on a fuzzy semigroup $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$ and $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. We define the fuzzy relation ρ^* on S as follows:

$((c,d),(r',s')) \in A \in \rho^*$ if and only if there exist $[x,t],[y,p] \in [S^{[e,\varepsilon]}]$ and $B \in \rho$ such that $((a,b),(r,s)) \in B, c = x \cdot a \cdot y, d = x \cdot b \cdot y, r' = (t \cdot_{xa} r) \cdot_{(x \cdot a)y} p$ and $s' = (t \cdot_{xa} r) \cdot_{(x \cdot a)y} p$.

Proposition 5.2. Let ρ and σ be fuzzy relations on a fuzzy semigroup $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$ and $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. Then:

(1) $\rho \subset \rho^*$

(2) $(\rho^*)^{-1} = (\rho^{-1})^*$

(3) If $\rho \subset \sigma$, then $\rho^* \subset \sigma^*$.

(4) $(\rho^*)^* = \rho^*$.

(5) $(\rho \cup \alpha)^* = \rho^* \cup \sigma^*$.

(6) $\rho = \rho^*$ if and only if ρ is both fuzzy left and fuzzy right compatible.

Proof. The proofs of (1), (2) and (3) are obvious.

(4) From (1) and (3), it is clear that $\rho^* \subset (\rho^*)^*$. Now let $((c,d),(r'',s'')) \in A \in (\rho^*)^*$. Then there exist $[x',t'],[y',p'] \in [S^{[e,\varepsilon]}]$ and $B \in \rho^*$ such that $(a',b'),(r',s') \in B, c = x' \cdot a' \cdot y', d = x' \cdot b' \cdot y', r'' = (t' \cdot_{x'a'} r') \cdot_{(x' \cdot a')y'} p'$ and $s'' = (t' \cdot_{x'b'} s') \cdot_{(x' \cdot b')y'} p'$. Since $B \in \rho^*$, there exist $[x,t],[y,p] \in [S^{[e,\varepsilon]}]$ and $C \in \rho$ such that $((a,b),(r,s)) \in C$, $a' = x \cdot a \cdot y, b' = x \cdot b \cdot y, r' = (t \cdot_{xa} r) \cdot_{(x \cdot a)y} p$ and $s' = (t \cdot_{xb} s) \cdot_{(x \cdot b)y} p$. Thus

$c = x' \cdot a' \cdot y' = x' \cdot (x \cdot a \cdot y) \cdot y' = (x' \cdot x) \cdot a \cdot (y \cdot y')$,

$d = x' \cdot b' \cdot y' = x' \cdot (x \cdot b \cdot y) \cdot y' = (x' \cdot x) \cdot b \cdot (y \cdot y')$,

$[x' \cdot x, t' \cdot_{x'x} t], [y' \cdot y, p' \cdot_{y'y} p] \in [S^{[e,\varepsilon]}]$,

$r'' = (t' \cdot_{x'x} t) \cdot_{(x' \cdot x)a} r \cdot_{(x' \cdot x \cdot a)(y' \cdot y)} (p' \cdot_{y'y} p)$,

$s'' = (t' \cdot_{x'x} t) \cdot_{(x' \cdot x)b} r \cdot_{(x' \cdot x \cdot b)(y' \cdot y)} (p' \cdot_{y'y} p)$,

So there exists $D \in \rho^*$ such that $((c,d),(r'',s'')) \in D$. Hence $(\rho^*)^* \subset \rho^*$.
Therefore $(\rho^*)^* = \rho^*$.

(5) From (3), it is clear that $\rho^* \subset (\rho \cup \sigma)^*$ and $\sigma^* \subset (\rho \cup \sigma)^*$. Then $\rho^* \cup \sigma^* \subset (\rho \cup \sigma)^*$. Now let $((c,d),(r',s')) \in A \in (\rho \cup \sigma)^*$. Then there exist $[x,t],[y,p] \in [S^{[e,\varepsilon]}]$ and $B \in \rho \cup \sigma$ such that $(a,b),(r,s) \in B, c = x \cdot a \cdot y, d = x \cdot b \cdot y, r' = (t \cdot_{xa} r) \cdot_{x \cdot ay} p$ and $s' = (t \cdot_{xb} s) \cdot_{(x \cdot b)y} p$. Thus there exist $C \in \rho^* \cup \sigma^*$ such that $((c,d),(r',s')) \in C$. So $(\rho \cup \sigma)^* \subset \rho^* \cup \sigma^*$. Hence $(\rho \cup \sigma)^* = \rho^* \cup \sigma^*$.

(6) ($\Rightarrow$): Suppose $\rho = \rho^*$. Let $((a,b),(r,s)) \in A \in \rho$ and let $[f,t] \in [S]$. Then clearly $[f,t] \in [S^{[e,\varepsilon]}]$. Since $\rho = \rho^*, a = f \cdot a \cdot e, b = f \cdot b \cdot e, r = (f \cdot_{fa} r) \cdot_{(f \cdot a)e} \varepsilon$ and $s = (t \cdot_{fb} s) \cdot_{(f \cdot b)e} \varepsilon$. Thus exist $B \in \rho$ such that

$((f \cdot a, f \cdot b), (t \cdot_{fa} r, t \cdot_{fb} s)) \in B$. So ρ is fuzzy left compatible. By the similar arguments, we can see that ρ is fuzzy right compatible.

($\Leftarrow$): Suppose ρ is both fuzzy left and fuzzy right compatible. Let $((c,d),(r',s') \in A \in \rho^*$. Then there exist $[x,t],[y,p] \in [S^{[e,\varepsilon]}]$ and $B \in \rho$ such that $((a,b),(r,s)) \in B, c = x \cdot a \cdot y, d = x \cdot b \cdot y, r' = (t \cdot_{xa} r) \cdot_{(x\cdot a)y} p$ and $s' = (t \cdot_{xb} s) \cdot_{(x\cdot b)y} p$. Thus, by the hypothesis, $A \in \rho$. So $\rho^* \subset \rho$. Hence, by (1), $\rho = \rho^*$. This completes the proof. □

Lemma 5.3. Let ρ be a fuzzy relation on a fuzzy semigroup $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$. If ρ is both fuzzy left and right compatible, then so is ρ^n for any $n \in \mathbf{N}$.

Proof. Suppose ρ is both fuzzy left and right compatible and $((a,r),(r,s)) \in A \in \rho^n$. Then there exist $(z_i, s_i) \in S \times I$ and $B_i \in \rho$ $(i = 1, 2, \cdots, n-1)$ such that $((a, z_1), (r, s_1)) \in B_1, \cdots, ((z_{n-1}, b), (s_{n-1}, s)) \in B_{n-1}$. Let $[x,t] \in [S]$. Then, by the hypothesis, there exist $C_i \in \rho$ and $D_i \in \rho$ $(i = 1, 2, \cdots, n-1)$ such that

$$((x \cdot a, x \cdot z_1), (t \cdot_{xa} r, t \cdot_{xz_1} s_1)) \in C_1, \cdots,$$
$$((x \cdot z_{n-1}, x \cdot b), (t \cdot_{xz_{n-1}}, t \cdot_{xb} s)) \in C_{n-1},$$
$$(a \cdot x, z_1 \cdot x)(r \cdot_{ax} t, s_1 \cdot_{z_1 x} t) \in D_1, \cdots,$$
$$((z_{n-1} \cdot x, b \cdot x), (s_{n-1} \cdot_{z_{n-1}x} t, s \cdot_{bx} t)) \in D_{n-1}.$$

Thus there exist $C \in \rho^n$ and $D \in \rho^n$ such that

$$((x \cdot a, x \cdot b), (t \cdot_{xa} r, t \cdot_{xb} s)) \in C$$

and

$$((a \cdot x, b \cdot x), (r \cdot_{ax} t, s \cdot_{bx} t)) \in D.$$

Hence ρ^n is both fuzzy left and right compatible. □

Theorem 5.4. Let ρ be a fuzzy relation on a fuzzy semigroup $(S, \odot)$, where $\odot = (\cdot_x, \cdot_{xy})$. Then $\widehat{\rho} = (\rho^*)^e$.

Proof. By Theorem 3.12, It is clear that $(\rho^*)^e \in \mathrm{FRel_E}(S)$ such that $\rho^* \subset (\rho^*)^e$. Moreover, by Proposition 5.2(1), $\rho \subset (\rho^*)^e$. Let $((a,b),(r,s)) \in A \in (\rho^*)^e$. Then there exist $n \in \mathbf{N}$ and $B \in \sigma^n$ such that $((a,b),(r,s)) \in B$, where $\sigma = \rho^* \cup (\rho^*)^{-1} \cup \triangle_X$. Thus, by (2) and (5) of Proposition 5.2

$$\sigma = \rho^* \cup (\rho^{-1})^* \cup \triangle_X^* = (\rho \cup \rho^{-1} \cup \triangle_X)^*.$$

So, (4) and (6) of Proposition 5.2, σ is both fuzzy left and right compatible, Then, by Lemma 5.3, so also is σ^n. Thus, for each $[x,t] \in [S]$, there exist $B, C \in \sigma^n \subset (\rho^*)^e$ such that

$$((x \cdot a, x \cdot b), (t \cdot_{xa} r, t \cdot_{xb} s)) \in B$$

and

$$((a \cdot x, b \cdot x), (r \cdot_{ax} t, s \cdot_{bx} t)) \in C.$$

So $(\rho^*)^e$ is a fuzzy congruence on S containing ρ.

Now let η be a fuzzy congruence on S containing ρ. Then, by (1) and (6) of Proposition 5.2, $\rho^* \subset \eta^* = \eta$. Since η is a fuzzy equivalence relation on S containing $\rho^*, (\rho^*)^e \subset \eta$. So $(\rho^*)^e$ is the smallest fuzzy congruence on S containing ρ. Hence $\widehat{\rho} = (\rho^*)^e$. □

We may rewrite Theorem 5.4 in more elementary terms. If $[c,r'],[d,s'] \in [S]$ such that

$$c = x \cdot a \cdot y, d = s \cdot b \cdot y, r' = (t \cdot_{xa} r) \cdot_{(x\cdot a)y} p$$

and

$$s' = (t \cdot_{xb} s) \cdot_{(x\cdot b)y} p$$

for some $[x,t]$, $[y,p] \in [S^{[e,\varepsilon]}]$, where either $((a,b), (r,s)) \in A \in \rho$ or $((b,a),(s,r)) \in B \in \rho$, then we say that $[c,r']$ *is connected to* $[d,s']$ *by an elementary* ρ*-transition.* Then we have the following alternative version of Theorem 5.4.

Theorem 5.5. Let ρ be a fuzzy relation on a fuzzy semigroup $(S, \odot)$, where $\odot = (\cdot, \cdot_{xy})$ and let $[a,r],[b,s] \in [S]$. Then $((a,b),(r,s)) \in A \in \widehat{\rho}$ if and only if either $[a,r] = [b,s]$ or for some $n \in \mathbf{N}$ there is a sequence

$$[a,r] = [z_1, r] \to [z_2, r_1] \to \cdots$$
$$\to [z_{n-1}, r_{n-2}] \to [z_n, s] = [b,s]$$

of elementary ρ-transitions connecting $[a,r]$ to $[b,s]$.

6. The lattice of fuzzy congruences

Let $(S, \odot)$ be a fuzzy semigroup, where $\odot = (\cdot, \cdot_{xy})$. Then, from Theorem 3.16, it is clear that $(\mathrm{FRel_E}(S), \wedge, \vee)$ is a complete lattice with $\triangle_S$ and $S \underline{\times} S$ as the least and the greatest element of $\mathrm{FRel_E}(S)$, where $\wedge$ and $\vee$ are operations on $\mathrm{FRel_E}(S)$ defined as follows : For any $\rho, \sigma \in \mathrm{FRel_E}(S)$,

$$\rho \wedge \sigma = \rho \cap \sigma \text{ and } \rho \vee \sigma = (\rho \cup \sigma)^e.$$

Similarly, if $\rho, \sigma \in$ FC(S), then so $\rho \cap \sigma, \rho \mathbin{\dot{\cup}} \sigma \in$ FC(S). Hence, the two operations $\wedge$ and $\vee$ on FC(S) are defined as follows: For any $\rho, \sigma \in$ FC(S)),

$$\rho \wedge \sigma = \rho \cap \sigma \text{ and } \rho \vee \sigma = \rho \mathbin{\dot{\cup}} \sigma.$$

Let $\rho, \sigma \in$ FC(S). Then, by (5) and (6) of Proposition 5.2, $(\rho \cup \sigma)^* = \rho^* \cup \sigma^* = \rho \cup \sigma$. So, by Theorem 5.4, $\rho \mathbin{\dot{\cup}} \sigma = (\rho \cup \sigma)^e$. Hence, the operation $\vee$ of ρ and σ in the lattice FC(S), $\wedge, \vee$) coincides with the operation $\vee$ of ρ and σ in the lattice $(\mathrm{FRel_E}(S), \wedge, \vee)$.

The following gives another description for the join $\rho \vee \sigma$ of two fuzzy congruences. This is the immediate result of Proposition 3.14 and Definition 4.6.

Lemma 6.1. Let $(S, \odot)$ be a fuzzy semigroup, where $\odot = (\cdot, \cdot_{xy})$. If $\rho, \sigma \in$ FC(S), then $\rho \vee \sigma = (\rho \circ \sigma)^\infty$.

The following is the immediate result of Corollary 3.15.

Lemma 6.2. Let $(S, \odot)$ be a fuzzy semigroup, where $\odot = (\cdot, \cdot_{xy})$. If $\rho, \sigma \in$ FC(S)) such that $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \vee \sigma = \rho \circ \sigma$.

Proposition 6.3. Let $(S, \odot)$ be a fuzzy group with $\odot = (\cdot, \cdot_{xy})$, where $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. Then $\rho \circ \sigma = \sigma \circ \rho$ for all $\rho, \sigma \in \mathrm{FC}(S)$.

Proof. Let$((x,y),(r,s)) \in A \in \rho \circ \sigma$. Then there exist $(z,t) \in S \times I$, $B \in \sigma$ and $C \in \rho$ such that

$$((x,z),(r,t)) \in B \text{ and } ((z,y),(t,s)) \in C\ .$$

Let $[e, \varepsilon] \in [S]$ be the fuzzy identity of $(S, \odot)$. Since $(S, \odot)$ is a fuzzy group, by Result 4.A and 4.B,

$$[x,r] \odot ([x^{-1}, r^{-1}] \odot [y,s]) = [y,s]$$

and

$$[y,s] \odot ([y^{-1}, s^{-1}] \odot [x,r]) = [x,r].$$

Then

$$((x,z),(r,t)) = ((y,z),(s,t)) \in B$$

and

$$((z,y),(t,s)) = ((z,x),(t,r)) \in C.$$

Thus there exists $D \in \rho \circ \sigma$ such that $((y,x),(s,r)) \in D$. By Result 3.A(6) and Definition 3.5

$$((x,y),(r,s)) \in D^{-1} \in (\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1} = \rho \circ \sigma.$$

So $\rho \circ \sigma \subset \sigma \circ \rho$. By similar arguments, we can see that $\sigma \circ \rho \subset \rho \circ \sigma$. This completes the proof. □

Definition 6.4[1]. A lattice $(L, \wedge, \vee)$ is said to be *modular* if for any $x, y, z \in L$ with $x \leq z$,

$$(x \vee y) \wedge z \leq x \vee (y \wedge z).$$

Theorem 6.5. Let $(S, \odot)$ be a fuzzy semigroup and let H be any sublattice of $(\mathrm{FC(S)}, \wedge, \vee)$ such that $\rho \circ \sigma = \sigma \circ \rho$ for all $\rho, \sigma \in H$. Then H is a modular lattice.

Proof. Let $\rho, \sigma, \eta \in H$ such that $\rho \subset \eta$. We will show that $(\rho \vee \sigma) \wedge \eta \subset (\sigma \wedge \eta)$. Since $\rho \circ \sigma = \sigma \circ \rho$ for all $\rho, \sigma \in H$, by Lemma 6.2, it is suffices to show that $(\rho \circ \sigma) \wedge \eta \subset \rho \circ (\sigma \wedge \eta)$. Let $((x,y),(r,s)) \in A \in (\rho \circ \sigma) \wedge \eta$. Then $((x,y),(r,s)) \in A \in \rho \circ \sigma$ and $((x,y),(r,s)) \in A \in \eta$. Since $\rho \circ \sigma = \sigma \circ \rho$, there exist $(z,t) \in S \times I, B \in \rho$ and $C \in \sigma$ such that

$$((x,z),(r,t)) \in B \text{ and } ((z,y),(t,s)) \in C.$$

Since $\rho \subset \eta$, there exists $D \in \eta$ such that $((z,y),(t,s)) \in D$. Thus there exists $E \in \sigma \wedge \eta$ such that $((z,y),(t,s)) \in E$. Since $((x,z),(r,t)) \in B \in \rho$, there exists $F \in \rho \circ (\sigma \wedge \eta)$ such that $((x,z),(r,s)) \in F$. So $(\rho \circ \sigma) \wedge \eta \subset \rho \circ (\sigma \wedge \eta)$. Hence H is a modular lattice.

The following is the immediate result of proposition 6.3 and Theorem 6.5.

Corollary 6.5. Let $(S, \odot)$ be a fuzzy group with $\odot = (\cdot, \cdot_{xy})$, where $r \cdot_{xy} s = 0$ only if $r = 0$ or $s = 0$. Then $(\mathrm{F}C(S), \wedge, \vee)$ is a modular lattice.

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Kul Hur, Su Youn Jang
Division of Mathematics and Informational Statistics,Wonkwang University,Iksan, Chonbuk, Korea 570-749.
E-mail : kulhur@wonkwang.ac.kr
suyoun123@yahoo.co.kr

Keon Chang Lee
Department of Computer Science, Dongshin Universuty, Naju, Cheonnam, Korea, 520-714
E-mail: Kclee@dsu.ac.kr