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THE ZERO-DISTRIBUTION AND THE ASYMPTOTIC BEHAVIOR OF A FOURIER INTEGRAL

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ABSTRACT. The zero-distribution of the Fourier integral

$$\int_{-\infty}^{\infty} Q(u)e^{P(u)+izu}du,$$

where P is a polynomial with leading term $-u^{2m}$ ($m \geq 1$) and Q an arbitrary polynomial, is described. To this end, an asymptotic formula for the integral is established by applying the saddle point method.

1. Introduction

Concerning the zeros of Fourier integrals, G. Pólya proved, among many other things, that all the zeros of the Fourier integral

$$\int_{-\infty}^{\infty} e^{-u^{2m}+izu}du \qquad (m = 1, 2, 3, \dots) \tag{1.1}$$

are real [11, 12]. If $m = 1$, it has no zeros at all, but if $m > 1$ it has infinitely many zeros. (See the remark after Theorem A below.) Recently, J. Kamimoto and the authors proved that all the zeros of (1.1) are simple [7]. This is a special property of the polynomials $-u^2, -u^4, \dots$. There are other polynomials with the same property. Pólya proved that all the zeros of

$$\int_{-\infty}^{\infty} e^{-u^{4m}+au^{2m}+bu^2+izu}du \qquad (m = 1, 2, 3, \dots;\ a \in \mathbb{R};\ b \geq 0)$$

are real [12, p.18], and it is known that if $m = 1$, or if $m \geq 2$ and $b > 0$, then all the zeros are simple. (See Theorem 3.10 of [5], Theorem 1.1 of [8] and Theorem 2.3 of [9].) N. G. de Bruijn proved that if $P(u)$ is a polynomial with leading term $-u^{2m}$ and $P'(iu)$ has real zeros only, then

$$\int_{-\infty}^{\infty} e^{P(u)+izu}du$$

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has real zeros only [3, Theorem 20], and it can be shown that all the zeros are simple. (See [7].) For general polynomials one cannot expect the same thing. For instance, all the zeros of

$$\int_{-\infty}^{\infty} e^{-u^{2m}+u+izu}du \qquad (m = 2, 3, \dots)$$

are distributed on the line $\operatorname{Im} z = 1$, because (1.1) has real zeros only. Nevertheless, it is natural to expect that if m is a positive integer and $P(u)$ is a polynomial with leading term $-u^{2m}$, then the zero-distribution of

$$\int_{-\infty}^{\infty} e^{P(u)+izu}du$$

is asymptotically equivalent to that of (1.1) in a suitable sense, because every polynomial is asymptotically equivalent to its leading term. The purpose of this paper is to show that this is in fact the case. The main result is this theorem.

Theorem A. *Let m be a positive integer, $P(u)$ a polynomial whose leading term is $-u^{2m}$ and $Q(u)$ an arbitrary polynomial which is not identically equal to zero. Let the entire function $f(z)$ be defined by*

$$f(z) = \int_{-\infty}^{\infty} Q(u)e^{P(u)+izu}du \qquad (z \in \mathbb{C}).$$

Then the following hold:

(1) *The order of $f(z)$ is $\frac{2m}{2m-1}$.*
(2) *For each $\epsilon > 0$ all but a finite number of the zeros of $f(z)$ lie in the set $|\operatorname{Im} z| \le \epsilon|\operatorname{Re} z|$.*
(3) *If $P(-u) = \overline{P(u)}$ for all $u \in \mathbb{R}$, then for each $\epsilon > 0$ all but a finite number of the zeros of $f(z)$ lie in the strip $|\operatorname{Im} z| \le \epsilon$.*
(4) *If $P(-u) = \overline{P(u)}$ and $Q(-u) = \overline{Q(u)}$ for all $u \in \mathbb{R}$, then all but a finite number of the zeros of $f(z)$ are real and simple.*

The general properties of entire functions that are needed in our proof of the results can be found in [2]. If $m = 1$, then, by a direct calculation, one can show that $f(z)$ has exactly d zeros, where d is the degree of $Q(u)$ (see Section 2 of this paper); and if $m \ge 2$, then the first assertion implies that the order of $f(z)$ is not an integer, and hence Hadamard's factorization theorem implies that $f(z)$ has infinitely many zeros. Suppose that $P(-u) = \overline{P(u)}$ and $Q(-u) = \overline{Q(u)}$ for all $u \in \mathbb{R}$. If $m \le 2$, then it can be shown that the number of non-real zeros of $f(z)$ does not exceed that of the polynomial $Q(iu)$ (see [3, p.224], [8, Section 3] and [10, Section 2]); and if $m \ge 3$, or $m = 2$ and $Q(iu)$ has non-real zeros, then the last assertion of this theorem, de Bruijn's theorem [3, Theorem 13] (see also [9, Theorem 2.3]) and Theorem 1.1 of [8] imply that there is a real constant Λ such that

$$\int_{-\infty}^{\infty} Q(u)e^{P(u)+\lambda u^2+izu}du$$

has non-real zeros if and only if $\lambda < \Lambda$. See also [10].

The last assertion of Theorem A is a direct consequence of the third one and the following theorem [9, Theorem 2.2] which solved a conjecture of de Bruijn [3, pp.199, 205].

Theorem B. *Let $F(u)$ be a complex-valued function defined on the real line; and suppose that $F(u)$ is integrable,*

$$F(u) = O\left(e^{-|u|^b}\right) \qquad (|u| \to \infty,\ u \in \mathbb{R}) \tag{1.2}$$

for some constant $b > 2$, and $F(-u) = \overline{F(u)}$ for all $u \in \mathbb{R}$. Suppose also that for each $\epsilon > 0$ all but a finite number of the zeros of the Fourier integral of $F(u)$ lie in the strip $|\mathrm{Im}\, z| \leq \epsilon$. Then for each $\lambda > 0$ all but a finite number of the zeros of the Fourier integral of $e^{\lambda u^2}F(u)$ are real and simple.

Proof that (3) *implies* (4). Suppose $P(-u) = \overline{P(u)}$ and $Q(-u) = \overline{Q(u)}$ for all $u \in \mathbb{R}$. Let λ be an arbitrary positive constant and put

$$F(u) = Q(u)e^{P(u)-\lambda u^2}.$$

We may assume, without loss of generality, that $m \geq 2$. Then the function $F(u)$ satisfies (1.2) with $b = 2m - 1 > 2$, and it is clear that $F(-u) = \overline{F(u)}$ for all $u \in \mathbb{R}$. Since $P_\lambda(u) = P(u) - \lambda u^2$ is a polynomial with leading term $-u^{2m}$ and $P_\lambda(-u) = \overline{P_\lambda(u)}$ for all $u \in \mathbb{R}$, (3) implies that for each $\epsilon > 0$ all but a finite number of the zeros of the Fourier integral of $F(u)$ lie in the strip $|\mathrm{Im}\, z| \leq \epsilon$. Hence, by Theorem B, all but a finite number of the zeros of the Fourier integral of $e^{\lambda u^2}F(u) = Q(u)e^{P(u)}$ are real and simple. □

The other assertions of Theorem A will be proved in Section 2. They are consequences of Theorem C stated below which describes the asymptotic behavior of the function $f(z)$. In order to state the theorem, we need some notation. Let the polynomial $P(u)$ be given by

$$P(u) = -u^{2m} + a_{2m-1}u^{2m-1} + \cdots + a_1 u + a_0.$$

Suppose that $\mathrm{Re}\, z \geq 0$ and write $z = re^{i\theta}$ with $r \geq 0$ and $-\pi/2 \leq \theta \leq \pi/2$. We put

$$R = \left(\frac{r}{2m}\right)^{\frac{1}{2m-1}} \quad \text{and} \quad \zeta = r^{\frac{1}{2m-1}} e^{\frac{i\theta}{2m-1}}. \tag{1.3}$$

Thus $z = \zeta^{2m-1}$, and we have

$$P'(u) + iz = -2mu^{2m-1} + (2m-1)a_{2m-1}u^{2m-2} + \cdots + a_1 + i\zeta^{2m-1}.$$

There is a positive constant r_1 such that if $r > r_1$, then the equation $P'(u)+iz = 0$ has exactly $2m-1$ (distinct and simple) roots in the complex u-plane. Suppose that $r > r_1$ and let u_j, $j \in [-m+1, m-1] \cap \mathbb{Z}$, denote the $2m-1$ roots of the equation $P'(u) + iz = 0$. By taking r_1 sufficiently large, we may assume that these roots are given by

$$u_j = (2m)^{\frac{-1}{2m-1}} e^{\frac{\pi i}{2m-1}\left(\frac{1}{2}+2j\right)} \zeta (1 + A_{j1}\zeta^{-1} + A_{j2}\zeta^{-2} + \cdots), \tag{1.4}$$

where the A_{jk}'s are constants independent of z, and the series converge absolutely and uniformly for $r > r_1$. Each u_j is an analytic function of z, which is defined for $r > r_1$ and $|\theta| \le \pi/2$. It is clear that for each j we have

$$u_j = Re^{\frac{i}{2m-1}\left(\frac{\pi}{2}+\theta+2j\pi\right)} \left(1 + O(R^{-1})\right) \tag{1.5}$$

and

$$\begin{aligned} P(u_j) + izu_j &= (2m-1)u_j^{2m} \left(1 + O\left(R^{-1}\right)\right) \\ &= (2m-1)R^{2m} e^{i\left(\frac{\pi}{2}+\theta+\frac{1}{2m-1}\left(\frac{\pi}{2}+\theta+2j\pi\right)\right)} \left(1 + O\left(R^{-1}\right)\right) \\ &= (2m-1)(2m)^{-\frac{2m}{2m-1}} e^{\frac{m+2j}{2m-1}\pi i} \zeta^{2m} \left(1 + O(|\zeta|^{-1})\right) \end{aligned} \tag{1.6}$$

for $r \to \infty$ and $|\theta| \le \pi/2$.

Theorem C. *Suppose $m \ge 2$. Let $Q(u)$ be a monic polynomial of degree d and $f(z)$ be as in Theorem A. Then, with*

$$A = i^d \sqrt{\frac{\pi}{m(2m-1)}} \left(Re^{\frac{i\theta}{2m-1}}\right)^{1-m+d} \quad \text{and} \quad B = i^{-d} e^{\frac{\pi i}{4m-2}(1-m+d)},$$

we have

$$\begin{aligned} f(z) = A \Big[&Be^{P(u_0)+izu_0} \left(1 + O\left(R^{-1}\right)\right) \\ &+ \bar{B} e^{P(u_{m-1})+izu_{m-1}} \left(1 + O\left(R^{-1}\right)\right) \Big] \quad (|\theta| \le \pi/2,\ r \to \infty), \end{aligned}$$

where R is defined in (1.3).

Several authors applied the saddle point method to obtain asymptotic formulas for the integral (1.1). See, for instance, [1, 4, 6, 13]. Theorem C is also proved by an application of the saddle point method (Section 3). In fact, we will prove a more precise formula (see (3.1) in Section 3).

2. Proof of (1), (2) and (3) of Theorem A

If $m = 1$, that is, if $P(u) = -u^2 + au + b$ for some constants a and b, then we have $f(z) = \sqrt{\pi} e^b Q(-iD) \exp\left(-(z - ia)^2/4\right)$, $D = d/dz$, and hence $f(z)$ has exactly d zeros, where d is the degree of the polynomial $Q(u)$, and it is clear that $f(z)$ is of order 2. This proves the theorem in the case when $m = 1$. From here on, we assume that $m \ge 2$. It is enough to prove the assertions in the right half plane $\operatorname{Re} z \ge 0$.

If we put

$$K(\theta, j) = (2m-1) \cos\left(\frac{\pi}{2} + \theta + \frac{1}{2m-1}\left(\frac{\pi}{2} + \theta + 2j\pi\right)\right),$$

then (1.6) implies that

$$\operatorname{Re}(P(u_j)+izu_j) = R^{2m}\left(K(\theta,j)+O(R^{-1})\right) \qquad (|\theta|\le \pi/2,\ r\to\infty).$$

Since $K(\theta,m-1) < K(\theta,0)$ for $-\pi/2 \le \theta < 0$ and $K(\theta,0) < K(\theta,m-1)$ for $0 < \theta \le \pi/2$, the first and the second assertions are immediate consequences of Theorem C.

To prove (3), suppose that $P(-u) = \overline{P(u)}$ for all real u. We will show the existence of positive constants β, C_1, C_2 and C_3 such that $0 < \beta < 1$ and

(2.1)
$$|\operatorname{Re}(P(u_{m-1})+izu_{m-1}) - \operatorname{Re}(P(u_0)+izu_0)| \ge C_1|y|\left(x^{\frac{1}{2m-1}} - C_2\right) - C_3 \qquad (x>1,\ |y|\le \beta x),$$

where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. It is clear that this inequality, Theorem C and (2) imply (3): Since $\beta > 0$, (2) implies that all but a finite number of the zeros of $f(z)$ lie in the set $\{z : |\operatorname{Im} z| \le \beta|\operatorname{Re} z|\}$; and (2.1) together with Theorem C imply that for every $\epsilon > 0$ there is a positive constant x_1 such that $f(z)$ does not vanish in the set $\{z : \operatorname{Re} z \ge x_1,\ \epsilon < |\operatorname{Im} z| \le \beta \operatorname{Re} z\}$.

We next prove inequality (2.1). Since $P(-u) = \overline{P(u)}$ for all real u, the coefficients $a_{2m-1}, a_{2m-3}, \ldots, a_1$ are purely imaginary and the coefficients $a_{2m-2}, a_{2m-4}, \ldots, a_0$ are real. If z is real, then the roots of the equation $P'(u)+iz=0$ are symmetrically located with respect to the imaginary axis in the complex u-plane. In particular $-\bar{u}_0 = u_{m-1}$ for real z, and we have

$$\overline{P(u_0)+izu_0} = P(-\bar{u}_0)+iz(-\bar{u}_0) = P(u_{m-1})+izu_{m-1} \qquad (z\in\mathbb{R}).$$

Hence, by (1.4), we have

$$(2.2)\quad P(u_0)+izu_0 = \sum_{k=0}^{\infty} B_k\zeta^{2m-k} \quad\text{and}\quad P(u_{m-1})+izu_{m-1} = \sum_{k=0}^{\infty}\bar{B}_k\zeta^{2m-k}$$

for some constants $B_0, B_1, B_2, \ldots$, and the series converge absolutely and uniformly for $r > r_1$. Thus

$$\begin{aligned}
&\operatorname{Re}(P(u_{m-1})+izu_{m-1}) - \operatorname{Re}(P(u_0)+izu_0)\\
(2.3)\qquad &= \operatorname{Re}\sum_{k=0}^{\infty}\left(\bar{B}_k - B_k\right)\zeta^{2m-k}\\
&= 2\left(\operatorname{Im} B_0 \operatorname{Im}\zeta^{2m} + \operatorname{Im} B_1\operatorname{Im}\zeta^{2m-1} + \cdots + \operatorname{Im} B_{2m-1}\operatorname{Im}\zeta\right) + O(1)
\end{aligned}$$

for $r > r_1$.

From (1.6) and (2.2), we see that $\operatorname{Im} B_0 > 0$. For a constant a, we define the function h_a by

$$h_a(s) = \sum_{n=1}^{\infty}(-1)^n\binom{a}{2n+1}s^{2n} \qquad (|s|<1).$$

If $z = x + iy$, with $x > 0$ and $-x < y < x$, then

$$\operatorname{Im} \zeta^{2m-k} = y x^{\frac{1-k}{2m-1}} \left(\frac{2m-k}{2m-1} + h_{\frac{2m-k}{2m-1}} \left(\frac{y}{x} \right) \right) \quad (k = 0, 1, 2, \ldots, 2m-1).$$

There is a constant β such that $0 < \beta < 1$ and

$$|s| \le \beta \quad \Rightarrow \quad \left| h_{\frac{2m}{2m-1}}(s) \right| < 1.$$

Suppose $z = x + iy$, $x > 1$ and $-\beta x \le y \le \beta x$. Then we have

$$\left| \operatorname{Im} \zeta^{2m} \right| > \frac{1}{2m-1} |y| x^{\frac{1}{2m-1}}$$

and

$$\left| \operatorname{Im} \zeta^{2m-k} \right| \le |y| \left(\frac{2m-k}{2m-1} + \sup_{|s| \le \beta} \left| h_{\frac{2m-k}{2m-1}}(s) \right| \right) \qquad (k = 1, 2, \ldots, 2m-1).$$

From these inequalities and (2.3), we obtain the desired result.

3. Proof of Theorem C

Suppose $r > r_1$ and $-\pi/2 \le \theta \le \pi/2$. Put $z = re^{i\theta}$, $J = \{j \in \mathbb{Z} : |j| \le m-1\}$ and $J^+ = \{0, 1, \ldots, m-1\}$. Let ρ_1 be a positive constant such that $P''(u), Q(u) \ne 0$ for $|u| > \rho_1$, and let $\mathcal{D} = \{\rho e^{i\phi} : \ \rho > \rho_1, \ -\pi/2 < \phi < 3\pi/2\}$. There is a unique analytic function $V(u)$ defined in $\mathcal{D}$ such that

$$P''(u) V(u)^2 = -2 \qquad (u \in \mathcal{D})$$

and

$$V(u) = \frac{1}{\sqrt{m(2m-1)}} u^{1-m} \left(1 + O\left(|u|^{-1} \right) \right) \qquad (u \in \mathcal{D}, \ |u| \to \infty).$$

From (1.5), we may assume that $u_j \in \mathcal{D}$ for $j \in J^+$. We put

$$v_j = (-1)^j V(u_j) \quad \text{and} \quad \varphi_j(z) = \sqrt{\pi} v_j Q(u_j) e^{P(u_j) + izu_j} \quad (j \in J^+).$$

Each v_j is an analytic function of z satisfying

$$\frac{P''(u_j)}{2} = -v_j^{-2}$$

and

$$v_j = \frac{1}{\sqrt{m(2m-1)}} R^{1-m} e^{\frac{i}{2m-1}\left((1-m)\theta + \left(\frac{1-m}{2} + j \right) \pi \right)} \left(1 + O(R^{-1}) \right)$$
$$(|\theta| \le \pi/2, \ r \to \infty).$$

Each φ_j is an analytic function of z and does not vanish in the region $r > r_1$, $|\theta| \le \pi/2$. Since $m \ge 2$, we have $\frac{2m}{2m-1} < 2$. Hence (1.6) implies that

$$\ln |\varphi_j(z)| = O\left(r^2 \right) \qquad (|\theta| \le \pi/2, \ r \to \infty).$$

We will prove that

$$\begin{aligned} f(z) =&\varphi_0(z)\left(1+O(R^{-2m})\right) \\ &+\varphi_{m-1}(z)\left(1+O(R^{-2m})\right) \quad (|\theta|\le \pi/2,\ r\to\infty). \end{aligned} \tag{3.1}$$

A straightforward calculation shows that this implies Theorem C.

Let $j\in J^+$ be arbitrary. There is a curve (continuous and piecewise smooth function) $\gamma_j:\mathbb{R}\to\mathbb{C}$ such that $\gamma_j(0)=u_j$ and

$$P(\gamma_j(s))+iz\gamma_j(s)=P(u_j)+izu_j-|s| \qquad (s\in\mathbb{R}). \tag{3.2}$$

We must have

$$\operatorname{Im}\,(P(\gamma_j(s))+iz\gamma_j(s))=\operatorname{Im}\,(P(u_j)+izu_j) \qquad (s\in\mathbb{R})$$

and

$$\lim_{s\to\infty}|\gamma_j(s)|=\lim_{s\to-\infty}|\gamma_j(s)|=\infty.$$

We also have

$$\lim_{s\to0}\frac{P''(u_j)(\gamma_j(s)-u_j)^2}{|P''(u_j)(\gamma_j(s)-u_j)^2|}=-1,$$

or equivalently

$$\lim_{s\to0}\frac{(\gamma_j(s)-u_j)^2}{|\gamma_j(s)-u_j|^2}=\frac{v_j^2}{|v_j|^2}.$$

We may assume, by replacing $\gamma_j(s)$ with $\gamma_j(-s)$ if necessary, that

$$\lim_{s\to0+}\frac{\gamma_j(s)-u_j}{|\gamma_j(s)-u_j|}=\frac{v_j}{|v_j|}. \tag{3.3}$$

If the values $\operatorname{Im}\,(P(u_j)+izu_j)$, $j\in J$, are all different, then the curves γ_j, $j\in J^+$, are uniquely determined by (3.2) and (3.3); and if $\theta\ne\frac{(2k+1)\pi}{4m}$ for all $k\in[-m,m-1]\cap\mathbb{Z}$, then $\operatorname{Im}\,(P(u_j)+izu_j)$, $j\in J$, are all different, whenever r becomes sufficiently large. Let α be a constant such that $0<\alpha<\frac{\pi}{4m}$, and let $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$, where

$$\begin{aligned} \mathcal{R}_1&=\{re^{i\theta}:r>r_1,\ |\theta|\le\alpha\},\\ \mathcal{R}_2&=\{re^{i\theta}:r>r_1,\ -\frac{\pi}{2}\le\theta\le-\frac{\pi}{2}+\alpha\} \quad \text{and}\\ \mathcal{R}_3&=\{re^{i\theta}:r>r_1,\ \frac{\pi}{2}-\alpha\le\theta\le\frac{\pi}{2}\}. \end{aligned}$$

We may assume, by taking r_1 sufficiently large, that for every $z\in\mathcal{R}$ the curves γ_j, $j\in J^+$, are uniquely determined. Using an elementary argument, one can prove that if r_1 is sufficiently large, then the following hold:

(1) If $z\in\mathcal{R}_1$,

$$\lim_{s\to\infty}\frac{\gamma_j(s)}{|\gamma_j(s)|}=e^{j\pi i/m} \quad\text{and}\quad \lim_{s\to-\infty}\frac{\gamma_j(s)}{|\gamma_j(s)|}=e^{(j+1)\pi i/m} \qquad (j\in J^+).$$

(2) If $z \in \mathcal{R}_2$,

$$\lim_{s\to\infty} \frac{\gamma_0(s)}{|\gamma_0(s)|} = 1 \quad \text{and} \quad \lim_{s\to-\infty} \frac{\gamma_0(s)}{|\gamma_0(s)|} = -1.$$

(3) If $z \in \mathcal{R}_3$,

$$\lim_{s\to\infty} \frac{\gamma_{m-1}(s)}{|\gamma_{m-1}(s)|} = 1 \quad \text{and} \quad \lim_{s\to-\infty} \frac{\gamma_{m-1}(s)}{|\gamma_{m-1}(s)|} = -1.$$

From here on, we assume the above three statements. Let ϵ be a positive constant such that $0 < \epsilon < \frac{\pi}{4m}$, and put

$$S_j = \left\{\rho e^{i\phi} : \rho > 0, \quad \frac{j\pi}{m} - \epsilon \le \phi \le \frac{j\pi}{m} + \epsilon\right\}.$$

If $z \in \mathcal{R}_1$, then there is a positive constant s_1 such that

$$s \ge s_1 \;\Rightarrow\; \gamma_j(s) \in S_j \quad \text{and} \quad s \le -s_1 \;\Rightarrow\; \gamma_j(s) \in S_{j+1}$$

hold for every $j \in J^+$; and since the leading term of $P(u)$ is $-u^{2m}$ and $0 < \epsilon < \frac{\pi}{4m}$, there are positive constants A and B such that

$$\left|Q(u)e^{P(u)+izu}\right| \le Ae^{-B|u|^{2m}} \qquad (u \in S_0 \cup S_1 \cup \cdots \cup S_m).$$

Hence, by Cauchy's theorem,

$$f(z) = \sum_{j=0}^{m-1} \int_{\gamma_j} Q(u)e^{P(u)+izu}du \qquad (z \in \mathcal{R}_1). \tag{3.4}$$

Similarly, we have

$$f(z) = \int_{\gamma_0} Q(u)e^{P(u)+izu}du \qquad (z \in \mathcal{R}_2), \tag{3.5}$$

and

$$f(z) = \int_{\gamma_{m-1}} Q(u)e^{P(u)+izu}du \qquad (z \in \mathcal{R}_3). \tag{3.6}$$

Now, we need a lemma whose proof will be given after the proof of (3.1).

Lemma. *For arbitrary $j \in J^+$ we have*

$$\int_{\gamma_j} Q(u)e^{P(u)+izu}du = \varphi_j(z)\left(1 + O\left(R^{-2m}\right)\right) \qquad (z \in \mathcal{R},\ r \to \infty).$$

From (3.5), (3.6) and the lemma, we have

$$f(z) = \begin{cases} \varphi_0(z)\left(1 + O(R^{-2m})\right), & (\theta = -\pi/2,\ r \to \infty) \\ \varphi_{m-1}(z)\left(1 + O(R^{-2m})\right), & (\theta = \pi/2,\ r \to \infty). \end{cases} \tag{3.7}$$

If we put

$$K(\theta, j) = (2m-1)\cos\left(\frac{\pi}{2} + \theta + \frac{1}{2m-1}\left(\frac{\pi}{2} + \theta + 2j\pi\right)\right),$$

then (1.6) implies that

$$\mathrm{Re}\,(P(u_j)+izu_j) = R^{2m}\left(K(\theta,j)+O(R^{-1})\right) \qquad (|\theta|\le\pi/2,\ r\to\infty).$$

We have $K(\theta,j) < \min\{K(\theta,0), K(\theta,m-1)\}$ for $|\theta|\le\alpha$ and $1\le j\le m-2$. Hence, by (3.4) and the lemma,

$$\begin{aligned} f(z) =& \varphi_0(z)\left(1+O(R^{-2m})\right) \\ &+ \varphi_{m-1}(z)\left(1+O(R^{-2m})\right) \qquad (z\in\mathcal{R}_1,\ r\to\infty). \end{aligned} \tag{3.8}$$

Now (3.1) will follow, once we show that

$$f(z) = \varphi_0(z)\left(1+O(R^{-2m})\right) \qquad (-\pi/2\le\theta\le-\alpha,\ r\to\infty) \tag{3.9}$$

and

$$f(z) = \varphi_{m-1}(z)\left(1+O(R^{-2m})\right) \qquad (\alpha\le\theta\le\pi/2,\ r\to\infty), \tag{3.10}$$

because $K(\theta,0) > K(\theta,m-1)$ for $-\pi/2\le\theta<0$ and $K(\theta,m-1) > K(\theta,0)$ for $0<\theta\le\pi/2$.

We prove (3.9) only: (3.10) is proved by the same way. We have $|\zeta| = (2m)^{1/(2m-1)}R$. Hence (3.9) is equivalent to the assertion that the analytic function

$$h_0(z) = \zeta^{2m}\left(\frac{f(z)}{\varphi_0(z)}-1\right)$$

is bounded in the region $\mathcal{R}_4 = \{re^{i\theta} : r>r_1,\ -\pi/2\le\theta\le-\alpha\}$. From (3.7), $h_0(z)$ is bounded on the ray $\theta=-\pi/2,\ r>r_1$. Since $K(-\alpha,0) > K(-\alpha,m-1)$, (3.8) implies that

$$f(z) = \varphi_0(z)\left(1+O(R^{-2m})\right) \qquad (\theta=-\alpha,\ r\to\infty).$$

Hence $h_0(z)$ is bounded on the ray $\theta=-\alpha,\ r>r_1$. Therefore $h_0(z)$ is bounded on the boundary of the region $\mathcal{R}_4$. It is known that the order of the entire function $f(z)$ is at most $\frac{2m}{2m-1}$ (<2). (For a proof, see [12, pp. 9–10].) Since

$$\ln|\varphi_0(z)| = O(r^2) \qquad (|\theta|\le\pi/2,\ r\to\infty),$$

it follows that

$$h_0(z) = O\left(e^{Cr^2}\right) \qquad (z\in\mathcal{R}_4,\ r\to\infty)$$

for some positive constant C. It is obvious that the two rays $\theta=-\pi/2,\ r>r_1$ and $\theta=-\alpha,\ r>r_1$ make an angle less than $\pi/2$. Therefore the Phragmén-Lindelöf theorem [2, p.4] implies that the function $h_0(z)$ is bounded throughout the region $\mathcal{R}_4$. This proves (3.9).

Proof of the Lemma. Let $j\in J^+$ be arbitrary. Put

$$s_j^- = \inf\left\{s\in\mathbb{R} : |\gamma_j(s)-u_j| \le R^{1-\frac{2m}{3}}\right\} \quad \text{and}$$
$$s_j^+ = \sup\left\{s\in\mathbb{R} : |\gamma_j(s)-u_j| \le R^{1-\frac{2m}{3}}\right\}.$$

We also put $\alpha_j = \gamma_j(s_j^-) - u_j$ and $\beta_j = \gamma_j(s_j^+) - u_j$. Thus

$$\int_{\gamma_j} Q(u)e^{P(u)+izu}du = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{u_j+\alpha_j}^{u_j+\beta_j} Q(u)e^{P(u)+izu}du,$$
$$I_2 = \int_{s_j^+}^{+\infty} Q(\gamma_j(s))e^{P(\gamma_j(s))+iz\gamma_j(s)}\gamma_j{}'(s)ds, \quad \text{and}$$
$$I_3 = \int_{-\infty}^{s_j^-} Q(\gamma_j(s))e^{P(\gamma_j(s))+iz\gamma_j(s)}\gamma_j{}'(s)ds.$$

If we put $u = u_j + w$, then

$$\begin{aligned} Q(u)e^{P(u)+izu} =& Q(u_j)e^{P(u_j)+izu_j}\,(1+F(w)) \\ & \times \left(1 + G(w) + G(w)^2H(w)\right) e^{P''(u_j)w^2/2}, \end{aligned}$$

where

$$F(w) = \sum_{n=1}^{\deg Q} \frac{Q^{(n)}(u_j)}{n!Q(u_j)}w^n, \quad G(w) = \sum_{n=3}^{2m} \frac{P^{(n)}(u_j)}{n!}w^n \quad \text{and}$$
$$H(w) = \frac{e^{G(w)} - 1 - G(w)}{G(w)^2};$$

and we have $P''(u_j)/2 = -v_j^{-2}$. Hence

$$\begin{aligned} I_1 =& Q(u_j)e^{P(u_j)+izu_j} \\ & \times \int_{\alpha_j}^{\beta_j} (1+F(w))\left(1+G(w)+G(w)^2H(w)\right)e^{-v_j^{-2}w^2}dw, \end{aligned}$$

and a straightforward calculation shows that

$$\begin{aligned} & \int_{\alpha_j}^{\beta_j} (1+F(w))\left(1+G(w)+G(w)^2H(w)\right)e^{-v_j^{-2}w^2}dw \\ =& \sqrt{\pi}v_j\left(1+O\left(R^{-2m}\right)\right) \qquad (|\theta| \le \pi/2,\ r \to \infty). \end{aligned}$$

Therefore

$$I_1 = \varphi_j(z)\left(1+O\left(R^{-2m}\right)\right) \qquad (|\theta| \le \pi/2,\ r \to \infty).$$

From (3.2), we have

$$I_2 = e^{P(u_j)+izu_j}\int_{s_j^+}^{\infty} Q(\gamma_j(s))\gamma_j{}'(s)e^{-s}ds.$$

Since $s_j^+ > 0$, $|u_j - \gamma_j(s_j^+)| = R^{1-\frac{2m}{3}}$ and

$$\begin{aligned} s_j^+ &= P(u_j) + izu_j - \left(P(\gamma_j(s_j^+)) + iz\gamma_j(s_j^+)\right) \\ &= -\sum_{n=2}^{2m} \frac{P^{(n)}(u_j)}{n!}(\gamma_j(s_j^+) - u_j)^n, \end{aligned}$$

it follows that

$$s_j^+ \sim m(2m-1)R^{2m/3} \qquad (z \in \mathcal{R},\ r \to \infty).$$

We may assume, by taking r_1 sufficiently large, that

$$|\gamma_j(s) - u_k| \geq 1 \qquad (z \in \mathcal{R},\ s \in \mathbb{R},\ k \in J \setminus \{j\}).$$

Then we have

$$\begin{aligned} |\gamma_j{}'(s)| &= \frac{1}{|P'(\gamma_j(s)) + iz|} \\ &= \frac{1}{2m}\prod_{k \in J} |\gamma_j(s) - u_k|^{-1} \leq \frac{R^{\frac{2m}{3}-1}}{2m} \qquad (z \in \mathcal{R},\ s \geq s_j^+). \end{aligned}$$

In particular,

$$\begin{aligned} |\gamma_j(s)| &= \left| u_j + \alpha_j + \int_{s_j^+}^{s} \gamma_j{}'(s)ds \right| \\ &\leq |u_j| + |\alpha_j| + \frac{s - s_j^+}{2m} R^{\frac{2m}{3}-1} \qquad (z \in \mathcal{R},\ s \geq s_j^+). \end{aligned}$$

Now, it is clear that we can find positive constants A and B such that

$$\left| \int_{s_j^+}^{\infty} Q(\gamma_j(s))\gamma_j{}'(s)e^{-s}ds \right| \leq Ae^{-R^B} \qquad (z \in \mathcal{R}).$$

Therefore we have

$$I_2 = \varphi_j(z)\, O\left(R^{-2m}\right) \qquad (z \in \mathcal{R},\ r \to \infty),$$

and the same argument gives

$$I_3 = \varphi_j(z)\, O\left(R^{-2m}\right) \qquad (z \in \mathcal{R},\ r \to \infty).$$

This proves the lemma. □

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