J. Korean Math. Soc. **44** (2007), No. 2, pp. 407–442

SPHERICAL SUBMANIFOLDS WITH FINITE TYPE SPHERICAL GAUSS MAP

Bang-Yen Chen and Huei-Shyong Lue

Abstract. The study of Euclidean submanifolds with finite type "classical" Gauss map was initiated by B.-Y. Chen and P. Piccinni in [11]. On the other hand, it was believed that for spherical submanifolds the concept of spherical Gauss map is more relevant than the classical one (see [20]). Thus the purpose of this article is to initiate the study of spherical submanifolds with finite type spherical Gauss map. We obtain several fundamental results in this respect. In particular, spherical submanifolds with 1-type spherical Gauss map are classified. From which we conclude that all isoparametric hypersurfaces of S^{n+1} have 1-type spherical Gauss map. Among others, we also prove that Veronese surface and equilateral minimal torus are the only minimal spherical surfaces with 2-type spherical Gauss map.

1. Introduction

Let M^n denote a Riemannian n-manifold with Laplacian operator Δ. A smooth map $\phi : M^n \to \mathbb{E}^N$ of M^n into the Euclidean N–space is said to be *of finite type* if it admits a finite spectral resolution:

$$\phi = c + \sum_{t=1}^{k} \phi_t, \tag{1.1}$$

where c is a constant vector in $\mathbb{E}^N$, ϕ_t's are non-constant $\mathbb{E}^N$-valued maps such that $\Delta\phi_t = \lambda_{p_t}\phi_t$ with $\lambda_{p_1} < \lambda_{p_2} < \cdots < \lambda_{p_k}$. Otherwise, it is said to be *of infinite type*. When the spectral resolution (1.1) contains exactly k non-constant terms, the map ϕ is called of k-type (see [6, 7, 8] for details).

Let $S^{N-1} \subset \mathbb{E}^N$ denote the unit hypersphere of $\mathbb{E}^N$ centered at the origin. A spherical finite type map $\phi : M \to S^{N-1} \subset \mathbb{E}^N$ of a Riemannian manifold M into S^{N-1} is called *mass-symmetric* if the vector c in its spectral resolution is the center of S^{N-1} (which is the origin of $\mathbb{E}^N$). Otherwise, ϕ is called non-mass-symmetric. When M is compact, ϕ is mass-symmetric if and only if we have $\int_M \phi * 1 = 0$.

Received August 30, 2005.

2000 *Mathematics Subject Classification.* Primary 53C40, 53C42, 53B25.

Key words and phrases. spherical Gauss map, finite type map, Clifford minimal torus, Veronese surface, equilateral torus.

Let $G(n,m)$ denote the Grassmannian consisting of linear n-subspaces of $\mathbb{E}^m$. Given an isometric immersion $\mathbf{x}: M^n \to \mathbb{E}^m$, the *classical Gauss map* $\nu^c : M^n \to G(n,m)$ associated with $\mathbf{x}$ is the map which carries each $p \in M^n$ to the linear n-subspace of $\mathbb{E}^m$ obtained by parallel displacement of the tangent space T_pM^n. Since $G(n,m)$ can be canonically imbedded in $\wedge^n\mathbb{E}^m = \mathbb{E}^{\binom{m}{n}}$, the classical Gauss map gives rise to a well-defined map from M^n into the Euclidean $\binom{m}{n}$-space $\mathbb{E}^{\binom{m}{n}}$.

The study of Euclidean submanifolds with finite type classical Gauss map was initiated by the first author and Piccinni in [11]. Since then many geometers have studied such submanifolds (see, for instance, [1, 2, 3, 4, 9, 10, 12, 13, 17]).

On the other hand, for a spherical submanifold M^n in S^{m-1}, Obata studied in [19] the map which carries $p \in M^n$ to the totally geodesic n-sphere of S^{m-1} determined by the tangent space T_pM^n; he also computed the induced metric on M^n via his map. Since a totally geodesic n-sphere S^n of S^{m-1} determines a unique linear $(n+1)$-space containing the totally geodesic S^n in $\mathbb{E}^m$, Obata's map can be extended to a map $\hat{\nu}$ of M^n into the Grassmannian $G(n+1,m)$ in a natural way, known as the spherical Gauss map. The composition $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G(n+1,m)$ in $\mathbb{E}^{\binom{m}{n+1}}$ is also called the *spherical Gauss map.*

The geometrical behavior of the classical and spherical Gauss maps are different. For instance, the classical Gauss map of every compact Euclidean submanifold is mass-symmetric; but the spherical Gauss map of a spherical compact submanifold is not mass-symmetric in general. It was believed that in the study of spherical submanifolds the spherical Gauss map is more relevant than the classical Gauss map (cf. for instance, [20]).

The main purpose of this article is to study spherical Gauss map of spherical submanifolds in the frame work of finite type theory. The main problem is

"To what extent does the type number of the spherical Gauss map determine the spherical submanifolds?"

In section 3, we provide some basic results on spherical Gauss map. As a by-product, we are able to extend a result of Lawson on minimal surfaces in spheres. In section 4, we classify spherical submanifolds with 1-type spherical Gauss map. This classification result implies that every isoparametric hypersurface of S^{n+1} has 1-type spherical Gauss map. In sections 5 and 6, we prove that the Veronese surface in S^4 is the only spherical minimal surface with mass-symmetric 2-type spherical Gauss map and the equilateral minimal torus in S^5 is the only minimal spherical surfaces with non-mass-symmetric 2-type spherical Gauss map. The last section provides more results on spherical Gauss map of spherical minimal submanifolds.

2. Preliminaries

Let M^n be a *submanifold* of a Riemannian manifold $\tilde{M}^m$, i.e., the Riemannian n-manifold M^n is isometrically immersed in the Riemannian m-manifold

$\tilde{M}^m$. If ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M^n and $\tilde{M}^m$, then the Gauss and Weingarten formulas of M^n in $\tilde{M}^m$ are given respectively by

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi \end{aligned} \tag{2.1}$$

for X, Y tangent to M^n and ξ normal to N, where h is the second fundamental form, D the normal connection, and A the shape operator. The second fundamental form h and the shape operator A are related by $g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi)$.

The mean curvature vector is $H = (1/n)\operatorname{tr} h$. The submanifold is called minimal (respectively, totally geodesic) if its mean curvature vector (respectively, second fundamental form) vanishes identically.

We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ such that, restricted to M^n, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M^n. We shall make use of the following convention on the ranges of indices unless mentioned otherwise:

$$1 \le A, B, C, \ldots \le m; \; 1 \le i, j, k, \ldots \le n; \; n+1 \le r, s, t, \ldots \le m.$$

Put $\tilde{\nabla}_X e_i = \sum_A \omega_i^A(X) e_A$ and $\tilde{\nabla}_X e_r = \sum_A \omega_r^A(X) e_A$. With respect to the frame field of M^n chosen above, let $\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^m$ be the field of dual frame. Then the structure equations are given by

$$\begin{aligned} d\omega^A &= -\sum_B \omega_B^A \wedge \omega^B, \quad \omega_B^A = -\omega_A^B, \\ d\omega_B^A &= -\sum_C \omega_C^A \wedge \omega_B^C + \Phi_B^A, \\ \Phi_B^A &= \frac{1}{2} \sum K_{BCD}^A \omega^C \wedge \omega^D, \quad K_{BCD}^A + K_{BDC}^A = 0. \end{aligned} \tag{2.2}$$

Restricting these forms to M^n we get $\omega^r = 0$. Since

$$0 = d\omega^r = -\sum_i \omega_i^r \wedge \omega^i,$$

Cartan's lemma yields $\omega_i^r = \sum_j h_{ij}^r \omega^j$ with $h_{ij}^r = h_{ji}^r$. The exterior differentiation of $\omega_i^r = \sum_j h_{ij}^r$ gives the equation of Codazzi:

$$\begin{aligned} h_{ij;k}^r &= h_{ik;j}^r, \\ h_{ij;k}^r &= e_k h_{ij}^r - \sum_\ell (h_{i\ell}^r \omega_j^\ell(e_k) + h_{j\ell}^r \omega_i^\ell(e_k)) + \sum_s h_{ij}^s \omega_s^r(e_k). \end{aligned} \tag{2.3}$$

From these we have the following Cartan's structure equations:

$$d\omega^i = -\sum_j \omega_j^i \wedge \omega^j, \quad d\omega_j^i = -\sum_k \omega_k^i \wedge \omega_j^k + \Omega_j^i, \tag{2.4}$$

$$\Omega_j^i = \frac{1}{2} \sum_{k,\ell} R_{jk\ell}^i \omega^k \wedge \omega^\ell, \quad R_{jk\ell}^i = K_{jk\ell}^i + \sum_r (h_{ik}^r h_{j\ell}^r - h_{i\ell}^r h_{jk}^r). \tag{2.5}$$

The Ricci tensor R_{jk} and the scalar curvature S of M^n are defined respectively by $R_{jk} = \sum_i R^i_{jik}$ and $S = \sum_i R_{ii}$. If the ambient space $\tilde{M}^m$ is the Euclidean m-space $\mathbb{E}^m$, then (2.5) implies that the scalar curvature S of M^n satisfies

$$S = n^2|H|^2 - ||h||^2, \tag{2.6}$$

where $|H|^2$ and $||h||^2$ are the squared norm of the mean curvature vector and of the second fundamental form of M^n in $\mathbb{E}^m$. In particular, if M^n is immersed in the unit hypersphere S^{m-1}, then (2.6) yields

$$S = n(n-1) + n^2|\hat{H}|^2 - ||\hat{h}||^2, \tag{2.7}$$

where $\hat{H}$ and $\hat{h}$ are the mean curvature vector and the second fundamental form of M^n in S^{m-1}, respectively. For M^n in S^{m-1}, we also have (cf. [5])

$$H = \hat{H} - \mathbf{x}, \quad h(X,Y) = \hat{h}(X,Y) - g(X,Y)\mathbf{x}. \tag{2.8}$$

A submanifold M is called *isotropic* if, at each given point $p \in M$, the length $|h(u,u)|$ is independent of the choice of the unit tangent vector $u \in T_pM$.

Theorem 2.1. [11] *Let $\phi : M \to \mathbb{E}^N$ be a smooth map from a compact Riemannian manifold M into the Euclidean N-space and let $\tau := \mathrm{div}(d\phi)$ be the tension field of ϕ. Then we have:*

(i) *ϕ is of finite type if and only if there is a non-trivial polynomial $Q(t)$ satisfying $Q(\Delta)\tau = 0$.*

(ii) *If ϕ is of finite type, there is a unique monic polynomial $P(t)$ of least degree, called the minimal polynomial, which satisfies $P(\Delta)\tau = 0$.*

(iii) *If ϕ is of finite type, then ϕ is of k-type if and only if the minimal polynomial P is of degree k.*

The same results hold if τ is replaced by $\phi - \phi_0$ with

$$\phi_0 = \left(\int_M \phi * 1\right) / (\mathrm{vol}(M)).$$

3. Some basic results on spherical Gauss map

Let V be an oriented k-plane in $\mathbb{E}^m$. If $\epsilon_1, \ldots, \epsilon_k$ is an oriented orthonormal basis of V, then $\epsilon_1 \wedge \cdots \wedge \epsilon_k$ is a decomposable k-vector of norm one which gives $\epsilon_1 \wedge \cdots \wedge \epsilon_k$ the orientation of V. Conversely, each decomposable k-vector of norm one determines a uniquely k-plane in $\mathbb{E}^m$. Hence, if $G(k,m)$ denotes the Grassmannian of oriented k-planes in $\mathbb{E}^m$, then $G(k,m)$ can be identified naturally with the decomposable k-vectors of norm one in $\mathbb{E}^{\binom{m}{k}} = \wedge^k\mathbb{E}^m$. In this way, we have natural inclusion of $G(k,m)$ in $\mathbb{E}^{\binom{m}{k}}$.

In this article, we shall always regard S^{m-1} as the unit hypersphere of $\mathbb{E}^m$ centered at the origin. Moreover, for an isometric immersion $\mathbf{x}$ of a Riemannian n-manifold M^n into S^{m-1}, we identify each tangent vector $v \in TM^n$ with the its image $d\mathbf{x}(v)$ under the differential $d\mathbf{x}$.

For each point $p \in M^n$, let $e_1, \dots, e_n$ be an orthonormal basis of T_pM^n. Then the $n+1$ vectors $\mathbf{x}, e_1, \dots, e_n$ determine a linear $(n+1)$-subspace in $\mathbb{E}^m$. The intersection of this linear subspace and S^{m-1} is the totally geodesic n-sphere determined by T_pM^n as in [19]. Thus, for a spherical immersion $\mathbf{x} : M^n \to S^{m-1}$, Obata's map can be considered as:

$$\nu : M^n \to G(n+1, m) \tag{3.1}$$

which carries each $p \in M^n$ to $\nu(p) = \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_n$.

Let $S^{\binom{m}{n+1}-1}$ be the unit hypersphere in $\mathbb{E}^{\binom{m}{n+1}}$ centered at the origin. Then $G(n+1, m)$ is a submanifold of $S^{\binom{m}{n+1}-1}$, which gives the natural inclusion:

$$\iota : G(n+1, m) \subset S^{\binom{m}{n+1}-1} \subset \mathbb{E}^{\binom{m}{n+1}}.$$

The spherical Gauss map $\hat{\nu} : M^n \to E^{\binom{m}{n+1}}$ associated with $\mathbf{x}$ is thus given by

$$\tilde{\nu} = \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_n : M^n \to G(n+1, m) \subset S^{\binom{m}{n+1}-1} \subset \mathbb{E}^{\binom{m}{n+1}}. \tag{3.2}$$

The associated map $\hat{\nu}$ of M^n in $S^{\binom{m}{n+1}-1}$ is also called the spherical Gauss map.

Let M be a Riemannian manifold and G a closed subgroup of the group $I(M)$ of isometries which acts transitively on M. An immersion f of M into another Riemannian manifold $\tilde{M}$ is called *G-equivariant* if there exists a homomorphism $\zeta : G \to I(\tilde{M})$ such that $f(a(p)) = \zeta(a)f(p)$ for each $a \in G$ and $p \in M$.

An important consequence of Theorem 2.1 is the following finiteness result.

Proposition 3.1. *If $\mathbf{x} : M \to S^{m-1} \subset \mathbb{E}^m$ is an equivariant isometric immersion of a compact homogeneous Riemannian n-manifold into S^{m-1}, then its spherical Gauss map $\tilde{\nu}$ is of finite type. Moreover, the type number of $\tilde{\nu}$ is at most $\binom{m}{n+1}$.*

Proof. Let τ be the tension field of $\tilde{\nu}$. Then $\tau, \Delta\tau, \dots, \Delta^{\binom{m}{n+1}}\tau$ are linearly dependent at a given point $u \in M$. Thus there is a polynomial $Q(t)$ of degree $\leq N$ satisfying $Q(\Delta)\tau = 0$ at u. Since $\mathbf{x}$ is equivariant, the group of isometries of the Euclidean space acts transitively on M as well as on the tangent bundle of M. Hence, it acts transitively on its spherical Gauss of map. Thus, we have $Q(\Delta)\tau = 0$ at each point in M. Therefore, Theorem 2.1 implies that the spherical Gauss map is of finite type. Since the degree of the minimal polynomial of Q is $\leq \binom{m}{n+1}$, the type number is at most $\binom{m}{n+1}$ according to theorem 2.1. □

Remark 3.1. Proposition 3.1 shows that there exist abundant examples of nice spherical submanifolds with finite type spherical Gauss map.

By differentiating $\tilde{\nu}$ in (3.2) we find

$$e_j\tilde{\nu} = \sum_{r,k} h^r_{jk}\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n. \tag{3.3}$$

From (3.3) and Gauss' equation, we see that the induced metric $\hat{g}$ on M^n via $\hat{\nu}$ is

$$\hat{g} = \sum_{j,k} \Big\{(n-1)\delta_{jk} + \sum_r (\operatorname{tr} h^r) h^r_{jk} - R_{jk}\Big\}\omega^j \otimes \omega^k. \tag{3.4}$$

Since the Laplacian of $\hat{\nu}$ is defined by

$$\Delta\tilde{\nu} = -\sum_i e_i e_i \tilde{\nu} + (\nabla_{e_i} e_i)\tilde{\nu}, \tag{3.5}$$

we obtain from (3.3) that

$$\begin{aligned}\Delta\tilde{\nu} &= nH \wedge e_1 \wedge \cdots \wedge e_n + ||h||^2 \tilde{\nu} \\ &\quad - n\sum_k \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge D_{e_k}\hat{H} \wedge e_{k+1} \wedge \cdots \wedge e_n \\ &\quad - \sum_{r,s,i,j,k} h^r_{ij} h^s_{ik}\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n.\end{aligned} \tag{3.6}$$

By applying (3.6), (2.8) and the last equation in (2.7), we obtain

$$\begin{aligned}\Delta\tilde{\nu} &= ||\hat{h}||^2 \tilde{\nu} + n\hat{H} \wedge e_1 \wedge \cdots \wedge e_n \\ &\quad - n\sum_k \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge D_{e_k}\hat{H} \wedge e_{k+1} \wedge \cdots \wedge e_n \\ &\quad + \sum_{r,s,j,k} R^r_{sjk}\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n.\end{aligned} \tag{3.7}$$

A map between two Riemannian manifolds is called *harmonic* if its tension field, $\tau = \operatorname{div}(d\phi)$, vanishes identically. For an isometric immersion, the tension field and the mean curvature vector field differ only by the dimension of the submanifold.

In [21], Ruh and Vilms proved that a Euclidean submanifold has parallel mean curvature vector if and only if its classical Gauss map is harmonic. On the other hand, for spherical Gauss map we have the following.

Proposition 3.2. *Let* $\mathbf{x} : (M^n, g) \to S^{m-1}$ *be an isometric immersion of a Riemannian n-manifold. Then we have:*

(a) *Obata's map* $\nu : (M^n, g) \to G(n+1, m)$ *is a harmonic map if and only if* $\mathbf{x} : (M^n, g) \to S^{m-1}$ *is a minimal immersion;*

(b) *The spherical Gauss map* $\tilde{\nu} : (M^n, g) \to \mathbb{E}^{\binom{m}{n+1}}$ *is harmonic if and only if* $\mathbf{x} : (M^n, g) \to S^{m-1}$ *is totally geodesic.*

Proof. Statement (a) follows from (3.7) and the fact that decomposable $(n+1)$-vectors of the forms:

$$e_r \wedge e_1 \wedge \cdots \wedge e_n, \quad \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_r \wedge \cdots \wedge e_n$$

provide a basis for the tangent space of $G(n+1, m)$. Statement (b) is an immediate consequence of (3.7). □

Lawson proved in [18] that the spherical Gauss map of minimal surfaces in S^3 are minimal surfaces in S^3 (possibly with branched points). An immediate application of Proposition 3.2 is the following extension of Lawson's result.

Proposition 3.3. *Let* $\mathbf{x} : (M^2, g) \to S^{m-1}$ *be a minimal surface without totally geodesic points. Then the Obata map* $\nu : (M^2, \hat{g}) \to G(3, m)$ *is a minimal isometric immersion (with respect to the induced metric* $\hat{g}$*).*

Proof. For the minimal isometric immersion $x : (M^2, \hat{g}) \to S^{m-1}$, (3.4) reduces to

$$\hat{g} = (1 - K)g, \tag{3.8}$$

where K is the Gauss curvature of (M^2, g). Since harmonicity is preserved under conformal change of metric, Proposition 3.2 implies that Obata's map ν is harmonic. Thus this immersion is a minimal immersion, since it is isometric. □

4. Spherical submanifolds with 1-type spherical Gauss map

In this section, we completely classify spherical submanifolds with 1-type spherical Gauss map.

Theorem 4.1. *A submanifold of* S^{m-1} *has mass-symmetric 1-type spherical Gauss map if and only if it is a minimal submanifold of* S^{m-1} *with constant scalar curvature and flat normal connection.*

Proof. If the spherical submanifold has mass-symmetric 1-type spherical Gauss map $\tilde{\nu}$, the $\Delta\tilde{\nu}$ and $\tilde{\nu}$ are proportional. Since the second and third terms on the right hand side of (3.7) are perpendicular to $\tilde{\nu}$, we see from (3.7) that $\tilde{\nu}$ is mass-symmetric 1-type if and only if $\hat{H} = R^r_{sjk} = 0$ and $||\hat{h}||^2$ is constant. Hence, by applying this and (2.7) we obtain the desired result. □

The standard imbedding of $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ in S^3 is called *Clifford minimal torus.*

Theorem 4.2. *A non-totally geodesic surface in* S^{m-1} *has mass-symmetric 1-type spherical Gauss map if and only if it is an open portion of the Clifford minimal torus (lying fully in a totally geodesic 3-sphere* $S^3 \subset S^{m-1}$*).*

Proof. It is easy to verify that the spherical Gauss map of the Clifford minimal torus satisfies $\Delta\tilde{\nu} = 2\tilde{\nu}$. Thus, it has mass-symmetric 1-type spherical Gauss map. The converse follows from Theorem 4.1 and the fact that the only minimal surfaces of S^{m-1} with constant Gauss curvature and flat normal connection are open portions of a totally geodesic 2-sphere or of the Clifford minimal torus. □

Theorem 4.3. *An n-dimensional submanifold of* S^{m-1} *has non-mass-symmetric 1-type spherical Gauss map if and only if it has constant scalar curvature and it is immersed in a totally geodesic* $(n+1)$*-sphere* $S^{n+1} \subset S^{m-1}$ *as a hypersurface with nonzero constant mean curvature.*

Proof. Let $\mathbf{x} : M^n \to S^{m-1}$ be an isometric immersion of a Riemannian n-manifold into S^{m-1}. If the spherical Gauss map $\tilde{\nu}$ of $\mathbf{x}$ is non-mass-symmetric 1-type, then we have $\Delta\tilde{\nu} = \lambda_p(\tilde{\nu} - c)$ for some vector c and real number λ_p. Thus we have

$$(\Delta\tilde{\nu})_j = \lambda_p(\tilde{\nu})_j, \tag{4.1}$$

where $(\cdot)_j = e_j(\cdot)$.

On the other hand, by a direct long computation, we obtain from (3.7) that

(4.2)
$$\begin{aligned}
&e_i(\Delta\tilde{\nu})\\
&= ||\hat{h}||_i^2\tilde{\nu} + ||\hat{h}||^2 \sum_{r,k} h_{ik}^r\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n\\
&+ 2nD_{e_i}\hat{H} \wedge e_1 \wedge \cdots \wedge e_n + n\sum_{r,k} h_{ik}^r \hat{H} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n\\
&- n\sum_{j\neq k} \omega_j^k(e_i)\mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge \underbrace{e_k}_{j-th} \wedge e_{j+1} \wedge \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n\\
&- n\sum_{j\neq k} h_{ij}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge \underbrace{e_r}_{j-th} \wedge e_{j+1} \wedge \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n\\
&+ n\sum \langle A_{D_{e_k}\hat{H}} e_i, e_k \rangle\, \tilde{\nu} - n\sum_k \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{D_{e_i}D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n\\
&+ \sum_{r,s,j,k} \left((e_iR_{sjk}^r)\mathbf{x} + R_{sjk}^r\, e_i\right) \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n\\
&+ \sum_{r,s} \sum_{j,k,\ell\neq} R_{sjk}^r \Big\{ h_{i\ell}^t\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \overbrace{e_t}^{\ell-th} \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n\\
&+ \omega_\ell^h(e_i)\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \overbrace{e_h}^{\ell-th} \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n \Big\}\\
&+ \sum_{r,s,j,k} (R_{sjk}^r \omega_s^t(e_i) - R_{sjk}^t \omega_s^r(e_i))\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_t}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n.
\end{aligned}$$

Case (a): $\hat{H} = 0$. In this case, equation (4.2) reduces to

$$\begin{aligned}
&e_i(\Delta\tilde{\nu})\\
&= ||\hat{h}||_i^2\tilde{\nu} + ||\hat{h}||^2 \sum_{r,k} h_{ik}^r\, \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n\\
&+ \sum_{r,s,j,k} \left((e_iR_{sjk}^r)\mathbf{x} + R_{sjk}^r e_i\right) \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n
\end{aligned} \tag{4.3}$$

$$+\sum_{r,s}\sum_{j,k,\ell\neq} R^r_{sjk}\Big\{h^t_{i\ell}\,\mathbf{x}\wedge e_1\wedge\cdots\wedge \underbrace{e_t}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n$$

$$+\sum_h \omega^h_\ell(e_i)\,\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_h}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\Big\}$$

$$+\sum_{r,s,j,k}(R^r_{sjk}\omega^t_s(e_i)-R^t_{sjk}\omega^r_s(e_i))\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n.$$

By comparing (3.3), (4.1) and (4.3), we get $||\hat{h}||^2_i = R^r_{sjk} = 0$. Thus, M^n has constant scalar curvature and flat normal connection. So, Theorem 4.1 implies that $\tilde{\nu}$ is mass-symmetric 1-type. This is a contradiction.

Case (b); $\hat{H}\neq 0$. Since the term $D_{e_i}\hat{H}\wedge e_1\wedge\cdots\wedge e_n$ appears only in $\tilde{\nabla}_{e_i}(\Delta\tilde{\nu})$ of (4.2), not in $e_i(\tilde{\nu})$, we know from (3.3), (4.1) and (4.2) that $D\hat{H}=0$. Thus, M^n has parallel nonzero mean curvature vector in S^{m-1}. So, it has nonzero constant mean curvature. Therefore, equation (4.2) reduces to

(4.4)

$$\begin{aligned}
&e_i(\Delta\tilde{\nu})\\
&= n\sum_{r,k} h^r_{ik}\hat{H}\wedge e_1\wedge\cdots\wedge e_{k-1}\wedge e_r\wedge e_{k+1}\wedge\cdots\wedge e_n\\
&\quad+||\hat{h}||^2_i\tilde{\nu}+||\hat{h}||^2\sum_{r,k} h^r_{ik}\,\mathbf{x}\wedge e_1\wedge\cdots\wedge e_{k-1}\wedge e_r\wedge e_{k+1}\wedge\cdots\wedge e_n\\
&\quad+\sum_{r,s,j,k}\{(e_iR^r_{sjk})\mathbf{x}+R^r_{sjk}e_i\}\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\\
&\quad+\sum_{r,s}\sum_{j,k,\ell\neq} R^r_{sjk}\Big\{h^t_{i\ell}\,\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\\
&\quad+\omega^h_\ell(e_i)\,\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_h}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\Big\}\\
&\quad+\sum_{r,s,j,k}(R^r_{sjk}\omega^t_s(e_i)-R^t_{sjk}\omega^r_s(e_i))\,\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n.
\end{aligned}$$

From (3.3), (4.1) and (4.4) we know that $||\hat{h}||$ and scalar curvature are constant. Also, we have

$$\tag{4.5}\begin{aligned}
&n\sum_{r,k}h^r_{ik}\hat{H}\wedge e_1\wedge\cdots\wedge e_{k-1}\wedge e_r\wedge e_{k+1}\wedge\cdots\wedge e_n\\
&+\sum_{r,s,j,k}R^r_{sjk}\,e_i\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n=0.
\end{aligned}$$

Put $\hat{H}=|\hat{H}|e_{n+1}$. It follows from (4.5) that $R^r_{sjk}=0$ for $r,s\geq n+2$ and $j,k=1,\ldots,n$. We also find $R^3_{sjk}=0$ from $D\hat{H}=0$. Thus, the normal connection of M^n in S^{m-1} is flat. Therefore, (4.5) yields

$$\sum_{r=n+2}^{m-1} \sum_{k} h^r_{ik} e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n = 0. \tag{4.6}$$

We see from (4.6) that the first normal space $\operatorname{Im} h$ is spanned by e_{n+1}. Therefore, by the reduction theorem of Erbarcher, we conclude that M^n is contained in a totally geodesic $S^{n+1} \subset S^{m-1}$.

Conversely, if M^n has constant scalar curvature and it lies in a totally geodesic $S^{n+1} \subset S^{m-1}$ with nonzero constant mean curvature, then (4.4) reduces to

$$e_i(\Delta\tilde{\nu}) = ||\hat{h}||^2 \sum_{r,k} h^r_{ik} \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n. \tag{4.7}$$

From this we have $e_i(\Delta\tilde{\nu}) = ||\hat{h}||^2 e_i(\tilde{\nu})$. Thus, $\Delta(\tilde{\nu} - c) = ||\hat{h}||^2 \tilde{\nu}$ for some nonzero vector c. Since $||\hat{h}||$ is constant, this shows that $\tilde{\nu}$ is non-mass-symmetric 1-type. □

Combining Theorems 4.1 and 4.3 yields the following remarkable result.

Corollary 4.1. *Every isoparametric hypersurface of S^{n+1} has 1-type spherical Gauss map.*

Remark 4.1. Theorem 4.1 and Theorem 4.3 determine completely spherical submanifolds with 1-type spherical Gauss map. On the other hand, Proposition 3.1, Theorem 4.1 and Theorem 4.3 imply that there exist rich examples of spherical submanifolds with higher type spherical Gauss map. In particular, most equivariant isometric immersions of compact homogeneous Riemannian spaces into S^{m-1} have high finite type spherical Gauss map.

Remark 4.2. Another implication of Theorems 4.1 and 4.3 is that finite type Euclidean submanifolds (cf. [6]) and spherical submanifolds with finite type spherical Gauss map are quite different. For instance, a submanifold in $\mathbb{E}^m$ is of 1-type if and only if it is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere of $\mathbb{E}^m$. On contrast, according to Theorems 4.1 and 4.3, most spherical minimal submanifolds (except those with constant scalar curvature and flat normal connection) have higher type spherical Gauss map. In the next two sections, we completely classify spherical minimal surfaces with 2-type spherical Gauss map.

5. Veronese surface and its spherical Gauss map

For a natural number k, the set of spherical harmonic polynomials of degree k of three variables is a $(2k+1)$-dimensional vector space. Consider the unit sphere S^{2k} in the vector space with the standard inner product. We have an isometric minimal immersion: $\psi_k : S^2(K) \to S^{2k}$ of the 2-sphere $S^2(K)$ of curvature $K = 2/(k(k+1))$ into S^{2k}, which is known as a Veronese-Borůvka sphere. The immersion $\psi_2 : S^2(\frac{1}{3}) \to S^4$ is known as the *Veronese surface.* Wallach [22] proved that any minimal surface of positive constant curvature K in a unit sphere is an open part of a Veronese-Borůvka sphere.

Let (x, y, z) be the natural coordinate system of $\mathbb{E}^3$ and $(u_1, \ldots, u_5)$ that of $\mathbb{E}^5$. The minimal immersion ψ_2 of $S^2(\frac{1}{3})$ into $S^4 \subset \mathbb{E}^5$ can be expressed explicitly as

$$\begin{gathered} u_1 = \frac{yz}{\sqrt{3}},\ u_2 = \frac{xz}{\sqrt{3}},\ u_3 = \frac{xy}{\sqrt{3}},\ u_4 = \frac{x^2 - y^2}{2\sqrt{3}}, \\ u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2) \end{gathered} \tag{5.1}$$

Theorem 4.1 implies that a totally geodesic S^2 in S^3 has 1-type spherical Gauss map. The next result provides a very simple characterization of Veronese surface in terms of 2-type spherical Gauss map.

Theorem 5.1. *A minimal surface M of S^{m-1} is an open portion of the Veronese surface (lying fully in a totally geodesic $S^4 \subset S^{m-1}$) if and only if it has mass-symmetric 2-type spherical Gauss map.*

Proof. First, assume that M^2 is an open portion of the Veronese surface defined by (5.1). If we choose the spherical coordinates:

$$\begin{gathered} x = \sqrt{3}\sin\Big(\frac{u}{\sqrt{3}}\Big)\cos\Big(\frac{v}{\sqrt{3}}\Big),\ y = \sqrt{3}\sin\Big(\frac{u}{\sqrt{3}}\Big)\sin\Big(\frac{v}{\sqrt{3}}\Big), \\ z = \sqrt{3}\cos\Big(\frac{u}{\sqrt{3}}\Big) \end{gathered} \tag{5.2}$$

on $S^2(\sqrt{3})$, then the metric tensor g and the Laplacian operator are

$$g = du^2 + \sin^2\Big(\frac{u}{\sqrt{3}}\Big)dv^2, \tag{5.3}$$

$$\Delta = \frac{1}{\sqrt{3}}\cot\Big(\frac{u}{\sqrt{3}}\Big)\frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2} + \csc^2\Big(\frac{u}{\sqrt{3}}\Big)\frac{\partial^2}{\partial v^2}. \tag{5.4}$$

Let us choose

$$e_1 = \frac{\partial}{\partial u},\ e_2 = \csc\Big(\frac{u}{\sqrt{3}}\Big)\frac{\partial}{\partial v},\ \hat{h}(e_1, e_1) = \frac{e_3}{\sqrt{3}},\ \hat{h}(e_1, e_2) = \frac{e_4}{\sqrt{3}}. \tag{5.5}$$

Then e_1, e_2, e_3, e_4 form an orthonormal frame field. Moreover, we have

$$\begin{gathered} h^3_{11} = h^4_{12} = \frac{1}{\sqrt{3}},\ h^3_{12} = h^4_{11} = -h^4_{22} = 0, \\ \omega^4_3 = 2\omega^2_1 = \frac{2}{\sqrt{3}}\cot\Big(\frac{u}{\sqrt{3}}\Big)\omega^2,\ ||\hat{h}||^2 = -2K^D = \frac{4}{3}. \end{gathered} \tag{5.6}$$

By applying (3.7), (5.6) and a long computation we find

$$\Delta\tilde{\nu} = \frac{4}{3}\tilde{\nu} - \frac{4}{3}\mathbf{x} \wedge e_3 \wedge e_4,\ \Delta^2\tilde{\nu} = \frac{32}{3}\tilde{\nu} - \frac{56}{3}\mathbf{x} \wedge e_3 \wedge e_4. \tag{5.7}$$

Thus, if we put

$$\tilde{\nu}_1 = \frac{4}{5}\tilde{\nu} + \frac{2}{5}\mathbf{x} \wedge e_3 \wedge e_4,\ \tilde{\nu}_3 = \frac{1}{5}\tilde{\nu} - \frac{2}{5}\mathbf{x} \wedge e_3 \wedge e_4, \tag{5.8}$$

we obtain $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_3$ with $\Delta\tilde{\nu}_1 = \frac{2}{3}\tilde{\nu}_1$, and $\Delta\tilde{\nu}_3 = 4\tilde{\nu}_3$, which implies that the spherical Gauss map is mass-symmetric and of 2-type.

Let us use the following convention on the ranges of indices:

$$1 \le i, j, k, \ell \le 2; \;\; 3 \le r, s, t \le 4; \;\; 5 \le \alpha, \beta, \gamma \le m-1.$$

Next, assume that $\mathbf{x} : M^2 \to S^{m-1}$ is a minimal surface with mass-symmetric 2-type spherical Gauss map.We choose $e_{n+1}, \dots, e_{m-1}, e_m$ in such way that

$$e_m = \mathbf{x}, \quad A_{e_5} = \cdots = A_{e_{m-1}} = 0. \tag{5.9}$$

Then $A_{r_m} = -I$ and $R^r_{s12} = 0$, $(r, s) \neq (3,4), (4,3)$. Thus, we get from (3.7) that

$$\begin{aligned} &\Delta\tilde{\nu} = ||\hat{h}||^2\tilde{\nu} + 2K^D\,\mathbf{x} \wedge e_3 \wedge e_4, \\ &K^D = \sum_j (h^3_{2j}h^4_{1j} - h^3_{1j}h^4_{2j}), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \Delta^2\tilde{\nu} = {} & (\Delta||\hat{h}||^2 + ||\hat{h}||^4)\tilde{\nu} + 2(K^D||\hat{h}||^2 + \Delta K^D)\,\mathbf{x} \wedge e_3 \wedge e_4 \\ & + 2K^D\Delta(\mathbf{x} \wedge e_3 \wedge e_4) - 2\sum ||\hat{h}||^2_j \tilde{\nabla}_{e_j}\tilde{\nu} \\ & - 4\sum K^D_j \tilde{\nabla}_{e_j}(\mathbf{x} \wedge e_3 \wedge e_4), \end{aligned} \tag{5.11}$$

where $||\hat{h}||^2_j = e_j||\hat{h}||^2$ and $K^D_j = e_j K^D$.

From (2.1) we have

$$\begin{aligned} e_j(\mathbf{x} \wedge e_3 \wedge e_4) = {} & e_j \wedge e_3 \wedge e_4 - \sum_i h^3_{ij}\,\mathbf{x} \wedge e_i \wedge e_4 \\ & - \sum_i h^4_{ij}\,\mathbf{x} \wedge e_3 \wedge e_i + \sum_\alpha \omega^\alpha_3(e_j)\,\mathbf{x} \wedge e_\alpha \wedge e_4 \\ & + \sum_\alpha \omega^\alpha_4(e_j)\,\mathbf{x} \wedge e_3 \wedge e_\alpha. \end{aligned} \tag{5.12}$$

Moreover, we obtain from (2.3) and (5.12) that

$$\begin{aligned} & \Delta(\mathbf{x} \wedge e_3 \wedge e_4) \\ & = 2K^D\tilde{\nu} \\ & + \Big(2 + ||\hat{h}||^2 + \sum_\alpha ||\omega^\alpha_3||^2 + \sum_\alpha ||\omega^\alpha_4||^2\Big)\,\mathbf{x} \wedge e_3 \wedge e_4 \\ & - 2\sum_{j,\alpha} \omega^\alpha_3(e_j)e_j \wedge e_\alpha \wedge e_4 - 2\sum_{j,\alpha} \omega^\alpha_4(e_j)e_j \wedge e_3 \wedge e_\alpha \\ & + \sum_i h^3_{ij;j}\,\mathbf{x} \wedge e_i \wedge e_4 - \sum_i h^4_{ij;j}\,\mathbf{x} \wedge e_i \wedge e_3 \end{aligned} \tag{5.13}$$

$$+2\sum_{j,\alpha}(h^3_{ij}\omega^\alpha_4(e_j)-h^4_{ij}\omega^\alpha_3(e_j))\mathbf{x}\wedge e_i\wedge e_\alpha$$
$$+\sum_{j,\alpha}\big(e_j(\omega^\alpha_3(e_j))\mathbf{x}\wedge e_4\wedge e_\alpha-e_j(\omega^\alpha_4(e_j))\mathbf{x}\wedge e_3\wedge e_\alpha\big)$$
$$-\sum_{j,\alpha}\big(\omega^4_3(e_j)\omega^\alpha_4(e_j)+\sum_\beta\omega^\beta_3(e_j)\omega^\beta_\alpha(e_j)\big)\mathbf{x}\wedge e_4\wedge e_\alpha$$
$$+\sum_{j,\alpha}\big(\omega^3_4(e_j)\omega^\alpha_3(e_j)+\sum_\beta\omega^\beta_4(e_j)\omega^\beta_\alpha(e_j)\big)\mathbf{x}\wedge e_3\wedge e_\alpha.$$

By combining (5.11), (5.12) and (5.13) and applying (2.3) of Codazzi and the minimality of M^2 in S^{m-1}, we obtain

$$
\begin{aligned}
\Delta^2\tilde{\nu}=&(\Delta||\hat{h}||^2+||\hat{h}||^4+4(K^D)^2)\tilde{\nu}+\Big\{2\Delta K^D\\
&+2K^D\Big(2||\hat{h}||^2+2+\sum_\beta\Big(||\omega^\beta_3||^2+||\omega^\beta_4||^2\Big)\Big)\Big\}\mathbf{x}\wedge e_3\wedge e_4\\
&-2\sum_{j,r}||\hat{h}||^2_j\big(h^r_{2j}\mathbf{x}\wedge e_1\wedge e_r-h^r_{1j}\,\mathbf{x}\wedge e_2\wedge e_r\big)\\
&+4\sum_{i,j}K^D_j\big(h^3_{ij}\mathbf{x}\wedge e_i\wedge e_4-h^4_{ij}\mathbf{x}\wedge e_i\wedge e_3-e_j\wedge e_3\wedge e_4\big)\\
&+4K^D\sum_{j,\alpha}(h^3_{ij}\omega^\alpha_4(e_j)-h^4_{ij}\omega^\alpha_3(e_j))\mathbf{x}\wedge e_i\wedge e_\alpha\\
&+4K^D\sum_{j,\alpha}\omega^\alpha_3(e_j)e_j\wedge e_4\wedge e_\alpha-4K^D\sum_{j,\alpha}\omega^\alpha_4(e_j)e_j\wedge e_3\wedge e_\alpha\\
&+2\sum_{j,\alpha}\Big\{K^D\big(\,(\nabla_{e_j}\omega^\alpha_3)\,e_j-\omega^4_3(e_j)\omega^\alpha_4(e_j)-\sum_\beta\omega^\beta_3(e_j)\omega^\beta_\alpha(e_j)\big)\\
&+2\,K^D_j\omega^\alpha_3(e_j)\Big\}\mathbf{x}\wedge e_4\wedge e_\alpha\\
&-2\sum_{j,\alpha}\Big\{K^D\big(\,(\nabla_{e_j}\omega^\alpha_4)\,e_j-\omega^3_4(e_j)\omega^\alpha_3(e_j)-\sum_\beta\omega^\beta_4(e_j)\omega^\beta_\alpha(e_j)\big)\\
&+2K^D_j\omega^\alpha_4(e_j)\Big\}\mathbf{x}\wedge e_3\wedge e_\alpha.
\end{aligned}\tag{5.14}
$$

If $\tilde{\nu}$ is mass-symmetric a 2-type, then we have $\tilde{\nu}=\tilde{\nu}_p+\tilde{\nu}_q$ with $\Delta\tilde{\nu}_p=\lambda_p\tilde{\nu}_p,\Delta\tilde{\nu}_q=\lambda_q\tilde{\nu}_q$, $\lambda_p\neq\lambda_q$. From this we get

$$\Delta^2\tilde{\nu}=(\lambda_p+\lambda_q)\Delta\tilde{\nu}-\lambda_p\lambda_q\tilde{\nu}.\tag{5.15}$$

Since $e_j\wedge e_3\wedge e_\alpha$, $e_j\wedge e_\alpha\wedge e_4$ appear only in $\Delta^2\tilde{\nu}$, and not in $\Delta\tilde{\nu}$ or in $\tilde{\nu}$, it follows from (5.14) and (5.15) that $K^D\omega^\alpha_3=K^D\omega^\alpha_4=0$. Hence, by continuity, we see that exactly one of the following two cases occurs:

(a) $K^D=0$ on M^2;

(b) $\omega^\alpha_3=\omega^\alpha_4=0$ and $K^D\neq 0$ on some non-empty open subset U of M^2.

Case (a): $K^D = 0$ *on* M^2. In this case, M^2 is a minimal surface of a totally geodesic $S^3 \subset S^{m-1}$. Thus (5.10) and (5.14) reduce to

$$\Delta\tilde{\nu} = ||\hat{h}||^2\tilde{\nu}, \tag{5.16}$$

$$\Delta^2\tilde{\nu} = \Delta||\hat{h}||^2\tilde{\nu} + ||\hat{h}||^4\tilde{\nu} + 2\sum_j e_j||\hat{h}||^2\{h^3_{1j}\,\mathbf{x}\wedge e_2\wedge e_3 - h^3_{2j}\mathbf{x}\wedge e_1\wedge e_3\}. \tag{5.17}$$

By applying (5.15), (5.16) and (5.17), we find

$$||\hat{h}||^2_1 h^3_{11} + ||\hat{h}||^2_2 h^3_{12} = ||\hat{h}||^2_1 h^3_{12} - ||\hat{h}||^2_2 h^3_{11} = 0. \tag{5.18}$$

Hence, $||\hat{h}||$ is constant. Thus, $\tilde{\nu}$ is of 1-type by Theorem 4.1, which is a contradiction.

Case (b) $\omega^\alpha_3 = \omega^\alpha_4 = 0, \alpha = 5, \dots, m-1$ *and* $K^D \neq 0$ *on* U. The first normal bundle $\operatorname{Im}\hat{h}$ on U is a rank 2 parallel subbundle of the normal bundle. Thus, U is a minimal surface in a totally geodesic $S^4 \subset S^{m-1}$ and (5.14) reduces to

$$\begin{aligned}\Delta^2\tilde{\nu} = {} & \{\Delta||\hat{h}||^2 + ||\hat{h}||^4 + 4(K^D)^2\}\tilde{\nu} \\ & + 4K^D\{1 + ||\hat{h}||^2\}\,\mathbf{x}\wedge e_3\wedge e_4 + 2\Delta K^D\,\mathbf{x}\wedge e_3\wedge e_4 \\ & + 2\sum_{j,r}||\hat{h}||^2_j(h^r_{1j}\,\mathbf{x}\wedge e_2\wedge e_r - h^r_{2j}\mathbf{x}\wedge e_1\wedge e_r) \\ & - 4\sum_j K^D_j\Big\{e_j\wedge e_3\wedge e_4 - \sum_i h^3_{ij}\mathbf{x}\wedge e_i\wedge e_4 \\ & - \sum_i h^4_{ij}\mathbf{x}\wedge e_3\wedge e_i\Big\}.\end{aligned} \tag{5.19}$$

Since $e_j\wedge e_3\wedge e_4$ appear only in $\Delta^2\tilde{\nu}$, (5.10), (5.15) and (5.19) imply that K^D is constant. Therefore (5.19) becomes

$$\begin{aligned}\Delta^2\tilde{\nu} = {} & \{\Delta||\hat{h}||^2 + ||\hat{h}||^4 + 4(K^D)^2\}\tilde{\nu} \\ & + 4K^D\{1 + ||\hat{h}||^2\}\,\mathbf{x}\wedge e_3\wedge e_4 \\ & + 2\sum_{j,r}||\hat{h}||^2_j(h^r_{1j}\,\mathbf{x}\wedge e_2\wedge e_r - h^r_{2j}\mathbf{x}\wedge e_1\wedge e_r).\end{aligned} \tag{5.20}$$

It follows from (5.10), (5.15) and (5.20) that U has constant curvature. Hence, a result of [16] implies that U is an open portion of the Veronese surface. So, by continuity, the whole M^2 is an open portion of the Veronese surface. □

6. Spectral characterization of equilateral minimal torus

Consider the map $\bar{\mathbf{y}} : \mathbb{E}^2 \to \mathbb{E}^6$ defined by

$$(6.1)\qquad \begin{aligned} \bar{\mathbf{y}}(s,t) = \frac{1}{\sqrt{3}}\Big(&\cos\Big(\frac{\sqrt{3}t+s}{\sqrt{2}}\Big), \sin\Big(\frac{\sqrt{3}t+s}{\sqrt{2}}\Big), \cos\Big(\frac{\sqrt{3}t-s}{\sqrt{2}}\Big),\\ &\sin\Big(\frac{\sqrt{3}t-s}{\sqrt{2}}\Big), \cos\sqrt{2}s, \sin\sqrt{2}s\Big). \end{aligned}$$

Then $\bar{\mathbf{y}}$ gives rise to an isometric immersion $\mathbf{y}$ from a flat torus $\bar{T}^2$ into $S^5 \subset \mathbb{E}^6$. The metric tensor on $\bar{T}^2$ induced from (6.1) is $g = ds^2 + dt^2$. It is easy to verify that $\Delta\bar{\mathbf{y}} = 2\bar{\mathbf{y}}$. Thus $\mathbf{y} : \bar{T}^2 \to S^5$ is a minimal flat torus $\bar{T}^2$ in S^5, which is known as the *equilateral minimal torus.* This minimal torus can also be expressed as:

$$(6.2)\qquad \bar{y}(\theta,\tau) = \frac{1}{\sqrt{3}}\big(\cos\theta, \sin\theta, \cos\tau, \sin\tau, \cos(\theta-\tau), \sin(\theta-\tau)\big),$$

with $\theta = (\sqrt{3}t + s)/\sqrt{2}$ and $\tau = (\sqrt{3}t - s)/\sqrt{2}$. However, the metric tensor on $\bar{T}^2$ induced from (6.2) is given by

$$g = \frac{2}{3}(d\theta^2 - d\theta d\tau + d\tau^2)$$

instead.

Theorem 6.1. *A minimal surface of S^{m-1} is an open portion of the equilateral minimal torus (lying fully in a totally geodesic $S^5 \subset S^{m-1}$) if and only if it has non-mass-symmetric 2-type spherical Gauss map.*

Proof. First, assume that M^2 is an open portion of the equilateral minimal torus in $S^5 \subset \mathbb{E}^6$ defined by (6.1). If we put

$$(6.3)\qquad \begin{aligned} &e_1 = \frac{\partial}{\partial s},\ e_2 = \frac{\partial}{\partial t},\ e_3 = \sqrt{2}\,(h(e_1,e_1) + \bar{y}),\\ &e_4 = \sqrt{2}\,h(e_1,e_2),\quad e_5 = D_{\partial/\partial s}e_3,\quad e_6 = \bar{y}, \end{aligned}$$

then e_3, e_4, e_5, e_6 are orthonormal normal vector fields. A direct computation gives

$$(6.4)\qquad \begin{aligned} &h^3_{11} = -h^3_{22} = h^4_{12} = \frac{1}{\sqrt{2}},\ h^3_{12} = h^4_{11} = h^4_{22} = h^\alpha_{ij} = 0,\\ &\omega^2_1 = \omega^4_3 = 0,\ \ \omega^5_3 = \omega^1,\ \ \omega^5_4 = -\omega^2,\ \alpha = 5,6;\ i,j = 1,2. \end{aligned}$$

It follows from (5.10), (5.15) and (6.4) that the spherical Gauss map satisfies

$$(6.5)\qquad \begin{aligned} &\Delta\tilde{\nu} = 2\tilde{\nu} - 2\,\bar{y}\wedge e_3\wedge e_4,\\ &\Delta^2\tilde{\nu} = 8\tilde{\nu} - 16\,\bar{y}\wedge e_3\wedge e_4 - 4e_1\wedge e_4\wedge e_5 - 4e_2\wedge e_3\wedge e_5. \end{aligned}$$

Put $c = \frac{1}{4}\big(\tilde{\nu} + \bar{y}\wedge e_3\wedge e_4 - e_1\wedge e_4\wedge e_5 - e_2\wedge e_3\wedge e_5\big)$. Then we see that c is a constant vector by differentiating c and applying (2.1), (3.3), and (6.4). If we put

$$\tilde{\nu}_1 = \tilde{\nu} + \frac{1}{3}\bar{y} \wedge e_3 \wedge e_4 - \frac{4}{3}c, \quad \tilde{\nu}_2 = \frac{1}{3}(c - \bar{y} \wedge e_3 \wedge e_4), \tag{6.6}$$

a straight-forward long computation yields $\tilde{\nu} = c + \tilde{\nu}_1 + \tilde{\nu}_2$ with $\Delta\tilde{\nu}_1 = 2\tilde{\nu}_1$ and $\Delta\tilde{\nu}_2 = 8\tilde{\nu}_2$. Thus, $\tilde{\nu}$ is non-mass-symmetric 2-type.

Conversely, assume M^2 has non-mass-symmetric 2-type spherical Gauss map. If we choose $e_1, e_2, e_3, \ldots, e_m$ as in the proof of Theorem 5.1 and use the same convention on the range of indices, we obtain (5.9), (5.11) and (5.14).

From (3.3) and (5.10), (5.14), and a very long computation, we obtain

$$\begin{aligned} e_k\tilde{\nu} &= \sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r), \\ e_k(\Delta\tilde{\nu}) &= ||\hat{h}||^2_k\tilde{\nu} + ||\hat{h}||^2 \sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r) \\ &\quad + 2K^D_k\, \mathbf{x} \wedge e_3 \wedge e_4 + 2K^D\Big\{ e_k \wedge e_3 \wedge e_4 - \sum_j h^3_{jk}\mathbf{x} \wedge e_j \wedge e_4 \\ &\quad + \sum_j h^4_{jk}\mathbf{x} \wedge e_j \wedge e_3 - \sum_\alpha \omega^\alpha_3(e_k)\mathbf{x} \wedge e_4 \wedge e_\alpha \\ &\quad + \sum_\alpha \omega^\alpha_4(e_k)\mathbf{x} \wedge e_3 \wedge e_\alpha \Big\} \end{aligned} \tag{6.7}$$

and

(6.8)

$$\begin{aligned} e_k(\Delta^2\tilde{\nu}) &= \{e_k(\Delta||\hat{h}||^2) + e_k(||\hat{h}||^4) + 8K^D K^D_k\}\tilde{\nu} \\ &\quad + 2\Big\{ e_k(\Delta K^D) + 2K^D_k + 2K^D_k||\hat{h}||^2 + K^D_k \sum_\beta \left(||\omega^\beta_3||^2 + ||\omega^\beta_4||^2\right) \\ &\quad + 2K^D||\hat{h}||^2_k + K^D \sum_\beta \left(||\omega^\beta_3||^2_k + ||\omega^\beta_4||^2_k\right) \Big\}\mathbf{x} \wedge e_3 \wedge e_4 \\ &\quad - 2\sum_{j,r} \left(||\hat{h}||^2_{jk}h^r_{2j} + ||\hat{h}||^2_j e_k(h^r_{2j})\right)\mathbf{x} \wedge e_1 \wedge e_r \\ &\quad + 2\sum_{j,r} \left(||\hat{h}||^2_{jk}h^r_{1j} + ||\hat{h}||^2_j e_k(h^r_{1j})\right)\mathbf{x} \wedge e_2 \wedge e_r \\ &\quad + 4\sum_j K^D_{jk}\Big\{ \sum_i h^3_{ij}\mathbf{x} \wedge e_i \wedge e_4 - \sum_i h^4_{ij}\mathbf{x} \wedge e_i \wedge e_3 \\ &\quad - \sum_\alpha \omega^\alpha_3(e_j)\mathbf{x} \wedge e_\alpha \wedge e_4 - \sum_\alpha \omega^\alpha_4(e_j)\mathbf{x} \wedge e_3 \wedge e_\alpha - e_j \wedge e_3 \wedge e_4 \Big\} \\ &\quad + 4\sum_j K^D_j\Big\{ \sum_i e_k(h^3_{ij})\mathbf{x} \wedge e_i \wedge e_4 - \sum_i e_k(h^4_{ij})\mathbf{x} \wedge e_i \wedge e_3 \end{aligned}$$

$$+\sum_{\alpha} e_k(\omega_3^\alpha(e_j))\mathbf{x}\wedge e_4\wedge e_\alpha-\sum_{\alpha} e_k(\omega_4^\alpha(e_j))\mathbf{x}\wedge e_3\wedge e_\alpha\Big\}$$

$$+2K_k^D\sum_{j,\alpha}\Big\{2(h_{ij}^3\omega_4^\alpha(e_j)-h_{ij}^4\omega_3^\alpha(e_j))\mathbf{x}\wedge e_i\wedge e_\alpha$$

$$+\big[\left(\nabla_{e_j}\omega_3^\alpha\right)e_j-\omega_3^4(e_j)\omega_4^\alpha(e_j)-\sum_{\beta}\omega_3^\beta(e_j)\omega_\alpha^\beta(e_j)\big]\mathbf{x}\wedge e_4\wedge e_\alpha$$

$$-\big[\left(\nabla_{e_j}\omega_4^\alpha\right)e_j-\omega_4^3(e_j)\omega_3^\alpha(e_j)-\sum_{\beta}\omega_4^\beta(e_j)\omega_\alpha^\beta(e_j)\big]\mathbf{x}\wedge e_3\wedge e_\alpha$$

$$+2\,\omega_3^\alpha(e_j)e_j\wedge e_4\wedge e_\alpha-2\omega_4^\alpha(e_j)e_j\wedge e_3\wedge e_\alpha\Big\}$$

$$+2K^D\sum_{j,\alpha}\Big\{2e_k\left(h_{ij}^3\omega_4^\alpha(e_j)-h_{ij}^4\omega_3^\alpha(e_j)\right)\mathbf{x}\wedge e_i\wedge e_\alpha$$

$$+e_k\big[\left(\nabla_{e_j}\omega_3^\alpha\right)e_j-\omega_3^4(e_j)\omega_4^\alpha(e_j)-\sum_{\beta}\omega_3^\beta(e_j)\omega_\alpha^\beta(e_j)\big]\mathbf{x}\wedge e_4\wedge e_\alpha\Big\}$$

$$-2K^D\sum_{j,\alpha}e_k\big[\left(\nabla_{e_j}\omega_4^\alpha\right)e_j-\omega_4^3(e_j)\omega_3^\alpha(e_j)-\sum_{\beta}\omega_4^\beta(e_j)\omega_\alpha^\beta(e_j)\big]\mathbf{x}\wedge e_3\wedge e_\alpha$$

$$+4K^D\sum_{j,\alpha}e_k(\omega_3^\alpha(e_j))e_j\wedge e_4\wedge e_\alpha-4K^D\sum_{j,\alpha}e_k(\omega_4^\alpha(e_j))e_j\wedge e_3\wedge e_\alpha$$

$$+(\Delta||\hat{h}||^2+||\hat{h}||^4+4(K^D)^2)\sum_{r}(h_{2k}^r\mathbf{x}\wedge e_1\wedge e_r-h_{1k}^r\mathbf{x}\wedge e_2\wedge e_r)$$

$$+2\Big\{\Delta K^D+K^D\big(2+2||\hat{h}||^2+\sum_{\beta}\left(||\omega_3^\beta||^2+||\omega_4^\beta||^2\right)\big)\Big\}$$

$$\times\Big\{e_k\wedge e_3\wedge e_4-\sum_{j}h_{jk}^3\mathbf{x}\wedge e_j\wedge e_4+\sum_{j}h_{jk}^4\mathbf{x}\wedge e_j\wedge e_3$$

$$-\sum_{\alpha}\omega_3^\alpha(e_k)\mathbf{x}\wedge e_4\wedge e_\alpha+\sum_{\alpha}\omega_4^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_\alpha\Big\}$$

$$-2\sum_{j,r}||\hat{h}||_j^2h_{2j}^r\Big\{e_k\wedge e_1\wedge e_r+\omega_1^2(e_k)\mathbf{x}\wedge e_2\wedge e_r+\sum_{s}h_{1k}^s\mathbf{x}\wedge e_s\wedge e_r$$

$$-h_{2k}^r\tilde{\nu}+\sum_{s}\omega_r^s(e_k)\mathbf{x}\wedge e_1\wedge e_s-\sum_{\alpha}\omega_r^\alpha(e_k)\mathbf{x}\wedge e_1\wedge e_\alpha\Big\}$$

$$+2\sum_{j,r}||\hat{h}||_j^2h_{1j}^r\Big\{e_k\wedge e_2\wedge e_r-\omega_1^2(e_k)\mathbf{x}\wedge e_1\wedge e_r+\sum_{s}h_{2k}^s\mathbf{x}\wedge e_s\wedge e_r$$

$$+h_{1k}^r\tilde{\nu}+\sum_{s}\omega_r^s(e_k)\mathbf{x}\wedge e_2\wedge e_s+\sum_{\alpha}\omega_r^\alpha(e_k)\mathbf{x}\wedge e_2\wedge e_\alpha\Big\}$$

$$+4\sum_{i,j}K_j^Dh_{ij}^3\Big\{e_k\wedge e_i\wedge e_4+\sum_{\ell}\omega_i^\ell(e_k)\mathbf{x}\wedge e_\ell\wedge e_4+h_{ik}^3\mathbf{x}\wedge e_3\wedge e_4$$

$$-\sum_{\ell}h_{k\ell}^4\mathbf{x}\wedge e_i\wedge e_\ell-\omega_3^4(e_k)\mathbf{x}\wedge e_i\wedge e_3+\sum_{\alpha}\omega_4^\alpha(e_k)\mathbf{x}\wedge e_i\wedge e_\alpha\Big\}$$

$$
\begin{aligned}
&-4\sum_{i,j} K_j^D h_{ij}^4 \Big\{ e_k \wedge e_i \wedge e_3 + h_{k\ell}^3 \mathbf{x} \wedge e_\ell \wedge e_i + \omega_3^4(e_k)\mathbf{x} \wedge e_i \wedge e_4 \\
&+ \sum_\alpha \omega_3^\alpha(e_k)\mathbf{x} \wedge e_i \wedge e_\alpha + \sum_\ell \omega_i^\ell(e_k)\mathbf{x} \wedge e_\ell \wedge e_3 - h_{ik}^4 \mathbf{x} \wedge e_3 \wedge e_4 \Big\} \\
&-4\sum_j K_j^D \Big\{ \sum_i \omega_j^i(e_k) e_i \wedge e_3 \wedge e_4 - \sum_i h_{ik}^3 e_j \wedge e_i \wedge e_4 \\
&- \sum_\alpha \omega_3^\alpha(e_k) e_j \wedge e_4 \wedge e_\alpha + \sum_i h_{ik}^4 e_j \wedge e_i \wedge e_3 + \sum_\alpha \omega_4^\alpha(e_k) e_j \wedge e_3 \wedge e_\alpha \Big\} \\
&+2\sum_{j,\alpha} \Big\{ K^D \big[\left(\nabla_{e_j}\omega_3^\alpha\right) e_j - \omega_3^4(e_j)\omega_4^\alpha(e_j) - \sum_\gamma \omega_3^\gamma(e_j)\omega_\alpha^\gamma(e_j) \big] \\
&+ 2K_j^D \omega_3^\alpha(e_j) \Big\} \Big\{ e_k \wedge e_4 \wedge e_\alpha - \sum_i h_{ik}^4 \mathbf{x} \wedge e_i \wedge e_\alpha - \omega_3^4(e_k)\mathbf{x} \wedge e_3 \wedge e_\alpha \\
&+ \sum_\beta \omega_4^\beta(e_k)\mathbf{x} \wedge e_\beta \wedge e_\alpha + \omega_3^\alpha(e_k)\mathbf{x} \wedge e_3 \wedge e_4 + \sum_\beta \omega_\alpha^\beta(e_k)\mathbf{x} \wedge e_4 \wedge e_\beta \Big\} \\
&-2\sum_{j,\alpha} \Big\{ K^D \big[\left(\nabla_{e_j}\omega_4^\alpha\right) e_j - \omega_4^3(e_j)\omega_3^\alpha(e_j) - \sum_\gamma \omega_4^\gamma(e_j)\omega_\alpha^\gamma(e_j) \big] \\
&+ 2K_j^D \omega_4^\alpha(e_j) \Big\} \Big\{ e_k \wedge e_3 \wedge e_\alpha - \sum_i h_{ik}^3 \mathbf{x} \wedge e_i \wedge e_\alpha \\
&+ \omega_3^4(e_k)\mathbf{x} \wedge e_4 \wedge e_\alpha - \omega_4^\alpha(e_k)\mathbf{x} \wedge e_3 \wedge e_4 + \sum_\beta \omega_3^\beta(e_k)\mathbf{x} \wedge e_\beta \wedge e_\alpha \\
&+ \sum_\beta \omega_\alpha^\beta(e_k)\mathbf{x} \wedge e_3 \wedge e_\beta \Big\} + 4K^D \sum_{j,\alpha} \left(h_{ij}^3 \omega_4^\alpha(e_j) - h_{ij}^4 \omega_3^\alpha(e_j) \right) \\
&\times \Big\{ e_k \wedge e_i \wedge e_\alpha + \sum_\ell \omega_i^\ell(e_k)\mathbf{x} \wedge e_\ell \wedge e_\alpha + \sum_r h_{ik}^r \mathbf{x} \wedge e_r \wedge e_\alpha \\
&- \omega_3^\alpha(e_k)\mathbf{x} \wedge e_i \wedge e_3 - \omega_4^\alpha(e_k)\mathbf{x} \wedge e_i \wedge e_4 + \sum_\beta \omega_\alpha^\beta(e_k)\mathbf{x} \wedge e_i \wedge e_\beta \Big\} \\
&+ 4K^D \sum_{j,\alpha} \omega_3^\alpha(e_j) \Big\{ \sum_i \omega_j^i(e_k) e_i \wedge e_4 \wedge e_\alpha + h_{jk}^3 e_3 \wedge e_4 \wedge e_\alpha \\
&+ \omega_3^\alpha(e_k) e_j \wedge e_3 \wedge e_4 + \sum_\beta \omega_\alpha^\beta(e_k) e_j \wedge e_4 \wedge e_\beta - \sum_i h_{ik}^4 e_j \wedge e_i \wedge e_\alpha \\
&- \omega_3^4(e_k) e_j \wedge e_3 \wedge e_\alpha - \sum_\beta \omega_4^\beta(e_k) e_j \wedge e_\alpha \wedge e_\beta \Big\} \\
&- 4K^D \sum_{j,\alpha} \omega_4^\alpha(e_j) \Big\{ \sum_i \omega_j^i(e_k) e_i \wedge e_3 \wedge e_\alpha - h_{jk}^4 e_3 \wedge e_4 \wedge e_\alpha \\
&- \sum_i h_{ik}^3 e_j \wedge e_i \wedge e_\alpha + \omega_3^4(e_k) e_j \wedge e_4 \wedge e_\alpha - \sum_\beta \omega_3^\beta(e_k) e_j \wedge e_\alpha \wedge e_\beta
\end{aligned}
$$

$$- \omega_4^\alpha(e_k)e_j \wedge e_3 \wedge e_4 + \sum_\beta \omega_\alpha^\beta(e_k)e_j \wedge e_3 \wedge e_\beta \Big\},$$

where $K^D_{jk} = e_k e_j K^D$ and $||\hat{h}||^2_{jk} = e_k e_j ||\hat{h}||^2$. Since M^2 has non-mass-symmetric 2-type spherical Gauss map, we have $\tilde{\nu} = c + \tilde{\nu}_1 + \tilde{\nu}_2$ with $\Delta\tilde{\nu}_1 = \lambda_p \tilde{\nu}_1$, $\Delta\tilde{\nu}_2 = \lambda_q \tilde{\nu}_2$, $\lambda_p \neq \lambda_q$ and nonzero vector c. Hence, we get

$$e_k(\Delta^2\tilde{\nu}) - (\lambda_p + \lambda_q)e_k(\Delta\tilde{\nu}) + \lambda_p\lambda_q e_k\tilde{\nu} = 0, \quad k = 1, 2. \tag{6.9}$$

If K^D vanishes identically on M^2, then M^2 lies in a totally geodesic $S^3 \subset S^{m-1}$. So, we may assume $m = 4$. Hence, (6.7) and (6.8) reduce to

(6.10)

$$\begin{aligned}
e_k\tilde{\nu} &= \sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r), \\
e_k(\Delta\tilde{\nu}) &= ||\hat{h}||^2_k \tilde{\nu} + ||\hat{h}||^2 (h^3_{2k}\mathbf{x} \wedge e_1 \wedge e_3 - h^3_{1k}\mathbf{x} \wedge e_2 \wedge e_3), \\
e_k(\Delta^2\tilde{\nu}) &= \{e_k(\Delta||\hat{h}||^2) + ||\hat{h}||^4_k\}\tilde{\nu} \\
&\quad - 2\sum_j \left(||\hat{h}||^2_{jk} h^3_{2j} + ||\hat{h}||^2_j e_k(h^3_{2j})\right)\mathbf{x} \wedge e_1 \wedge e_3 \\
&\quad + 2\sum_j \left(||\hat{h}||^2_{jk} h^3_{1j} + ||\hat{h}||^2_j e_k(h^3_{1j})\right)\mathbf{x} \wedge e_2 \wedge e_3 \\
&\quad + \left(\Delta||\hat{h}||^2 + ||\hat{h}||^4\right)(h^3_{2k}\mathbf{x} \wedge e_1 \wedge e_3 - h^3_{1k}\mathbf{x} \wedge e_2 \wedge e_3) \\
&\quad + 2\sum_j ||\hat{h}||^2_j h^3_{2j}\{\delta_{2k}e_1 \wedge e_2 \wedge e_3 - \omega_1^2(e_k)\mathbf{x} \wedge e_2 \wedge e_3 + h^3_{2k}\tilde{\nu}\} \\
&\quad + 2\sum_j ||\hat{h}||^2_j h^3_{1j}\{\delta_{1k}e_1 \wedge e_2 \wedge e_3 - \omega_1^2(e_k)\mathbf{x} \wedge e_1 \wedge e_3 + h^3_{1k}\tilde{\nu}\}.
\end{aligned}$$

We see from (6.10) that $e_1 \wedge e_2 \wedge e_3$ appears in $e_k(\Delta^2\tilde{\nu})$, not in $e_k(\Delta\tilde{\nu})$ or in $e_k\tilde{\nu}$. Thus, (6.9) and (6.10) imply that $||\hat{h}||$ is constant. So, according to Theorem 4.1, $\tilde{\nu}$ is mass-symmetric 1-type which is a contradiction. Since M^2 is analytic, those imply $K^D \neq 0$ *on an open dense subset* U *of* M^2.

From $K^D \neq 0$ on U, we may choose $e_1, e_2, e_3, \ldots, e_{m-1}$ on U such that $A_{e_3}, A_{e_4} \neq 0, A_{e_5} = \cdots = A_{e_{m-1}} = h^3_{12} = 0$, $D_{e_1}e_3$ lies in the linear subspace spanned by e_4, e_5 and $D_{e_2}e_3$ lies in the linear subspace spanned by e_4, e_5, e_6. From these we have

$$\omega_3^6(e_1) = \omega_3^7 = \cdots = \omega_3^{m-1} = h^3_{12} = h^\alpha_{jk} = 0, \quad \alpha = 5, \ldots, m-1, \tag{6.11}$$

$$K^D = -2h^3_{11}h^4_{12} \neq 0, \quad h^3_{11} = -h^3_{22} \neq 0 \tag{6.12}$$

on the dense open subset U.

We see from (6.7) and (6.8) that $e_1 \wedge e_2 \wedge e_\alpha$ appears in $e_k(\Delta^2\tilde{\nu})$, not in $e_k(\Delta\tilde{\nu})$ or in $e_k\tilde{\nu}$. Thus, (6.8) and (6.9) yield

$$\begin{aligned}&\sum_j (h^3_{ij}\omega^\alpha_4(e_j) - h^4_{ij}\omega^\alpha_3(e_j))e_k \wedge e_i \wedge e_\alpha \\ &= \sum_{i,j} \omega^\alpha_3(e_j)h^4_{ik}e_j \wedge e_i \wedge e_\alpha - \sum_{i,j} \omega^\alpha_4(e_j)h^3_{ik}e_j \wedge e_i \wedge e_\alpha\end{aligned} \tag{6.13}$$

for $k = 1, 2;\ \alpha = 5, \ldots, m-1$.

In view of (6.11), we find from (6.13) that

$$h^4_{11}\omega^\alpha_3(e_2) - h^4_{12}\omega^\alpha_3(e_1) = h^3_{11}\omega^\alpha_4(e_2), \tag{6.14}$$

$$h^4_{11}\omega^\alpha_3(e_1) + h^4_{12}\omega^\alpha_3(e_2) = h^3_{11}\omega^\alpha_4(e_1) \tag{6.15}$$

for $\alpha = 5, \ldots, m-1$. Combining these with $\omega^6_3(e_1) = 0$ gives

$$h^4_{11}\omega^6_3(e_2) = h^3_{11}\omega^6_4(e_2), \quad h^4_{12}\omega^6_3(e_2) = h^3_{11}\omega^6_4(e_1), \tag{6.16}$$

which implies that

$$h^4_{11}\omega^6_4(e_1) = h^4_{12}\omega^6_4(e_2). \tag{6.17}$$

On the other hand, we see from (6.7) and (6.8) that $e_3 \wedge e_4 \wedge e_\alpha$ appears only in $e_k(\Delta^2\tilde{\nu})$, not in $e_k(\Delta\tilde{\nu})$ or in $e_k\tilde{\nu}$. Thus, (6.8) and (6.9) give

$$\sum_j \omega^\alpha_3(e_j)h^3_{jk} + \sum_j \omega^\alpha_4(e_j)h^4_{jk} = 0 \tag{6.18}$$

for $k = 1, 2$ and $\alpha = 5, \ldots, m-1$. Hence, in view of (6.11), we find

$$\omega^\alpha_3(e_1)h^3_{11} + \omega^\alpha_4(e_1)h^4_{11} + \omega^\alpha_4(e_2)h^4_{12} = 0, \tag{6.19}$$

$$\omega^\alpha_3(e_2)h^3_{11} - \omega^\alpha_4(e_1)h^4_{12} + \omega^\alpha_4(e_2)h^4_{11} = 0 \tag{6.20}$$

for $\alpha = 5, \ldots, m-1$. In particular, we have $\omega^6_4(e_1)h^4_{11} + \omega^6_4(e_2)h^4_{12} = 0$. Combining this with (6.11) and (6.17) gives

$$h^4_{11}\omega^6_4(e_1) = 0, \quad \omega^6_3(e_1) = \omega^6_4(e_2) = 0. \tag{6.21}$$

By substituting the second equation of (6.21) into (6.16) and (6.20), we obtain

$$h^4_{11}\omega^6_3(e_2) = 0, \quad \omega^6_3(e_2)h^3_{11} = \omega^6_4(e_1)h^4_{12}. \tag{6.22}$$

Also we obtain from (6.11), (6.19) and (6.20) that

$$\omega^\alpha_4(e_1)h^4_{11} + \omega^\alpha_4(e_2)h^4_{12} = \omega^\alpha_4(e_1)h^4_{12} - \omega^\alpha_4(e_2)h^4_{11} = 0$$

for $\alpha = 7, \ldots, m-1$. Combining this with (6.11) gives

$$\omega^7_3 = \cdots = \omega^{m-1}_3 = \omega^7_4 = \cdots = \omega^{m-1}_4 = 0. \tag{6.23}$$

We see from (6.7) and (6.8) that $e_j \wedge e_5 \wedge e_6$ appears in $e_k(\Delta^2\tilde{\nu})$, not in $e_k(\Delta\tilde{\nu})$ or in $e_k\tilde{\nu} = 0$. Thus, we obtain from (6.8), (6.9), (6.11) and (6.21) that

$$\omega^5_4(e_1)\omega^6_3(e_2) = \omega^6_4(e_1)\omega^5_3(e_2). \tag{6.24}$$

Now, we divide the proof into two cases:

Case (i): $h^4_{11} \neq 0$. From (6.21)-(6.23) we obtain $\omega^\alpha_3 = \omega^\alpha_4 = 0$ for $\alpha = 6, \dots, m-1$. Also, from (6.14), (6.15), (6.19) and (6.20), we have

$$\begin{aligned} &h^3_{11}\omega^5_4(e_2) - h^4_{11}\omega^5_3(e_2) + h^4_{12}\omega^5_3(e_1) = 0, \\ &h^3_{11}\omega^5_4(e_1) - h^4_{11}\omega^5_3(e_1) - h^4_{12}\omega^5_3(e_2) = 0, \\ &h^3_{11}\omega^5_3(e_1) + h^4_{11}\omega^5_4(e_1) + h^4_{12}\omega^5_4(e_2) = 0, \\ &h^3_{11}\omega^5_3(e_2) + h^4_{11}\omega^5_4(e_2) - h^4_{12}\omega^5_4(e_1) = 0. \end{aligned} \tag{6.25}$$

After solving this linear system we obtain $\omega^5_3 = \omega^5_4 = 0$. Consequently, we have

$$\omega^\alpha_3 = \omega^\alpha_4 = 0, \ \ \alpha = 5, \dots, m-1. \tag{6.26}$$

Since the first normal bundle, $\operatorname{Im} \hat{h}$, is spanned by e_3, e_4 on U, (6.26) implies that the first normal bundle is a parallel subbundle of the normal bundle. Hence, the reduction theorem of Erbarcher implies that M^2 lies in a totally geodesic $S^4 \subset S^{m-1}$. So, without loss of generality, we may assume that $m = 5$.

Now, by comparing the coefficients of $e_1 \wedge e_2 \wedge e_3$ and of $e_1 \wedge e_2 \wedge e_4$, we obtain from (6.7), (6.8) and (6.9) that

$$\begin{aligned} ||\hat{h}||^2_1 h^3_{11} = 4K^D_1 h^4_{12} - 4K^D_2 h^4_{11}, \ \ ||\hat{h}||^2_2 h^3_{11} = 4K^D_1 h^4_{11} + 4K^D_2 h^4_{12}, \\ ||\hat{h}||^2_1 h^4_{11} + ||\hat{h}||^2_2 h^4_{12} = 4K^D_2 h^3_{11}, \ \ ||\hat{h}||^2_1 h^4_{12} - ||\hat{h}||^2_2 h^4_{11} = 4K^D_1 h^3_{11}. \end{aligned}$$

After solving this system we obtain $K^D_1 = K^D_2 = 0$. Thus, the normal curvature K^D is a nonzero constant. Therefore, equations (6.7) and (6.8) reduce to

$$\begin{aligned} e_k\tilde{\nu} = &\sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r), \\ e_k(\Delta\tilde{\nu}) = &\, ||\hat{h}||^2_k \tilde{\nu} + ||\hat{h}||^2 \sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r) \\ &+ 2K^D \Big\{ e_k \wedge e_3 \wedge e_4 - \sum_j h^3_{jk}\mathbf{x} \wedge e_j \wedge e_4 + \sum_j h^4_{jk}\mathbf{x} \wedge e_j \wedge e_3 \Big\}, \\ e_k(\Delta^2\tilde{\nu}) = &\Big\{ e_k(\Delta||\hat{h}||^2) + ||\hat{h}||^4_k + 2\sum_{i,j,r} ||\hat{h}||^2_j h^r_{ij} h^r_{ik} \Big\} \tilde{\nu} \\ &- 2\sum_{j,r} \big(||\hat{h}||^2_{jk} h^r_{2j} + ||\hat{h}||^2_j e_k(h^r_{2j}) \big) \mathbf{x} \wedge e_1 \wedge e_r \\ &+ 2\sum_{j,r} \Big(||\hat{h}||^2_{jk} h^r_{1j} + ||\hat{h}||^2_j e_k(h^r_{1j}) \Big) \mathbf{x} \wedge e_2 \wedge e_r \\ &+ (\Delta||\hat{h}||^2 + ||\hat{h}||^4 + 4(K^D)^2) \sum_r (h^r_{2k}\mathbf{x} \wedge e_1 \wedge e_r - h^r_{1k}\mathbf{x} \wedge e_2 \wedge e_r) \end{aligned}$$

$$\begin{aligned}
&+4K^D||\hat{h}||_k^2\,\mathbf{x}\wedge e_3\wedge e_4+4K^D(1+2||\hat{h}||^2)\\
&\times\Big\{e_k\wedge e_3\wedge e_4-\sum_j h^3_{jk}\mathbf{x}\wedge e_j\wedge e_4+\sum_j h^4_{jk}\mathbf{x}\wedge e_j\wedge e_3\Big\}\\
&-2\sum_{j,r}||\hat{h}||_j^2h^r_{2j}\Big\{e_k\wedge e_1\wedge e_r+\omega_1^2(e_k)\mathbf{x}\wedge e_2\wedge e_r\\
&+\sum_s\omega_r^s(e_k)\mathbf{x}\wedge e_1\wedge e_s+\sum_s h^s_{1k}\mathbf{x}\wedge e_s\wedge e_r\Big\}\\
&+2\sum_{j,r}||\hat{h}||_j^2h^r_{1j}\Big\{e_k\wedge e_2\wedge e_r-\omega_1^2(e_k)\mathbf{x}\wedge e_1\wedge e_r\\
&+\sum_s\omega_r^s(e_k)\mathbf{x}\wedge e_2\wedge e_s+\sum_s h^s_{2k}\mathbf{x}\wedge e_s\wedge e_r\Big\}.
\end{aligned}\tag{6.27}$$

Now, by comparing the coefficients of $e_k\wedge e_3\wedge e_4$ in (6.27), we obtain from (6.9) and (6.27) that $4||\hat{h}||^2=\lambda_p+\lambda_q-2$, which is constant. Hence, M^2 has constant Gauss curvature. Hence, M^2 is an open piece of Veronese surface according to a result of Kenmotsu [16]. This contradicts to Theorem 5.1.

Case (ii): $h^4_{11}=0$. We obtain from (6.14), (6.15), (6.19) and (6.20) that

$$\begin{aligned}
&h^3_{11}\omega_4^5(e_2)+h^4_{12}\omega_3^5(e_1)=h^3_{11}\omega_4^5(e_1)-h^4_{12}\omega_3^5(e_2)=0,\\
&h^3_{11}\omega_3^5(e_1)+h^4_{12}\omega_4^5(e_2)=h^3_{11}\omega_3^5(e_2)-h^4_{12}\omega_4^5(e_1)=0,\\
&h^3_{11}\omega_4^6(e_1)-h^4_{12}\omega_3^6(e_2)=h^3_{11}\omega_3^6(e_2)-h^4_{12}\omega_4^6(e_1)=0.
\end{aligned}\tag{6.28}$$

Since $h^3_{11},h^4_{12}\neq 0$, solving this system gives

$$\omega_3^5(e_1)=\pm\omega_4^5(e_2),\ \ \omega_3^5(e_2)=\pm\omega_4^5(e_1),\ \ \omega_3^6(e_2)=\pm\omega_4^6(e_1).\tag{6.29}$$

If $\omega_3^5=\omega_4^5=\omega_3^6=\omega_4^6=0$, then by applying the same argument as Case (i), we know that M^2 is a minimal surface of constant Gauss curvature of a totally geodesic $S^4\subset S^{m-1}$, which leads to the same contradiction as Case (i). Hence, at least one of $\omega_3^5,\omega_3^6,\omega_4^5,\omega_4^6$ is nonzero. Therefore, (6.28) and (6.29) yield $h^3_{11}=\pm h^4_{12}$. So, after replacing e_4 by $-e_4$ if necessary, we find

$$\begin{aligned}
&h^3_{11}=h^4_{12}\neq 0,\ h^3_{12}=h^4_{11}=h^4_{22}=\omega_3^6(e_1)=\omega_4^6(e_2)=0,\\
&\omega_3^5(e_1)=-\omega_4^5(e_2),\ \ \omega_3^5(e_2)=\omega_4^5(e_1),\ \ \omega_3^6(e_2)=\omega_4^6(e_1),\\
&||\hat{h}||^2=4(h^3_{11})^2=-2K^D.
\end{aligned}\tag{6.30}$$

By considering $e_1\wedge e_2\wedge e_r$, we obtain from (6.7), (6.8), (6.9) and (6.30) that

$$\begin{aligned}
0=&\sum_{j,r}K_j^D(h^r_{1j}\delta_{1k}+h^r_{2j}\delta_{2k})e_1\wedge e_2\wedge e_r\\
&+\sum_i K_k^Dh^4_{ik}e_k\wedge e_i\wedge e_3-\sum_j K_j^Dh^3_{kk}e_j\wedge e_k\wedge e_4\\
&-\sum_{i,j}K_j^D\{h^3_{ij}e_k\wedge e_i\wedge e_4-h^4_{ij}e_k\wedge e_i\wedge e_3\}.
\end{aligned}\tag{6.31}$$

Combining this with (6.30) gives $K_1^D = K_2^D = 0$. Thus, K^D, h_{11}^3, h_{12}^4 are constant. So, it follows from the last equation in (6.30) that the Gauss curvature of M^2 is also constant. Thus, we have $K \geq 0$ (cf. [15]). From these, we find

$$h(e_1, e_1) = ce_3, \quad h(e_1, e_2) = ce_4, \quad h(e_2, e_2) = -ce_3, \tag{6.32}$$

for a nonzero constant c. By applying (6.32) and the equation of Codazzi, we find

$$\omega_3^4 = 2\omega_1^2. \tag{6.33}$$

By differentiating (6.33) and applying (6.30) and structure equations, we have

$$\sum_\alpha ||\omega_3^\alpha||^2 = \sum_\alpha ||\omega_4^\alpha||^2 = 6c^2 - 2 = \frac{3}{2}||\hat{h}||^2 - 2. \tag{6.34}$$

Hence, after using (6.34) and the constancy of K (or equivalently $||\hat{h}||$) and K^D, we know that (6.7) and (6.8) reduce to

$$\begin{aligned}
e_k\tilde{\nu} = &\sum_r (h_{2k}^r \mathbf{x} \wedge e_1 \wedge e_r - h_{1k}^r \mathbf{x} \wedge e_2 \wedge e_r), \\
e_k(\Delta\tilde{\nu}) = &\,||\hat{h}||^2 \sum_r (h_{2k}^r \mathbf{x} \wedge e_1 \wedge e_r - h_{1k}^r \mathbf{x} \wedge e_2 \wedge e_r) \\
&- ||\hat{h}||^2 \Big\{ e_k \wedge e_3 \wedge e_4 - h_{kk}^3 \mathbf{x} \wedge e_k \wedge e_4 + \sum_j h_{jk}^4 \mathbf{x} \wedge e_j \wedge e_3 \\
&- \sum_\alpha \omega_3^\alpha(e_k)\mathbf{x} \wedge e_4 \wedge e_\alpha + \sum_\alpha \omega_4^\alpha(e_k)\mathbf{x} \wedge e_3 \wedge e_\alpha \Big\}, \\
e_k(\Delta^2\tilde{\nu}) = &\, -2||\hat{h}||^2 \sum_{j,\alpha} \left(h_{ij}^3 e_k \omega_4^\alpha(e_j) - h_{ij}^4 e_k \omega_3^\alpha(e_j) \right) \mathbf{x} \wedge e_i \wedge e_\alpha \\
&- ||\hat{h}||^2 \sum_{j,\alpha} e_k \big(\left(\nabla_{e_j} \omega_3^\alpha\right) e_j - \omega_3^4(e_j)\omega_4^\alpha(e_j) \\
&- \sum_\beta \omega_3^\beta(e_j)\omega_\alpha^\beta(e_j) \big) \mathbf{x} \wedge e_4 \wedge e_\alpha \\
&+ ||\hat{h}||^2 \sum_{j,\alpha} e_k \big(\left(\nabla_{e_j} \omega_4^\alpha\right) e_j - \omega_4^3(e_j)\omega_3^\alpha(e_j) \\
&- \sum_\beta \omega_4^\beta(e_j)\omega_\alpha^\beta(e_j) \big) \mathbf{x} \wedge e_3 \wedge e_\alpha \\
&- 2||\hat{h}||^2 \sum_{j,\alpha} e_k(\omega_3^\alpha(e_j)) e_j \wedge e_4 \wedge e_\alpha
\end{aligned} \tag{6.35}$$

$$+2\|\hat{h}\|^2\sum_{j,\alpha} e_k(\omega_4^\alpha(e_j))e_j\wedge e_3\wedge e_\alpha$$

$$+2\|\hat{h}\|^4\sum_r(h_{2k}^r\mathbf{x}\wedge e_1\wedge e_r-h_{1k}^r\mathbf{x}\wedge e_2\wedge e_r)$$

$$-2\|\hat{h}\|^2\left(6c^2-1+\|\hat{h}\|^2\right)\Big\{e_k\wedge e_3\wedge e_4-h_{kk}^3\mathbf{x}\wedge e_k\wedge e_4$$

$$+\sum_j h_{jk}^4\mathbf{x}\wedge e_j\wedge e_3-\sum_\alpha\omega_3^\alpha(e_k)\mathbf{x}\wedge e_4\wedge e_\alpha+\sum_\alpha\omega_4^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_\alpha\Big\}$$

$$-\|\hat{h}\|^2\sum_{j,\alpha}\Big(\left(\nabla_{e_j}\omega_3^\alpha\right)e_j-\omega_3^4(e_j)\omega_4^\alpha(e_j)-\sum_\gamma\omega_3^\gamma(e_j)\omega_\alpha^\gamma(e_j)\Big)$$

$$\times\Big\{e_k\wedge e_4\wedge e_\alpha-\sum_i h_{ik}^4\mathbf{x}\wedge e_i\wedge e_\alpha-\omega_3^4(e_k)\mathbf{x}\wedge e_3\wedge e_\alpha$$

$$+\sum_\beta\omega_4^\beta(e_k)\mathbf{x}\wedge e_\beta\wedge e_\alpha+\omega_3^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_4+\sum_\beta\omega_\alpha^\beta(e_k)\mathbf{x}\wedge e_4\wedge e_\beta\Big\}$$

$$+\|\hat{h}\|^2\sum_{j,\alpha}\Big(\left(\nabla_{e_j}\omega_4^\alpha\right)e_j-\omega_4^3(e_j)\omega_3^\alpha(e_j)-\sum_\gamma\omega_4^\gamma(e_j)\omega_\alpha^\gamma(e_j)\Big)$$

$$\times\Big\{e_k\wedge e_3\wedge e_\alpha-\sum_i h_{ik}^3\mathbf{x}\wedge e_i\wedge e_\alpha+\omega_3^4(e_k)\mathbf{x}\wedge e_4\wedge e_\alpha$$

$$+\sum_\beta\omega_3^\beta(e_k)\mathbf{x}\wedge e_\beta\wedge e_\alpha-\omega_4^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_4+\sum_\beta\omega_\alpha^\beta(e_k)\mathbf{x}\wedge e_3\wedge e_\beta\Big\}$$

$$-2\|\hat{h}\|^2\sum_{j,\alpha}\left(h_{ij}^3\omega_4^\alpha(e_j)-h_{ij}^4\omega_3^\alpha(e_j)\right)\Big\{e_k\wedge e_i\wedge e_\alpha+\sum_\ell\omega_i^\ell(e_k)\mathbf{x}\wedge e_\ell\wedge e_\alpha$$

$$+\sum_r h_{ik}^r\mathbf{x}\wedge e_r\wedge e_\alpha-\omega_3^\alpha(e_k)\mathbf{x}\wedge e_i\wedge e_3-\omega_4^\alpha(e_k)\mathbf{x}\wedge e_i\wedge e_4$$

$$+\sum_\beta\omega_\alpha^\beta(e_k)\mathbf{x}\wedge e_i\wedge e_\beta\Big\}-2\|\hat{h}\|^2\sum_{j,\alpha}\omega_3^\alpha(e_j)\Big\{\sum_i\omega_j^i(e_k)e_i\wedge e_4\wedge e_\alpha$$

$$+h_{jk}^3e_3\wedge e_4\wedge e_\alpha+\omega_3^\alpha(e_k)e_j\wedge e_3\wedge e_4+\sum_\beta\omega_\alpha^\beta(e_k)e_j\wedge e_4\wedge e_\beta$$

$$-\sum_i h_{ik}^4e_j\wedge e_i\wedge e_\alpha-\omega_3^4(e_k)e_j\wedge e_3\wedge e_\alpha-\sum_\beta\omega_4^\beta(e_k)e_j\wedge e_\alpha\wedge e_\beta\Big\}$$

$$+2\|\hat{h}\|^2\sum_{j,\alpha}\omega_4^\alpha(e_j)\Big\{\sum_i\omega_j^i(e_k)e_i\wedge e_3\wedge e_\alpha-h_{jk}^4e_3\wedge e_4\wedge e_\alpha$$

$$-h_{kk}^3e_j\wedge e_k\wedge e_\alpha+\omega_3^4(e_k)e_j\wedge e_4\wedge e_\alpha-\sum_\beta\omega_3^\beta(e_k)e_j\wedge e_\alpha\wedge e_\beta$$

$$-\omega_4^\alpha(e_k)e_j\wedge e_3\wedge e_4+\sum_\beta\omega_\alpha^\beta(e_k)e_j\wedge e_3\wedge e_\beta\Big\}.$$

By considering $e_1 \wedge e_3 \wedge e_4$ and $e_2 \wedge e_3 \wedge e_4$, (6.9) and (6.35) give

$$2\big(6c^2 - 1 + ||\hat{h}||^2\big)e_k \wedge e_3 \wedge e_4 + 2\sum_{j,\alpha} \omega_3^\alpha(e_j)\omega_3^\alpha(e_k)e_j \wedge e_3 \wedge e_4$$
$$+ 2\sum_{j,\alpha} \omega_4^\alpha(e_j)\omega_4^\alpha(e_k)e_j \wedge e_3 \wedge e_4 = (\lambda_p + \lambda_q)e_k \wedge e_3 \wedge e_4.$$

Combining this with (6.30) and (6.32) gives

$$\lambda_p + \lambda_q = 8||\hat{h}||^2 - 6. \tag{6.36}$$

It follows from (6.9) and (6.35) that

(6.37)
$$\begin{aligned} 0 = {} & 2\sum_{j,\alpha} \big(e_k(\omega_4^\alpha(e_j)) + \omega_3^\alpha(e_j)\omega_3^4(e_k)\big)e_j \wedge e_3 \wedge e_\alpha \\ & + \sum_{j,\alpha} \Big[\left(\nabla_{e_j}\omega_4^\alpha\right) e_j + \omega_3^4(e_j)\omega_3^\alpha(e_j) + \sum_\gamma \omega_4^\gamma(e_j)\omega_\gamma^\alpha(e_j)\Big] e_k \wedge e_3 \wedge e_\alpha \\ & + 2\sum_{i,j,\alpha} \omega_4^\alpha(e_j)\omega_j^i(e_k)e_i \wedge e_3 \wedge e_\alpha + 2\sum_{j,\alpha,\beta} \omega_4^\alpha(e_j)\omega_\alpha^\beta(e_k)e_j \wedge e_3 \wedge e_\beta \end{aligned}$$

and

(6.38)
$$\begin{aligned} 0 = {} & 2\sum_{j,\alpha} e_k(\omega_3^\alpha(e_j))e_j \wedge e_4 \wedge e_\alpha - 2\sum_{j,\alpha} \omega_4^\alpha(e_j)\omega_3^4(e_k)e_j \wedge e_4 \wedge e_\alpha \\ & + \sum_{j,\alpha} \Big[\left(\nabla_{e_j}\omega_3^\alpha\right) e_j - \omega_3^4(e_j)\omega_4^\alpha(e_j) - \sum_\gamma \omega_3^\gamma(e_j)\omega_\alpha^\gamma(e_j)\Big] e_k \wedge e_4 \wedge e_\alpha \\ & + 2\sum_{i,j,\alpha} \omega_3^\alpha(e_j)\omega_j^i(e_k)e_i \wedge e_4 \wedge e_\alpha + 2\sum_{j,\alpha,\beta} \omega_3^\alpha(e_j)\omega_\alpha^\beta(e_k)e_j \wedge e_4 \wedge e_\beta. \end{aligned}$$

Now, by considering $e_1 \wedge e_3 \wedge e_6, e_2 \wedge e_3 \wedge e_6, e_1 \wedge e_4 \wedge e_6$, and $e_2 \wedge e_4 \wedge e_6$ respectively, we obtain from (6.37) and (6.38) that

$$\begin{aligned} e_2(\omega_3^6(e_2)) = {} & -\omega_3^5(e_2)\omega_5^6(e_2), \;\; \omega_3^6(e_2)(\omega_1^2(e_1) + \omega_3^4(e_1)) \\ = {} & \omega_3^5(e_1)\omega_5^6(e_1), \end{aligned} \tag{6.39}$$

$$\begin{aligned} e_1(\omega_3^6(e_2)) = {} & -\omega_3^5(e_2)\omega_5^6(e_1), \;\; \omega_3^6(e_2)(\omega_1^2(e_2) + \omega_3^4(e_2)) \\ = {} & \omega_3^5(e_1)\omega_5^6(e_2). \end{aligned} \tag{6.40}$$

Similarly, by considering $e_i \wedge e_r \wedge e_5$ for $i = 1, 2$ and $r = 3, 4$, we find

$$e_2(\omega_3^5(e_2)) = \omega_3^6(e_2)\omega_5^6(e_2) - \omega_3^5(e_1)(\omega_1^2(e_2) + \omega_3^4(e_2)), \tag{6.41}$$

$$e_1(\omega_3^5(e_2)) = \omega_3^6(e_2)\omega_5^6(e_1) - \omega_3^5(e_1)(\omega_1^2(e_1) + \omega_3^4(e_1)), \tag{6.42}$$

$$e_2(\omega_3^5(e_1)) = \omega_3^5(e_2)(\omega_1^2(e_2) + \omega_3^4(e_2)), \tag{6.43}$$

$$e_1(\omega_3^5(e_1)) = \omega_3^5(e_2)(\omega_1^2(e_1) + \omega_3^4(e_1)). \tag{6.44}$$

Also, by differentiating ω_5^6 and applying (6.30), we find

$$e_2(\omega_5^6(e_1)) - e_1(\omega_5^6(e_2)) = \omega_1^2(e_1)\omega_5^6(e_1) + \omega_1^2(e_2)\omega_5^6(e_2) + 2\omega_3^5(e_1)\omega_3^6(e_2). \tag{6.45}$$

Now, by applying (6.30), (6.39)-(6.44), and a direct computation, we have

$$\begin{aligned}
&\sum_j (h_{jk}^3\omega_3^\alpha(e_j) + h_{jk}^4\omega_4^\alpha(e_j)) = \sum_j \left(h_{ij}^3\omega_4^\alpha(e_j) - h_{ij}^4\omega_3^\alpha(e_j)\right) = 0,\\
&\sum_j \left\{ \left(\nabla_{e_j}\omega_3^\alpha\right) e_j - \omega_3^4(e_j)\omega_4^\alpha(e_j) - \sum_\beta \omega_3^\beta(e_j)\omega_\alpha^\beta(e_j)\right\} = 0,\\
&\sum_j \left\{ \left(\nabla_{e_j}\omega_4^\alpha\right) e_j - \omega_4^3(e_j)\omega_3^\alpha(e_j) - \sum_\beta \omega_4^\beta(e_j)\omega_\alpha^\beta(e_j)\right\} = 0
\end{aligned} \tag{6.46}$$

for $i, k = 1, 2$; $\alpha = 5, 6$. Thus, in views of (6.24), (6.30), (6.37), (6.38) and (6.46), equation (6.35) reduces to

(6.47)

$$\begin{aligned}
e_k\tilde{\nu} &= \sum_r (h_{2k}^r \mathbf{x}\wedge e_1\wedge e_r - h_{1k}^r\mathbf{x}\wedge e_2\wedge e_r),\\
e_k(\Delta\tilde{\nu}) &= \|\hat{h}\|^2 \sum_r (h_{2k}^r \mathbf{x}\wedge e_1\wedge e_r - h_{1k}^r\mathbf{x}\wedge e_2\wedge e_r)\\
&\quad - \|\hat{h}\|^2 \Big\{(\delta_{1k}e_1\wedge e_3\wedge e_4 + \delta_{2k}e_2\wedge e_3\wedge e_4)\\
&\quad + \sum_j (h_{jk}^4 - h_{kk}^3\delta_{jk})\mathbf{x}\wedge e_j\wedge e_3\\
&\quad - \sum_\alpha \omega_3^\alpha(e_k)\mathbf{x}\wedge e_4\wedge e_\alpha + \sum_\alpha \omega_4^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_\alpha\Big\},\\
e_k(\Delta^2\tilde{\nu}) &= 2\|\hat{h}\|^4 \sum_r (h_{2k}^r \mathbf{x}\wedge e_1\wedge e_r - h_{1k}^r\mathbf{x}\wedge e_2\wedge e_r)\\
&\quad + 2\|\hat{h}\|^2(2+\|\hat{h}\|^2)\Big\{h_{kk}^3\mathbf{x}\wedge e_k\wedge e_4 - \sum_j h_{jk}^4\mathbf{x}\wedge e_j\wedge e_3\\
&\quad - e_k\wedge e_3\wedge e_4 + \sum_\alpha \omega_3^\alpha(e_k)\mathbf{x}\wedge e_4\wedge e_\alpha - \sum_\alpha \omega_4^\alpha(e_k)\mathbf{x}\wedge e_3\wedge e_\alpha\Big\}\\
&\quad - 2\|\hat{h}\|^2(\delta_{1k}e_1\wedge e_3\wedge e_4 + \delta_{2k}e_2\wedge e_3\wedge e_4).
\end{aligned}$$

Now, by considering the coefficients of $\mathbf{x}\wedge e_1\wedge e_3$, we obtain from (4.2) and (6.47) that

$$4\|\hat{h}\|^4 - 2(\lambda_p + \lambda_q - 2)\|\hat{h}\|^2 + \lambda_p\lambda_q = 0.$$

Combining this with (6.36) yields

$$\lambda_p + \lambda_q = 8\|\hat{h}\|^2 - 6, \quad \lambda_p\lambda_q = (9\|\hat{h}\|^2 - 10)\|\hat{h}\|^2. \tag{6.48}$$

Case (i): $K > 0$. In this case, a result of Wallach [22] implies that there exists an integer $k \geq 2$ such that M^2 is an open part of the k-th standard immersion

$\psi_k : S\big(\frac{2}{k(k+1)}\big) \to S^{2k} \subset S^{m-1}$ of the 2-sphere of constant curvature $\frac{2}{k(k+1)}$. Without loss generality, we may assume $M = S\big(\frac{2}{k(k+1)}\big)$.

It is known that eigenvalues of the Laplacian Δ on $S\big(\frac{2}{k(k+1)}\big)$ are given by

$$\lambda_i = \frac{2i(i+1)}{k(k+1)}, \quad i = 0, 1, 2, \dots. \tag{6.49}$$

On the other hand, the equation of Gauss gives

$$||\hat{h}||^2 = \frac{2(k-1)(k+2)}{k(k+1)}. \tag{6.50}$$

Substituting (6.50) into (6.48) yields

$$\lambda_p, \lambda_q = \frac{5k(k+1) - 16 \mp \sqrt{112 + k(1+k)(9k^2+9k-56)}}{k(k+1)}. \tag{6.51}$$

By comparing (6.49) and (6.51) we find

$$\begin{aligned} 5k(k+1) - \sqrt{112 + k(1+k)(9k^2+9k-56)} &= 2i(i+1) + 16, \\ 5k(k+1) + \sqrt{112 + k(1+k)(9k^2+9k-56)} &= 2j(j+1) + 16 \end{aligned} \tag{6.52}$$

for some natural numbers i, j, which is impossible unless $k = 2$, $i = 1$ and $j = 3$. It follows from $k = 2$ that the minimal surface is the Veronese surface. So, according to Theorem 4.1 the spherical Gauss map is mass-symmetric. This is a contradiction.

Case (ii): $K = 0$. In this case, equation of Gauss yields $||\hat{h}||^2 = 2$. Hence, we obtain from (6.48) that $\lambda_p = 2$ and $\lambda_q = 8$. Combining these with (6.30) yields

$$\begin{aligned} &h^3_{11} = h^4_{12} = \frac{1}{\sqrt{2}},\ h^3_{12} = h^4_{11} = h^4_{22} = \omega^6_3(e_1) = \omega^6_4(e_2) = 0, \\ &\omega^5_3(e_1) = -\omega^5_4(e_2),\ \omega^5_3(e_2) = \omega^5_4(e_1),\ \omega^6_3(e_2) = \omega^6_4(e_1), \\ &K^D = -1, \ \ ||\hat{h}||^2 = 2. \end{aligned} \tag{6.53}$$

Since $K = 0$, there exists an integer $\ell \geq 2$ such that M^2 is an open portion of a flat minimal torus T^2 of a totally geodesic $S^{2\ell-1} \subset S^m$ (see, [14]). Also, we see from Theorem 4.1 that $\ell \geq 3$. We many assume that the T^2 is $\mathbb{E}^2/\Lambda$, where Λ is the lattice in $\mathbb{E}^2$ defined by

$$\Lambda = \big\{\big(2n_1\pi u, 2n_2\pi v + 2h\pi w\big) : n_1, n_2 \in \mathbb{Z}\big\} \tag{6.54}$$

for real numbers u, v, w with $u, v > 0$. It is known that the dual lattice of Λ is

$$\Lambda^* = \Big\{\Big(\frac{m_1}{2\pi u} - \frac{m_2 w}{2\pi u v}, \frac{m_2}{2\pi v}\Big) : m_1, m_2 \in \mathbb{Z}\Big\}, \tag{6.55}$$

the spectrum of $T^2 = \mathbb{E}^2/\Lambda$ is

$$\Big\{\Big(\frac{m_1}{u} - \frac{m_2 w}{uv}\Big)^2 + \Big(\frac{m_2}{v}\Big)^2 : m_1, m_2 \in \mathbb{Z}\Big\}, \tag{6.56}$$

and the eigenspace $V(\lambda)$ of Δ with eigenvalue λ is

$$\text{Span}\Big\{\cos\Big(\frac{\alpha s}{u}+\frac{nt}{v}\Big),\sin\Big(\frac{\alpha s}{u}+\frac{nt}{v}\Big):\Big(\frac{\alpha}{u}\Big)^2+\Big(\frac{n}{v}\Big)^2=\lambda\Big\},\quad \alpha=m_1-m_2\frac{w}{v}. \tag{6.57}$$

Put $\mathcal{L}=\{1,2,\dots,\ell\}$ with $\ell\geq 3$. Since T^2 is a minimal surface of $S^{2\ell-1}$, we have $\Delta\mathbf{x}=2\mathbf{x}$. We may assume that $\mathbf{x}:T^2\to S^{2\ell-1}\subset\mathbb{E}^{2\ell}$ is given by

$$\mathbf{x}(s,t)=\Big(\mu_i\sin(\bar\alpha_i s+\bar p_i t),\mu_i\cos(\bar\alpha_i s+\bar p_i t)\Big)_{i\in\mathcal{L}} \tag{6.58}$$

with $\mu_i>0,\bar p_i\geq 0$ and $\bar\alpha_i\in\mathbf{R}$ satisfying $\bar\alpha_i^2+\bar p_i^2=2$. From (6.58) we get

$$\begin{aligned}\frac{\partial\mathbf{x}}{\partial s}&=\Big(\bar\alpha_i\mu_i\cos(\bar\alpha_i s+\bar p_i t),-\bar\alpha_i\mu_i\sin(\bar\alpha_i s+\bar p_i t)\Big)_{i\in\mathcal{L}},\\ \frac{\partial\mathbf{x}}{\partial t}&=\Big(\bar p_i s\mu_i\cos(\bar\alpha_i s+\bar p_i t),-\bar p_i\mu_i\sin(\bar\alpha_i s+\bar p_i t)\Big)_{i\in\mathcal{L}}.\end{aligned} \tag{6.59}$$

$\partial\mathbf{x}/\partial s$ and $\partial\mathbf{x}/\partial t$ are orthonormal vector fields on T^2.

Since $\mathbf{x}$ is isometric, (6.59) gives

$$\sum_i\mu_i^2=\sum_i\mu^2\bar\alpha_i^2=\sum_i\mu_i^2\bar p_i^2=1,\ \sum_i\mu_i^2\bar\alpha_i^2\bar p_i^2=0. \tag{6.60}$$

It follows from the definition of $\tilde\nu$, (6.58), (6.59), and properties of $\wedge$ that each coordinate function of $\tilde\nu$ in $\mathbb{E}^{\binom{m}{3}}$ is a multiple of one of the following functions:

$$\begin{aligned}&\cos(\bar\alpha_i s+\bar p_i t),\ \ \sin(\bar\alpha_i s+\bar p_i t)\sin(\bar\alpha_j s+\bar p_j t)\sin(\bar\alpha_k s+\bar p_k t),\\ &\sin(\bar\alpha_i s+\bar p_i t),\ \ \sin(\bar\alpha_i s+\bar p_i t)\sin(\bar\alpha_j s+\bar p_j t)\cos(\bar\alpha_k s+\bar p_k t),\\ &\sin(\bar\alpha_i s+\bar p_i t)\cos(\bar\alpha_j s+\bar p_j t)\cos(\bar\alpha_k s+\bar p_k t),\\ &\cos(\bar\alpha_i s+\bar p_i t)\cos(\bar\alpha_j s+\bar p_j t)\cos(\bar\alpha_k s+\bar p_k t)\end{aligned} \tag{6.61}$$

for distinct $i,j,k\in\mathcal{L}$. Thus, each coordinate function of $\tilde\nu$ in $\mathbb{E}^{\binom{m}{3}}$ is a linear combination of

$$\begin{aligned}&\cos((\bar\alpha_i+\bar\alpha_j+\bar\alpha_k)s+(\bar p_i+\bar p_j+\bar p_k)t),\\ &\sin((\bar\alpha_i+\bar\alpha_j+\bar\alpha_k)s+(\bar p_i+\bar p_j+\bar p_k)t),\\ &\cos((\bar\alpha_i+\bar\alpha_j-\bar\alpha_k)s+(\bar p_i+\bar p_j-\bar p_k)t),\\ &\sin((\bar\alpha_i+\bar\alpha_j-\bar\alpha_k)s+(\bar p_i+\bar p_j-\bar p_k)t),\\ &\cos(\bar\alpha_i s+\bar p_i t),\ \sin(\bar\alpha_i s+\bar p_i t)\end{aligned} \tag{6.62}$$

for distinct $i,j,k\in\mathcal{L}$.

Since we have a spectral resolution: $\tilde\nu=c+\tilde\nu_1+\tilde\nu_2$ with $\Delta\tilde\nu_1=2\tilde\nu_1$, $\Delta\tilde\nu_2=8\tilde\nu_2$, and $0\neq c\in\mathbb{E}^{\binom{m}{3}}$, the above observation gives the conditions:

$$(\bar\alpha_i+\bar\alpha_j+\bar\alpha_k)^2+(\bar p_i+\bar p_j+\bar p_k)^2,\ (\bar\alpha_i+\bar\alpha_j-\bar\alpha_k)^2+(\bar p_i+\bar p_j-\bar p_k)^2\\ \in\{0,2,8\}$$

for distinct $i, j, k \in \mathcal{L}$. By combining this with $\bar{\alpha}_i^2 + \bar{p}_i^2 = 2$, we find

$$\gamma_{ij} + \gamma_{ik} + \gamma_{jk}, \ \gamma_{ij} - \gamma_{ik} - \gamma_{jk} \in \{-3, -2, 1\} \tag{6.63}$$

for distinct $i, j, k \in \mathcal{L}$, where $\gamma_{ij} = \bar{\alpha}_i\bar{\alpha}_j + \bar{p}_i\bar{p}_j$. Condition (6.63) yields

$$\gamma_{ij} \in \{-3, -\tfrac{5}{2}, -2, -1, -\tfrac{1}{2}, 1\} \tag{6.64}$$

for distinct $i, j \in \mathcal{L}$. From condition (6.64) we find

$$\gamma_{ij} + \gamma_{k\ell} \in \{-6, -\tfrac{11}{2}, -5, -\tfrac{9}{2}, -4, -\tfrac{7}{2}, -3, -\tfrac{5}{2} - 2, -\tfrac{3}{2}, -1, 0, \tfrac{1}{2}, 2\}. \tag{6.65}$$

We divide the proof of *Case* (ii) into two cases.

Case (ii-a): *Two of γ_{ij} are equal.* Without loss of generality, we may assume that either (1) $\gamma_{12} = \gamma_{13}$ or (2) $\gamma_{12} = \gamma_{34} \neq \gamma_{13}$.

Case (ii-a-1): $\gamma_{12} = \gamma_{13}$. In this case, (6.64) and condition $\gamma_{12} - \gamma_{13} - \gamma_{23} \in \{-3, -2, 1\}$ from (6.63) give

$$\gamma_{23} = -1 \ \text{ and } \ \gamma_{12} = \gamma_{13} \in \{-3, -\tfrac{5}{2}, -2, -1, -\tfrac{1}{2}, 1\}. \tag{6.66}$$

If $\gamma_{12} = \gamma_{13} \in \{-3, -\frac{5}{2}, -2\}$, we obtain from (6.66) that $\gamma_{12} + \gamma_{13} + \gamma_{23} \leq -5$ which contradicts to (6.63). So, this is impossible.

If $\gamma_{12} = \gamma_{13} = -\frac{1}{2}$ holds, then (6.66) yields $\gamma_{23} - \gamma_{12} - \gamma_{13} = 0$, which contradicts to (6.63). Thus, this is also impossible. Consequently, we must have either (1) $\gamma_{12} = \gamma_{13} = -1$ or (2) $\gamma_{12} = \gamma_{13} = 1$.

First, let us assume that $\ell \geq 4$.

If $\gamma_{12} = \gamma_{13} = -1$ holds, then condition $\gamma_{12} + \gamma_{14} + \gamma_{24} \in \{-3, -2, 1\}$ gives $\gamma_{14} + \gamma_{24} \in \{-2, -1, 2\}$. Also, we obtain from $\gamma_{12} - \gamma_{14} - \gamma_{24} \in \{-3, -2, 1\}$ that $\gamma_{14} + \gamma_{24} \in \{-2, 1, 2\}$. By combining these we obtain $\gamma_{14} + \gamma_{24} \in \{-2, 2\}$.

Similarly, using $\gamma_{12} = \gamma_{23} = -1$ and $\gamma_{13} = \gamma_{23} = -1$ we obtain $\gamma_{14} + \gamma_{34} \in \{-2, 2\}$ and $\gamma_{24} + \gamma_{34} \in \{-2, 2\}$, respectively. Thus, we know that $\gamma_{14} + \gamma_{24}$, $\gamma_{14} + \gamma_{34}$ and $\gamma_{24} + \gamma_{34}$ belong to $\{-2, 2\}$. After solving this system under the two side conditions: (6.63) and (6.64), we obtain either $\gamma_{14} = \gamma_{24} = \gamma_{34} = 1$ or $\gamma_{14} = \gamma_{24} = \gamma_{34} = -1$.

If $\gamma_{12} = \gamma_{13} = -1$ holds, we have either

(a) $\gamma_{ij} = -1$, $1 \leq i < j \leq 4$; or

(b) $\gamma_{12} = \gamma_{13} = \gamma_{23} = -1$ and $\gamma_{14} = \gamma_{24} = \gamma_{34} = 1$.

Next, assume that $\gamma_{12} = \gamma_{13} = 1$ holds. If $\gamma_{14} \in \{-3, -\frac{5}{2}, -2\}$, then condition $\gamma_{12} - \gamma_{14} - \gamma_{24} \in \{-3, -2, 1\}$ from (6.63) implies $\gamma_{24} \geq 2$. This contradicts to (6.64).

If $\gamma_{14} = -\frac{1}{2}$, then condition $\gamma_{12} - \gamma_{14} - \gamma_{24} \in \{-3, -2, 1\}$ from (6.63) and (6.64) imply $\gamma_{24} \in \{\frac{1}{2}, \frac{7}{2}, \frac{9}{2}\}$, which contradicts to (6.64).

If $\gamma_{14} = 1$, condition $\gamma_{12} + \gamma_{14} + \gamma_{24} \in \{-3, -2, 1\}$ and (6.64) give $\gamma_{24} = -1$. From $\gamma_{23} - \gamma_{12} - \gamma_{13} \in \{-3, -2, 1\}$ and (6.64) we get $\gamma_{23} = -1$. Similarly, from $\gamma_{34} - \gamma_{24} - \gamma_{23} \in \{-3, -2, 1\}$ and (6.64), we get $\gamma_{34} = -1$. Therefore, we have $\gamma_{12} = \gamma_{13} = \gamma_{14} = 1$ and $\gamma_{23} = \gamma_{24} = \gamma_{34} = -1$.

If $\gamma_{14} = -1$, then $\gamma_{12} - \gamma_{14} - \gamma_{24} \in \{-3, -2, 1\}$ and (6.64) yield $\gamma_{24} = 1$. Also, (6.64) and $\gamma_{13} - \gamma_{14} - \gamma_{34} \in \{-3, -2, 1\}$ from (6.63) give $\gamma_{34} = 1$. Hence, we obtain $\gamma_{12} = \gamma_{13} = \gamma_{24} = \gamma_{34} = 1$, $\gamma_{23} = \gamma_{14} = -1$. Therefore, under $\gamma_{12} = \gamma_{13}$, one of the following two cases must occurs:

(A): $\ell \geq 4$ *and* $(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34})$ *is one of the following:*

$$\begin{aligned}&(-1,-1,-1,-1,-1,-1), \quad (-1,-1,1,-1,1,1),\\ &(1,1,-1,-1,1,1), \quad (1,1,1,-1,-1,-1);\end{aligned} \tag{6.67}$$

(B): $\ell = 3$ *and* $\gamma_{12} = \gamma_{13} = 1$ *or* $\gamma_{12} = \gamma_{13} = -1$.

Now, we claim that (A) cannot happen. This can be seen as follows:

If $(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34})$ is given by the first or second case, then we get $\gamma_{12} = \gamma_{13} = \gamma_{23} = -1$. In this case, we find $\bar\alpha_1\bar\alpha_2 + \bar p_1\bar p_2 = \bar\alpha_1\bar\alpha_3 + \bar p_1\bar p_3 = \bar\alpha_2\bar\alpha_3 + \bar p_2\bar p_3 = -1$. After solving this for $\bar\alpha_1$, we get

$$\bar\alpha_1 = \frac{\sqrt{(1+\bar p_1\bar p_2)(1+\bar p_1\bar p_3)}}{\sqrt{-(1+\bar p_2\bar p_3)}} \quad \text{or} \quad = \frac{\sqrt{-(1+\bar p_1\bar p_2)(1+\bar p_1\bar p_3)}}{\sqrt{1+\bar p_2\bar p_3}}. \tag{6.68}$$

But this is impossible, since $\bar p_1, \bar p_2, \bar p_3$ are ≥ 0 and $\bar\alpha_1$ is a real number.

Now, assume $(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}) = (1,1,-1,-1,1,1)$. We have

$$\bar\alpha_1\bar\alpha_2 + \bar p_1\bar p_2 = \bar\alpha_1\bar\alpha_3 + \bar p_1\bar p_3 = 1, \tag{6.69}$$
$$\bar\alpha_2\bar\alpha_3 + \bar p_2\bar p_3 = \bar\alpha_1\bar\alpha_4 + \bar p_1\bar p_4 = -1,$$
$$\bar\alpha_2\bar\alpha_4 + \bar p_2\bar p_4 = \bar\alpha_3\bar\alpha_4 + \bar p_3\bar p_4 = 1. \tag{6.70}$$

After solving system (6.69), we obtain

$$\begin{aligned}(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3, \bar\alpha_4) = \pm\Bigg(&\frac{\sqrt{(\bar p_1\bar p_2 - 1)(1-\bar p_1\bar p_3)}}{\sqrt{1+\bar p_2\bar p_3}},\\ &\frac{-\sqrt{(\bar p_1\bar p_2 - 1)(1+\bar p_2\bar p_3)}}{\sqrt{1-\bar p_2\bar p_3}}, \frac{\sqrt{(1-\bar p_1\bar p_3)(1+\bar p_2\bar p_3)}}{\sqrt{\bar p_1\bar p_2 - 1}},\\ &\frac{-\sqrt{1+\bar p_2\bar p_3}(1+\bar p_1\bar p_4)}{\sqrt{(1-\bar p_1\bar p_2)(\bar p_1\bar p_3 - 1)}}\Bigg)\end{aligned} \tag{6.71}$$

Substituting this into (6.70) gives $(\bar p_1 + \bar p_2)(\bar p_3 + \bar p_4) = (\bar p_1 + \bar p_3)(\bar p_2 + \bar p_4) = 0$. Thus, one of the following four cases occurs:

(α) $\bar p_3 = \bar p_2 = -\bar p_1$;
(β) $\bar p_4 = \bar p_1 = -\bar p_3$
(γ) $\bar p_3 = \bar p_2 = -\bar p_4$;
(δ) $\bar p_4 = \bar p_1 = -\bar p_2$;

If (α) occurs (respectively, (β), (γ), or (δ) occurs), we obtain

$$\bar\alpha_1 = \pm\sqrt{-(1+\bar p_1^2)} \Bigg(\text{respectively, } \pm\sqrt{-(1+\bar p_3^2)}, \pm\frac{\sqrt{-(1+\bar p_1^2)(1+\bar p_1\bar p_4)}}{\sqrt{1+\bar p_4^2}} \text{ or } \pm\frac{\sqrt{-(1+\bar p_2\bar p_3)(1+\bar p_2^2)}}{\sqrt{1+\bar p_2\bar p_3}}\Bigg),$$

which is a contradiction since $\bar\alpha_1$ is a real number.

If $(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}) = (1,1,1,-1,-1,-1)$ holds, then we get

$$\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}_1\bar{p}_2 = \bar{\alpha}_1\bar{\alpha}_3 + \bar{p}_1\bar{p}_3 = \bar{\alpha}_1\bar{\alpha}_4 + \bar{p}_1\bar{p}_4 = 1, \tag{6.72}$$

$$\bar{\alpha}_2\bar{\alpha}_3 + \bar{p}_2\bar{p}_3 = \bar{\alpha}_2\bar{\alpha}_4 + \bar{p}_2\bar{p}_4 = \bar{\alpha}_3\bar{\alpha}_4 + \bar{p}_3\bar{p}_4 = -1. \tag{6.73}$$

After solving this system, we get

$$\bar{\alpha}_2 = \frac{\sqrt{(1+\bar{p}_2\bar{p}_3)(1+\bar{p}_2\bar{p}_4)}}{\sqrt{-(1+\bar{p}_3\bar{p}_4)}}.$$

But this is impossible since $\bar{\alpha}_2$ is a real number. So, (A) cannot happen. Consequently, *under the assumption:* $\gamma_{12} = \gamma_{13}$, *we must have* $\ell = 3$ *and* $\gamma_{12} = \gamma_{13} = \pm 1$.

Case (ii-a-2): $\gamma_{12} = \gamma_{34} \neq \gamma_{13}$. This case occurs only when $\ell \geq 4$.

In view of (6.64), we divide this into six cases:

Case (ii-a-2.1): $\gamma_{12} = \gamma_{34} = -3$. In this case, $\gamma_{12} + \gamma_{13} + \gamma_{23} \in \{-3,-2,1\}$ implies $\gamma_{13} + \gamma_{23} \in \{0,1,4\}$. Combining this with (6.65) gives $\gamma_{13} + \gamma_{23} = 0$. In view of (6.64), we obtain $\gamma_{23} = 1$ and $\gamma_{13} = -1$ from $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$. Thus, we find $\gamma_{23} - \gamma_{13} - \gamma_{12} = 5$ which contradicts to (6.63).

Case (ii-a-2.2): $\gamma_{12} = \gamma_{34} = -\frac{5}{2}$. Condition $\gamma_{12} + \gamma_{13} + \gamma_{23} \in \{-3,-2,1\}$ gives $\gamma_{13} + \gamma_{23} \in \{-\frac{1}{2}, \frac{1}{2}, \frac{7}{2}\}$. Combining this with (6.65) yields $\gamma_{13} + \gamma_{23} = \frac{1}{2}$. Thus, we obtain $\gamma_{23} = 1$ and $\gamma_{13} = -\frac{1}{2}$ from $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$ using (6.64). From these we find $\gamma_{23} - \gamma_{13} - \gamma_{12} = 4$ which is impossible.

Case (ii-a-2.3): $\gamma_{12} = \gamma_{34} = -2$. Condition $\gamma_{12} + \gamma_{13} + \gamma_{23} \in \{-3,-2,1\}$ gives $\gamma_{13} + \gamma_{23} \in \{-1,0,3\}$. Combining this with (6.65) yields $\gamma_{13} + \gamma_{23} = -1$ or 0.

If $\gamma_{13} + \gamma_{23} = -1$ occurs, $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$ gives $\gamma_{23} \in \{0, \frac{3}{2}, 2\}$ which is impossible due to (6.64).

If $\gamma_{13} + \gamma_{23} = 0$ occurs, $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$ gives $\gamma_{23} \in \{\frac{1}{2}, 2, \frac{5}{2}\}$, which is also impossible due to (6.64).

Case (ii-a-2.4): $\gamma_{12} = \gamma_{34} = -1$. In this case, $\gamma_{12} + \gamma_{13} + \gamma_{23} \in \{-3,-2,1\}$ yields $\gamma_{13} + \gamma_{23} \in \{-2,-1,2\}$.

If $\gamma_{13} + \gamma_{23} = -2$, condition $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$ implies that $(\gamma_{23}, \gamma_{13}) = (-1,-1)$, $(-\frac{1}{2}, -\frac{3}{2})$, or $(1,-3)$. From these we have $\gamma_{23} - \gamma_{13} - \gamma_{12} \in \{1,2,5\}$. Combining this with (6.63) yields $\gamma_{12} = \gamma_{13} = -1$, which contradicts to the assumption of *Case* (ii-a-1.2).

If $\gamma_{13}+\gamma_{23} = -1$ holds, $\gamma_{13}-\gamma_{23}-\gamma_{12} \in \{-\frac{5}{2}, -\frac{3}{2}, \frac{3}{2}\}$ yields $\gamma_{23} \in \{-\frac{1}{2}, 1, \frac{3}{2}\}$. Comparing this with (6.64) gives $(\gamma_{23}, \gamma_{13}) = (-\frac{1}{2}, -\frac{1}{2})$ or $(1,-2)$. The first case reduces to Case (ii-a-1) after making the interchanging: $1 \leftrightarrow 3$. If the later case holds, we find $\gamma_{23} - \gamma_{13} - \gamma_{12} = 4$ which contradicts to (6.63).

If $\gamma_{13} + \gamma_{23} = 2$ holds, $\gamma_{13} - \gamma_{23} - \gamma_{12} \in \{-3,-2,1\}$ implies that $\gamma_{23} = 1, \frac{5}{2}$, or 3. Comparing this with (6.64) gives $\gamma_{23} = \gamma_{13} = 1$. Thus, this case also reduces to *Case* (ii-a-1) after making the interchanging: $1 \leftrightarrow 3$.

Case (ii-a-2.5): $\gamma_{12}=\gamma_{34}=-\frac{1}{2}$. Condition $\gamma_{12}+\gamma_{13}+\gamma_{23} \in \{-3,-2,1\}$ gives $\gamma_{13} + \gamma_{23} \in \{-\frac{5}{2}, -\frac{3}{2}, \frac{3}{2}\}$. Combining this with (6.65) yields $\gamma_{13} + \gamma_{23} = -\frac{5}{2}$

or $-\frac{3}{2}$. If $\gamma_{13}+\gamma_{23} = -\frac{5}{2}$ holds, we obtain $(\gamma_{23},\gamma_{13}) = (-\frac{3}{2},-1)$ $(0,-\frac{5}{2})$, or $(\frac{1}{2},-3)$ from $\gamma_{13}-\gamma_{23}-\gamma_{12} \in \{-3,-2,1\}$. From these we know that $\gamma_{23}-\gamma_{13}-\gamma_{12} = 0, 3$, or 4, which contradicts to (6.63).

If $\gamma_{13}+\gamma_{23} = -\frac{3}{2}$ holds, we get from $\gamma_{13}-\gamma_{23}-\gamma_{12} \in \{-3,-2,1\}$ that $(\gamma_{23},\gamma_{13}) = (1,-\frac{5}{2})$ $(\frac{1}{2},-2)$, or $(-1,-\frac{1}{2})$. Thus, we get $\gamma_{23}-\gamma_{13}-\gamma_{12} = 4, 3$, or 0, respectively. This contradicts to (6.63), too.

Case (ii-a-2.6): $\gamma_{12} = \gamma_{34} = 1$. In this case, $\gamma_{12}+\gamma_{13}+\gamma_{23} \in \{-3,-2,1\}$ gives $\gamma_{13}+\gamma_{23} = -4$, -3, or 0. Thus we get $\gamma_{12}-\gamma_{13}-\gamma_{23} = 5$, 4 or 1. Comparing this with (6.63) shows that $\gamma_{13}+\gamma_{23} = 0$.

If $\gamma_{13} > 0$ holds, we get $\gamma_{13} = 1$ according to (6.64). This contradicts to the assumption: $\gamma_{12} \neq \gamma_{13}$ of *Case* (ii-a-1.2). Hence, we have $\gamma_{13} < 0$ instead. Thus, according to $\gamma_{13} = -\gamma_{23}$ and (6.64), we have $\gamma_{23} = 1$. So, we get $\gamma_{23} = -\gamma_{13} = 1$. Consequently, we have $\gamma_{12} = \gamma_{23} = 1$. Hence, this case reduces to *Case* (ii-a-1) after making the interchanging: $1 \leftrightarrow 2$.

Case (ii-b): *γ_{ij} are distinct for $1 \le i < j \le \ell$.* Since $\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{23}$, γ_{24}, and γ_{34} are distinct, (6.64) implies $\ell \le 4$.

If $\ell = 4$, then without loss of generality, we may put $\gamma_{12} = -3$. Then there exist $i,j,k,\ell \in \{1,2,3,4\}$ such that $(\gamma_{ij},\gamma_{k\ell}) = (-2,-1)$ according to (6.64). So, we have $\gamma_{12}+\gamma_{ij}+\gamma_{k\ell} = -6 \notin \{-3,-2,1\}$ which contradicts to (6.63). Consequently, we know that *Case* (ii-b) *occurs only under* the condition: $\ell = 3$.

Without loss of generality, we may assume that $\gamma_{12} < \gamma_{13} < \gamma_{23}$.

If $\gamma_{23} < 0$, we obtain from (6.64) that $\gamma_{12}+\gamma_{13}+\gamma_{23} \le -\frac{7}{2}$ which contradicts (6.63). Hence, we must have $\gamma_{23} = 1$ and $\gamma_{12}+\gamma_{13} \in \{-4,-3\}$. Thus, by (6.64), we know that $(\gamma_{12},\gamma_{13})$ is $(-3,-1)$, $(-\frac{5}{2},-\frac{1}{2})$ or $(-2,-1)$. In all cases, we have $\gamma_{23}-\gamma_{13}-\gamma_{12} \in \{4,5\}$, which contradicts to (6.63). Hence, *Case* (ii-b) is impossible.

In summary, we have $\ell = 3$ and $\gamma_{12} = \gamma_{13} = 1$ or $\gamma_{12} = \gamma_{13} = -1$.

If $\gamma_{12} = \gamma_{13} = -1$ holds, by considering $\gamma_{23}-\gamma_{13}-\gamma_{12}$, we obtain from (6.63) and (6.64) that $\gamma_{23} = 1$. Hence, we get $\gamma_{23}-\gamma_{12}-\gamma_{13} = 3$ which is impossible due to (6.63). Thus, we find $\gamma_{12} = \gamma_{13} = 1$. So, by using $\gamma_{23}+\gamma_{13}+\gamma_{12} \in \{-3,-2,1\}$, we obtain from (6.63) and (6.64) that $\gamma_{23} = -1$. Consequently, we have $(\gamma_{12},\gamma_{13},\gamma_{23}) = (1,1,-1)$. Since $\ell = 3$, it follows from (6.58) that $T^2 \subset S^5 \subset \mathbb{E}^6$. Thus, we may assume the immersion of T^2 in $\mathbb{E}^6$ is

$$\begin{aligned}\mathbf{x}(s,t) = \Big(&\mu_1\sin(\bar{\alpha}_1 s+\bar{p}_1 t), \mu_1\cos(\bar{\alpha}_1 s+\bar{p}_1 t),\\ &\mu_2\sin(\bar{\alpha}_2 s+\bar{p}_2 t), \mu_2\cos(\bar{\alpha}_2 s+\bar{p}_2 t),\\ &\mu_3\sin(\bar{\alpha}_3 s+\bar{p}_3 t), \mu_3\cos(\bar{\alpha}_3 s+\bar{p}_3 t)\Big).\end{aligned} \tag{6.74}$$

We may assume $\bar{p}_3 = 0$ by applying a suitable rotation on the (s,t)-plane if necessary. From the discussions above, we know that $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{p}_1, \bar{p}_2$ satisfy

$$\begin{aligned}&\bar{\alpha}_1\bar{\alpha}_2+\bar{p}_1\bar{p}_2 = \bar{\alpha}_1\bar{\alpha}_3 = -\bar{\alpha}_2\bar{\alpha}_3 = 1,\\ &\bar{\alpha}_1^2+\bar{p}_1^2 = \bar{\alpha}_2^2+\bar{p}_2^2 = \bar{\alpha}_3^2 = 2.\end{aligned} \tag{6.75}$$

After solving this system we obtain

$$\sqrt{2}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{p}_1, \bar{p}_2) = \pm\Big(1, -1, 2, \sqrt{3}, \sqrt{3}\Big). \tag{6.76}$$

Hence, up to rigid motions, the immersion of T^2 in $S^5 \subset \mathbb{E}^6$ takes the form:

$$\begin{aligned} \mathbf{x}(s,t) = \Big(&\mu_1 \cos\frac{1}{\sqrt{2}}\big(\sqrt{3}t + s\big), \mu_1 \sin\frac{1}{\sqrt{2}}\big(\sqrt{3}t + s\big), \\ &\mu_2 \cos\frac{1}{\sqrt{2}}\big(\sqrt{3}t - s\big), \mu_2 \sin\frac{1}{\sqrt{2}}\big(\sqrt{3}t - s\big), \mu_3 \cos\sqrt{2}s, \mu_3 \sin\sqrt{2}s\Big) \end{aligned} \tag{6.77}$$

for some positive numbers μ_1, μ_2, μ_3. Finally, by applying (6.60), we obtain $\mu_1 = \mu_2 = \mu_3 = \frac{1}{\sqrt{3}}$. Consequently, the flat minimal torus T^2 in S^5 is congruent to the equilateral minimal torus defined by (6.1). □

7. Some additional results on spherical Gauss map

The next result provides some answers to the following question:
Question: *When the spherical Gauss map of a spherical minimal surface is minimal ?*

Proposition 7.1. *If* $\mathbf{x} : (M^2, g) \to S^{m-1}$ *is a non-totally geodesic minimal surface, then its spherical Gauss map* $\hat{\nu} : (M^2, \hat{g}) \to S^{N-1}$ *with* $N = \binom{m}{3}$ *is a minimal surface of* S^{N-1} *if and only if any one of the following three conditions holds:*

(1) M^2 *is a minimal surface of a totally geodesic 3-sphere* $S^3 \subset S^{m-1}$.
(2) $\mathbf{x} : (M^2, g) \to S^{m-1}$ *has flat normal connection.*
(3) $\hat{\nu} : (M^2, \hat{g}) \to S^{N-1}$ *has parallel mean curvature vector.*

Proof. Let $\mathbf{x} : (M^2, g) \to S^{m-1}$ be a non-totally geodesic minimal surface and let $\hat{\Delta}$ be the Laplacian operator of $(M^2, \hat{g})$. Then the totally geodesic points are isolated. If $\hat{\nu} : (M^2, \hat{g}) \to S^{N-1}$ of $\mathbf{x}$ is minimal in S^{N-1}, then condition (3) holds trivially.

Now, let us choose $e_3, \ldots, e_{m-1}$ such that $h^r_{ij} = 0$ for $r = 5, \ldots, m-1$. Then we have (5.10). Since the induced metric $\hat{g} = (1 - K)g$ via $\hat{\nu}$ is conformal to the original metric g on M^2 according to (3.8), we obtain from (5.10) that

$$\hat{\Delta}\tilde{\nu} = -2\tilde{\nu} + \left(\frac{2K^D}{1-K}\right)\mathbf{x} \wedge e_3 \wedge e_4. \tag{7.1}$$

Thus, the mean curvature vector $\tilde{H}$ of $\tilde{\nu} : (M^2, \hat{g}) \to S^{N-1}$ is

$$\tilde{H} = \left(\frac{2K^D}{1-K}\right)\mathbf{x} \wedge e_3 \wedge e_4. \tag{7.2}$$

Hence, $\hat{\nu} : M^2 \to S^{N-1}$ is minimal if and only if condition (2) holds.

Conditions (1) and (2) are equivalent, since minimal surfaces in S^{m-1} with flat normal connection are minimal surfaces lying in a totally geodesic $S^3 \subset S^{m-1}$.

Now, assume that condition (3) holds. Then we have

$$\tilde{\nabla}_X \tilde{H} = \langle \tilde{\nabla}_X \tilde{H}, f_1 \rangle f_1 + \langle \tilde{\nabla}_X \tilde{H}, f_2 \rangle f_2, \tag{7.3}$$

where f_1, f_2 are orthonormal vector fields of $(\tilde{\nu}(M), \hat{g})$. From (7.2) we have

$$\begin{aligned} \tilde{\nabla}_X \tilde{H} &= 2X\Big(\frac{K^D}{1-K}\Big) \mathbf{x} \wedge e_3 \wedge e_4 \\ &+ \frac{2K^D}{1-K}\{X \wedge e_3 \wedge e_4 - \mathbf{x} \wedge A_{e_r}\mathbf{x} \wedge e_4 \\ &- \mathbf{x} \wedge e_3 \wedge A_{e_4}X + \mathbf{x} \wedge D_X e_3 \wedge e_4 + \mathbf{x} \wedge e_3 \wedge D_X e_4\} \end{aligned} \tag{7.4}$$

for $X \in TM^2$, which shows that f_1, f_2 are linear combinations of

$$\mathbf{x} \wedge e_r \wedge e_2, \quad \mathbf{x} \wedge e_1 \wedge e_r, \quad r = 3, \ldots, m-1.$$

Thus, (7.3) and (7.4) yield $K^D = 0$, i.e., the normal connection of $\mathbf{x}$ is flat. Therefore, we obtain condition (1). □

The next result provides a general property of the spherical Gauss map for spherical minimal surfaces.

Proposition 7.2. *If (M^2, g) is a minimal surface of S^{m-1}, then the mean curvature vector $\tilde{H}$ of $\hat{\nu} : (M, \hat{g}) \to S^{\binom{m}{3}-1}$ satisfies $|\hat{H}| \leq 2$, with the equality holding if and only if the immersion is isotropic.*

Proof. As we did in the proof of Theorem 4.1, let us choose $e_3, \ldots, e_{m-1}$ with $h^r_{ij} = 0$ for $r = 5, \ldots, m-1$. Let us also choose e_1, e_2 which diagonalize the shape operator A_{e_3}. Since we have $K = 1 - \frac{1}{2}||\hat{h}||^2$ from Gauss' equation, we find

$$\begin{aligned} &(1-K)^2 - (K^D)^2 \\ &= \{(h^3_{11})^2 + (h^4_{11})^2 + (h^4_{12})^2\}^2 - 4(h^3_{11}h^4_{12})^2 \\ &= \{(h^3_{11})^2 - (h^4_{12})^2\}^2 + (h^4_{11})^4 + 2(h^4_{11})^2((h^3_{11})^2 + (h^4_{12})^2). \end{aligned} \tag{7.5}$$

So, we obtain $|\tilde{H}| \leq 2$ by (7.1), with equality holding if and only if $|h^3_{11}| = |h^4_{12}|$ and $h^4_{11} = 0$. The later conditions are equivalent to the isotropy of M^2 in S^{m-1}. □

Finally, we give the following.

Proposition 7.3. *If $\mathbf{x} : (M^3, g) \to S^4$ is minimal without totally geodesic points, then the scalar curvature of $(M^3, \hat{g})$ equals to 6.*

Proof. We can identify $\tilde{\nu}$ as the unit normal vector field e_4 of M^3 in S^4. Let $h_{ij} =: h^4_{ij}$ and (h^{ij}) the inverse matrix of (h_{ij}). If we put $f_i = \sum_j h^{ij} e_j$, then we get $(de_4)(f_i) = -e_i$. Thus, f_1, f_2, f_3 form an orthonormal basis on $(M^3, \hat{g})$.

Since $\langle dx, f_i\rangle = \sum_j h^{ij}\omega^j$, the matrix (h^{ij}) is the matrix of the second fundamental form of $\hat{\nu} : M^3 \to S^4$ with respect to f_1, f_2, f_3. Thus, from the equation of Gauss we know that the curvature tensor $\hat{R}$ of $(M^3, \hat{g})$ satisfies

$$\hat{R}^i_{jk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} + h^{ik}h^{j\ell} - h^{i\ell}h^{jk}. \tag{7.6}$$

Hence, the scalar curvature $\hat{S}$ of $(M^3, \hat{g})$ is

$$\hat{S} = 6 + \Big(\sum_j h^{jj}\Big)^2 - \sum_{i,j}(h^{ij})^2.$$

If e_1, e_2, e_3 diagonalize (h_{ij}), then we find

$$\hat{S} = 6 + \frac{2\sum_j h_{jj}}{h_{11}h_{22}h_{33}} = 6.$$

This proves the proposition. □

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BANG-YEN CHEN
DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824–1027, U.S.A.
E-mail address: bychen@math.msu.edu

HUEI-SHYONG LUE
DEPARTMENT OF COMPUTER SCIENCES AND INFORMATION ENGINEERING
YUANPEI INSTITUTE OF SCIENCE AND TECHNOLOGY
HSINCHU, TAIWAN
E-mail address: hslue@math.nthu.edu.tw