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SUPER-REPLICABLE FUNCTIONS $\mathcal{N}(j_{1,N})$ AND PERIODICALLY VANISHING PROPERTY

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Abstract. We find the super-replication formulae which would be a generalization of replication formulae. And we apply the formulae to derive periodically vanishing property in the Fourier coefficients of the Hauptmodul $\mathcal{N}(j_{1,12})$ as a super-replicable function.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod N ($N = 1, 2, \ldots$). Since the group $\Gamma_1(N)$ acts on $\mathfrak{H}$ by linear fractional transformations, we get the modular curve $X_1(N) = \Gamma_1(N)\backslash\mathfrak{H}^*$, as a projective closure of the smooth affine curve $\Gamma_1(N)\backslash\mathfrak{H}$, with genus $g_{1,N}$. Here, $\mathfrak{H}^*$ denotes the union of $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{Q})$.

Ishida and Ishii showed in [11] that for $N \geq 7$, the function field $K(X_1(N))$ is generated over $\mathbb{C}$ by the modular functions $X_2(z,N)^{\epsilon_N \cdot N}$ and $X_3(z,N)^N$, where $X_r(z,N) = e^{2\pi i \frac{(r-1)(N-1)}{4N}} \prod_{s=0}^{N-1} \frac{K_{r,s}(z)}{K_{1,s}(z)}$ and ϵ_N is 1 or 2 according as N is odd or even. Here, $K_{r,s}(z)$ is a Klein form of level N for integers r and s not both congruent to 0 mod N. On the other hand, since the genus $g_{1,N} = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$ ([25], [15]), the function field $K(X_1(N))$ in this case is a rational function field $\mathbb{C}(j_{1,N})$ for some modular function $j_{1,N}$ (Table 3, Appendix).

The element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $\Gamma_1(N)$ takes z to $z+1$, and in particular a modular function f in $K(X_1(N))$ is periodic. Thus it can be written as a Laurent series in $q = e^{2\pi i z}$ ($z \in \mathfrak{H}$), which is called a *q-series* (or *q-expansion*) of f. We call f *normalized* if its q-series starts with $q^{-1} + 0 + a_1 q + a_2 q^2 + \cdots$. By a *Hauptmodul* t we mean the normalized generator of a genus zero function field $K(X_1(N))$ and we write $t = q^{-1} + 0 + \sum_{k\geq 1} H_k q^k$ for its q-series.

For a Fuchsian group Γ, let $\overline{\Gamma}$ denote the inhomogeneous group of Γ ($= \Gamma/\pm I$). Let $\Gamma_0(N)$ be the Hecke subgroup given by $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0$

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mod $N\}$. Also, let $t = \mathcal{N}(j_{1,N})$ be the Hauptmodul of $\Gamma_1(N)$ and $X_n(t)$ be a unique polynomial in t of degree n such that $X_n(t) - \frac{1}{n}q^{-n}$ belongs to the maximal ideal of the local ring $\mathbb{C}[[q]]$. Polynomials with this property are known as the Faber polynomials ([5], Chapter 4). Write $X_n(t) = \frac{1}{n}q^{-n} + \sum_{m\geq 1} H_{m,n}q^m$.

When $\overline{\Gamma}_1(N) = \overline{\Gamma}_0(N)$, $\mathcal{N}(j_{1,N})$ becomes a replicable function, that is, it satisfies the following replication formulae

$$(*) \qquad H_{a,b} = H_{c,d} \quad \text{whenever } ab = cd \text{ and } (a,b) = (c,d)$$

([1], [3], [22]). Given a replicable function f the *n-plicate* of f is defined iteratively by

$$f^{(n)}(nz) = -\sum_{\substack{ad=n\\0\leq b<d}}{}' f^{(a)}\left(\frac{az+b}{d}\right) + nX_n(f)$$

where the primed sum means that the term with $a = n$ is omitted ([3]). We call f *completely replicable* if f is a replicable function with rational integer coefficients and has only a finite number of distinct replicates, which are themselves replicable functions. According to [1] there are, excluding the trivial cases $q^{-1} + aq$, 326 completely replicable functions of which 171 are monstrous functions, i.e., modular functions whose q-series coincide with the Thompson series $T_g(q) = \sum_{n\in\mathbb{Z}} \mathrm{Tr}(g|V_n)q^n$ for some element g of the monster simple group M whose order is approximately $8 \cdot 10^{53}$. Here we observe that $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is the infinite dimensional graded representation of M constructed by Frenkel *et al.* ([8], [9]). Furthermore, in [3] Cummins and Norton showed that if f is replicable, it can be determined only by the 12 coefficients of its first 23 ones.

If $\overline{\Gamma}_1(N) \neq \overline{\Gamma}_0(N)$, unlike those replicable functions mentioned above, we show in §3 that the Fourier coefficients of $X_n(t)$ with $t = \mathcal{N}(j_{1,N})$ $(N \neq 7, 9)$ satisfy a twisted formula (10) by a character ψ (see Corollary 11). Here we note that when we work with the Thompson series, it is reduced to replication formulae in $(*)$ by viewing ψ as the trivial character. Thus in this sense it gives a more general class of modular functions, which we propose to call $\mathcal{N}(j_{1,N})$ a *super-replicable* function.

There would be certain similarity between some of replicable functions and super-replicable ones as follows. We derived in [19] the following self-recursion formulas for the Fourier coefficients of $\mathcal{N}(j_{1,N})$ without the aid of its 2-plicate when $N = 2, 6, 8, 10, 12$: for $k \geq 1$,

$$\begin{aligned} H_{4k-1} = \frac{H_{2k-1}}{2} &+ 2\sum_{1\leq j\leq k-1} H_{2j}H_{4k-2j-2} \\ &+ \alpha \cdot H_{4k-2} - \frac{{H_{2k-1}}^2}{2} - \sum_{1\leq j\leq 2k-2} H_j H_{4k-j-2} \end{aligned}$$

$$
\begin{aligned}
H_{4k} &= -\beta \cdot H_{4k-2} - \sum_{1\le j<2k-1} H_j H_{2(2k-j-1)} \\
H_{4k+1} &= \frac{H_{2k}}{2} + 2\sum_{1\le j<k} H_{2j}H_{4k-2j} + \alpha \cdot H_{4k} + \frac{{H_{2k}}^2}{2} - \sum_{1\le j<2k} H_j H_{4k-j} \\
H_{4k+2} &= -\beta \cdot H_{4k} - \sum_{1\le j<2k} H_j H_{2(2k-j)},
\end{aligned}
$$

where $\alpha = -\mathcal{N}(j_{1,N})(\frac{1+N/2}{N})$ and $\beta = -\mathcal{N}(j_{1,N})(\frac{1}{N/2})$. Furthermore, we verified in [20] that the above recursion can be also applied to 14 monstrous functions of even levels (including $\mathcal{N}(j_{1,2})$ and $\mathcal{N}(j_{1,6})$) which are Thompson series of type $2B$, $6C$, $6E$, $6F$, $10B$, $10E$, $14B$, $18C$, $18D$, $22B$, $30C$, $30G$, $42C$, $46AB$ (these are all replicable functions) and one monster-like function of type $18e$ (for the definition of monster-like function, we refer to [6]). Therefore the Hauptmoduln mentioned above which have self-recursion formulas can be determined just by the first four coefficients H_1, H_2, H_3 and H_4 without the aid of 2-plicate. What is more interesting would be the fact that there seems to be a connection between super-replicable functions and infinite dimensional Lie superalgebras. That is, considering the arguments from Borcherds [2], Kang [12] and Koike [22] we have believed that the super-replication formulae in (10) might suggest the existence of certain infinite dimensional Lie superalgebra whose denominator identity implies such formulae. Meanwhile, Kang et al. [14] recently showed that there is indeed such a Lie algebra as follows. Let $\widehat{\Gamma}$ be a free abelian group of finite rank and Γ be a countable (usually infinite) sub-semigroup in $\widehat{\Gamma}$ such that every element in Γ can be written as a sum of elements of Γ in only finitely many ways. We consider a $\Gamma \times \mathbb{Z}_2$-graded Lie superalgebra ([13]) with a product identity of the form

$$
\prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \exp\left(-\sum_{k\ge1}\frac{1}{k}\nu^{(k)}(\alpha,a)E^{k(\alpha,a)}\right) = 1 - \sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2} \zeta(\beta,b)E^{(\beta,b)},
$$

where $\nu^{(k)}(\alpha,a), \zeta(\beta,b) \in \mathbb{Z}$ and $E^{(\lambda,a)}$ are the basis elements of the semi-group algebra $\mathbb{C}[\Gamma\times\mathbb{Z}_2]$ with the multiplication given by $E^{(\lambda,a)}E^{(\mu,b)} = E^{(\lambda+\mu,a+b)}$ for $\lambda,\mu \in \Gamma$ and $a,b \in \mathbb{Z}_2$. When $\mathrm{rank}(\widehat{\Gamma}) = 2$, $\widehat{\Gamma}= \mathbb{Z}\times\mathbb{Z}$ and $\Gamma= \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$, they proved that one can get the product identity of the form

$$
\prod_{m,n=1}^{\infty} \exp\left(-\sum_{k\ge1}\frac{1}{k}\nu^{(k)}(m,n)p^{km}q^{kn}\right) = 1 - \sum_{m,n=1}^{\infty}\zeta(m,n)p^m q^n
$$

with $p = E^{(1,0)}$ and $q = -E^{(0,1)}$, from which one derives a characterization of the super-replication formulae. Moreover, they further computed the supertraces of the Monster Lie superalgebras associated with super-replicable functions.

Lastly, as an application of super-replication formulae we consider the following periodically vanishing property. Many of monstrous functions, for example, Thompson series of type $4B$, $4C$, $4D$, $6F$, $8B$, $8C$, $8D$, $8E$, $8F$, $9B$, etc have periodically vanishing properties among the Fourier coefficients (see the Table 1 in [24]). This result must be known to experts, but we could not find a reference. Hereby we describe it in Theorem 13. Meanwhile, as for the case of super-replicable functions, we see from the Appendix, Table 4 that only the Haupmodul $\mathcal{N}(j_{1,12})$ seems to have such property. To this end, we shall first derive in §4 an identity (24) which is analogous to the "2^k-plication formula" ([7], [22]) satisfied by replicable functions. And, combining this with the super-replication formulae we are able to verify that the Fourier coefficients H_m of $\mathcal{N}(j_{1,12})$ vanish whenever $m \equiv 4 \mod 6$ (Corollary 19).

Through the article we adopt the following notations:

- $S_{\Gamma_1(N)}$ the set of $\Gamma_1(N)$-inequivalent cusps
- $q_h = e^{2\pi i z/h}$, $z \in \mathfrak{H}$
- $f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = f\left(\frac{az+b}{cz+d}\right)$
- $f(z) = g(z) + O(1)$ means that $f(z) - g(z)$ is bounded as z goes to $i\infty$.

2. Hauptmodul of $\Gamma_1(12)$

In this section we investigate the generalities of the modular function $j_{1,12}$ which is under primary consideration and construct the Hauptmodul $\mathcal{N}(j_{1,12})$. We also examine some number theoretic property of $\mathcal{N}(j_{1,12})$. As for more arithmetic properties, we refer to [10].

Lemma 1. *Let $\frac{a}{c}$ and $\frac{a'}{c'}$ be fractions in lowest terms. Then $\frac{a}{c}$ is $\Gamma_1(N)$-equivalent to $\frac{a'}{c'}$ if and only if $\pm\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a+nc \\ c \end{smallmatrix}\right) \mod N$ for some $n \in \mathbb{Z}$.*

Proof. Straightforward. □

Using the above lemma we can check that the cusps

$$0,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ 1/6,\ 1/8,\ 1/9,\ \infty,\ 5/12$$

are $\Gamma_1(12)$-inequivalent. But from [15] we know that the cardinality of $S_{\Gamma_1(12)}$ is 10, whence

$$S_{\Gamma_1(12)} = \{0,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ 1/6,\ 1/8,\ 1/9,\ \infty,\ 5/12\}.$$

For later use we are in need of calculating the widths of the cusps of $\Gamma_1(12)$.

Lemma 2. *Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp where $(a,c) = 1$. Then the width of a/c in $X_1(N)$ is given by $N/(c,N)$ if $N \neq 4$.*

Proof. If $N \mid 4$, the statement is obvious. Hence, we assume that N does not divide 4, i.e., $N \neq 1, 2, 4$. First, choose b and d such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Let

h be the width of the cusp a/c. Then h is the smallest positive integer such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \pm\Gamma_1(N).$$

Thus we have

$$\begin{pmatrix} 1-cah & * \\ -c^2h & 1+cah \end{pmatrix} \in \pm\Gamma_1(N).$$

If $\begin{pmatrix} 1-cah & * \\ -c^2h & 1+cah \end{pmatrix}$ is an element of $-\Gamma_1(N)$, by taking trace $2 \equiv -2 \mod N$; hence $N \mid 4$. Thus when $N \neq 1, 2, 4$, $\begin{pmatrix} 1-cah & * \\ -c^2h & 1+cah \end{pmatrix} \in \Gamma_1(N)$. This condition is equivalent to saying that

$$h \in \frac{N}{(c^2, N)}\mathbb{Z} \bigcap \frac{N}{(ca, N)}\mathbb{Z} = \frac{N}{(c, N)}\mathbb{Z}.$$

□

We then have the following table of inequivalent cusps of $\Gamma_1(12)$:

Table 1.

cusp	∞	0	1/2	1/3	1/4	1/5	1/6	1/8	1/9	5/12
width	1	12	6	4	3	12	2	3	4	1

Recall the Jacobi theta functions θ_2, θ_3, and θ_4 defined by

$$\theta_2(z) = \sum_{n\in\mathbb{Z}} q_2^{(n+\frac{1}{2})^2}$$
$$\theta_3(z) = \sum_{n\in\mathbb{Z}} q_2^{n^2}$$
$$\theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n q_2^{n^2}$$

for $z \in \mathfrak{H}$. We have the following transformation formulas ([28] pp.218-219).

(1) $$\theta_2(z+1) = e^{\frac{1}{4}\pi i}\theta_2(z)$$

(2) $$\theta_3(z+1) = \theta_4(z)$$

(3) $$\theta_4(z+1) = \theta_3(z)$$

(4) $$\theta_2\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_4(z)$$

(5) $$\theta_3\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_3(z)$$

(6) $$\theta_4\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_2(z).$$

Lemma 3. *Let k be an odd positive integer and N be a multiple of 4. Then for $F(z) \in M_{\frac{k}{2}}(\widetilde{\Gamma}_0(N))$ and $m \geq 1$, $F(mz) \in M_{\frac{k}{2}}(\widetilde{\Gamma}_0(mN), \chi_m)$ with $\chi_m(d) = \left(\frac{m}{d}\right)$ and $(d, m) = 1$.*

Proof. [31], Proposition 1.3. $\square$

Put $j_{1,12}(z) = \theta_3(2z)/\theta_3(6z)$.

Theorem 4. *(a) $\theta_3(2z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(4))$ and $\theta_3(6z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(12), \chi_3)$.*

(b) $K(X_1(12))$ is equal to $\mathbb{C}(j_{1,12}(z))$. $j_{1,12}$ takes the following value at each cusp: $j_{1,12}(\infty) = 1$, $j_{1,12}(0) = \sqrt{3}$, $j_{1,12}(\frac{1}{2}) = 0$ (a simple zero), $j_{1,12}(\frac{1}{3}) = i$, $j_{1,12}(\frac{1}{4}) = \sqrt{3}i$, $j_{1,12}(\frac{1}{5}) = -\sqrt{3}$, $j_{1,12}(\frac{1}{6}) = \infty$ (a simple pole), $j_{1,12}(\frac{1}{8}) = -\sqrt{3}i$, $j_{1,12}(\frac{1}{9}) = -i$, $j_{1,12}(\frac{5}{12}) = -1$.

Proof. For the first part, we recall that ([21], p.184)

$$\theta_3(2z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(4)).$$

Then by Lemma 3 we immediately get that

$$\theta_3(6z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(12), \chi_3).$$

By the assertion (a), it is clear that $j_{1,12}(z) \in K(X_1(12))$. Thus for (b), it is enough to show that $j_{1,12}(z)$ has only one simple zero and one simple pole on the curve $X_1(12)$. As is well-known, $\theta_3(z)$ never vanishes on $\mathfrak{H}$. Hence we are forced to investigate the zeroes and poles of $j_{1,12}$ at each cusp of $\Gamma_1(12)$. Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(i) $s = \infty$:

$$j_{1,12}(\infty) = \lim_{z \to i\infty} \frac{\theta_3(2z)}{\theta_3(6z)} = \lim_{q \to 0} \frac{1 + 2q + 2q^4 + \cdots}{1 + 2q^3 + 2q^{12} + \cdots} = 1.$$

(ii) $s = 0$:

$$\begin{aligned} j_{1,12}(0) &= \lim_{z \to i\infty} \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_S \\ &= \lim_{z \to i\infty} \frac{\sqrt{-i\frac{z}{2}}\ \theta_3(\frac{z}{2})}{\sqrt{-i\frac{z}{6}}\ \theta_3(\frac{z}{6})} \quad \text{by (5)} \\ &= \sqrt{3}. \end{aligned}$$

(iii) $s = \frac{1}{2}$: We Observe that $(ST^{-2}S)\infty = \left(\begin{smallmatrix} -1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\infty = \frac{1}{2}$.

Considering the identities

$$\begin{aligned}
\theta_3(2z)^2|_{[S]_1} &= z^{-1}\theta_3\left(-\frac{2}{z}\right)^2 = z^{-1}\{\left(-\frac{iz}{2}\right)^{\frac{1}{2}}\theta_3\left(\frac{z}{2}\right)\}^2 \quad \text{by (5)}\\
&= -\frac{i}{2}\theta_3\left(\frac{z}{2}\right)^2\\
\theta_3(2z)^2|_{[ST^{-2}]_1} &= -\frac{i}{2}\theta_3\left(\frac{z}{2}\right)^2|_{[T^{-2}]_1} = -\frac{i}{2}\theta_4\left(\frac{z}{2}\right)^2 \quad \text{by (3)}\\
\theta_3(2z)^2|_{[ST^{-2}S]_1} &= -\frac{i}{2}\theta_4\left(\frac{z}{2}\right)^2|_{[S]_1} = -\frac{i}{2}z^{-1}\{(-2iz)^{\frac{1}{2}}\theta_2(2z)\}^2 \quad \text{by (6)}\\
&= -\theta_2(2z)^2,
\end{aligned}$$

we get that

$$\begin{aligned}
\theta_3(2z)^2|_{s=\frac{1}{2}} &= \lim_{z\to i\infty}\theta_3(2z)^2|_{[ST^{-2}S]_1} = \lim_{z\to i\infty} -\theta_2(2z)^2\\
&= \lim_{z\to i\infty} -2^2\ q_2(1+q^2+q^6+q^{12}+\cdots)^2\\
&\quad (\text{since } \theta_2(z) = 2q_8(1+q+q^3+\cdots))\\
&= 0 \ \ (\text{a triple zero}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\theta_3(2z)^2|_{\left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right)_1} &= \frac{1}{\sqrt{3}}\ \theta_3(2z)^2|_{\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)_1\left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right)_1}\\
&= \frac{1}{\sqrt{3}}\ \theta_3(2z)^2|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)_1\left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right)_1\left(\begin{smallmatrix}1&-1\\0&3\end{smallmatrix}\right)_1}\\
&= \frac{1}{\sqrt{3}}\ \theta_3(2z)^2|_{\left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right)_1\left(\begin{smallmatrix}1&-1\\0&3\end{smallmatrix}\right)_1}\\
&= -\frac{1}{\sqrt{3}}\ \theta_2(2z)^2|_{\left(\begin{smallmatrix}1&-1\\0&3\end{smallmatrix}\right)_1}\\
&= -3^{-1}\ \theta_2\left(2\cdot\frac{z-1}{3}\right)^2 = -3^{-1}e^{-\frac{\pi i}{3}}\ q_6(1+\cdots)^2,
\end{aligned}$$

so that $\theta_3(6z)^2$ has a simple zero at $\frac{1}{2}$. Thus $j_{1,12}^2 = \frac{\theta_3(2z)^2}{\theta_3(6z)^2}$ has a double zero at $\frac{1}{2}$, whence $j_{1,12}$ has a simple zero at $\frac{1}{2}$.

(iv) $s=\frac{1}{3}$: $(ST^{-3}S)\infty = \frac{1}{3}$. First we recall that

$$\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_S = \sqrt{3}\ \frac{\theta_3(\frac{z}{2})}{\theta_3(\frac{z}{6})} \quad \text{by (5).}$$

Observe that $\theta_2(2z) = \frac{1}{2}(\theta_3(\frac{z}{2}) - \theta_4(\frac{z}{2}))$ and $\theta_3(2z) = \frac{1}{2}(\theta_3(\frac{z}{2}) + \theta_4(\frac{z}{2}))$. From these identities we can write $\theta_3(\frac{z}{2}) = \theta_2(2z) + \theta_3(2z)$ and $\theta_3(\frac{z}{6}) = \theta_2(\frac{2}{3}z) +$

$\theta_3(\frac{2}{3}z)$. Then we have that

$$\begin{aligned}\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{ST^{-3}} &= \sqrt{3}\cdot\left.\frac{\theta_2(2z)+\theta_3(2z)}{\theta_2(\frac{2}{3}z)+\theta_3(\frac{2}{3}z)}\right|_{T^{-3}} \\ &= \sqrt{3}\cdot\frac{(e^{-\frac{\pi i}{4}})^6\theta_2(2z)+\theta_3(2z)}{(e^{-\frac{\pi i}{4}})^2\theta_2(\frac{2}{3}z)+\theta_3(\frac{2}{3}z)} \quad \text{by (1), (2) and (3),} \\ &= \sqrt{3}\cdot\frac{i\theta_2(2z)+\theta_3(2z)}{-i\theta_2(\frac{2}{3}z)+\theta_3(\frac{2}{3}z)}\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{ST^{-3}S} &= \sqrt{3}\cdot\left.\frac{i\theta_2(2z)+\theta_3(2z)}{-i\theta_2(\frac{2}{3}z)+\theta_3(\frac{2}{3}z)}\right|_{S} \\ &= \sqrt{3}\cdot\frac{i\theta_4(\frac{z}{2})+\theta_3(\frac{z}{2})}{-i\sqrt{3}\theta_4(\frac{3}{2}z)+\sqrt{3}\theta_3(\frac{3}{2}z)} \quad \text{by (4) and (5)}\end{aligned}$$

which goes to $\frac{i+1}{-i+1}=i$ as $z\to i\infty$, so that

$$j_{1,12}\left(\frac{1}{3}\right)=i.$$

(v) $s=\frac{1}{4}$: $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)\infty=\frac{1}{4}$. In this case we use the following well-known fact from [21] p.148 : For $\gamma\in\Gamma_0(4)$ and $z\in\mathfrak{H}$,

$$\Theta(\gamma z)=\left(\frac{c}{d}\right)\sqrt{\left(\frac{-1}{d}\right)}^{-1}\sqrt{cz+d},$$

where $\Theta(z)=\theta_3(2z)$. Then

$$\begin{aligned}\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)} &= \frac{\Theta(\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)z)}{\Theta(\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)z)} = \frac{\Theta(\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)z)}{\Theta(\left(\begin{smallmatrix}3&2\\4&3\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-2\\0&3\end{smallmatrix}\right)z)} \\ &= \frac{\sqrt{4z+1}\ \Theta(z)}{\left(\frac{4}{3}\right)i^{-1}\sqrt{4\cdot\frac{z-2}{3}+3}\ \Theta(\left(\begin{smallmatrix}1&-2\\0&3\end{smallmatrix}\right)z)} \\ &= \sqrt{3}i\ \frac{\Theta(z)}{\Theta\left(\frac{z-2}{3}\right)}\end{aligned}$$

which tends to $\sqrt{3}i$ when z goes to $i\infty$. Therefore

$$j_{1,12}\left(\frac{1}{4}\right)=\sqrt{3}i.$$

(vi) $s = \frac{1}{5}$: Because $\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ sends ∞ to $\frac{1}{5}$,

$$\begin{aligned} j_{1,12}\left(\frac{1}{5}\right) &= \lim_{z\to i\infty} \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} \\ &= \lim_{z\to i\infty} -\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} \qquad \text{by (a)} \\ &= -\,j_{1,12}(0) = -\sqrt{3}. \end{aligned}$$

(vii) $s = \frac{1}{6}$: Observe that $(ST^{-6}S)\infty = \frac{1}{6}$.
Considering the identities

$$\begin{aligned} \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_S &= \sqrt{3}\cdot\frac{\theta_3(\frac{z}{2})}{\theta_3(\frac{z}{6})} \quad \text{by (5)} \\ \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{ST^{-6}} &= \sqrt{3}\cdot\left.\frac{\theta_3(\frac{z}{2})}{\theta_3(\frac{z}{6})}\right|_{T^{-6}} = \sqrt{3}\cdot\frac{\theta_4(\frac{z}{2})}{\theta_4(\frac{z}{6})} \quad \text{by (2) and (3)} \\ \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{ST^{-6}S} &= \sqrt{3}\cdot\left.\frac{\theta_4(\frac{z}{2})}{\theta_4(\frac{z}{6})}\right|_S = \sqrt{3}\cdot\frac{\sqrt{-2iz}\ \theta_2(2z)}{\sqrt{-6iz}\ \theta_2(6z)} \quad \text{by (6),} \end{aligned}$$

we have that

$$\begin{aligned} j_{1,12}\left(\frac{1}{6}\right) &= \lim_{z\to i\infty} \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{ST^{-6}S} = \lim_{z\to i\infty}\frac{\theta_2(2z)}{\theta_2(6z)} \\ &= \lim_{z\to i\infty}\frac{2\ q_8^2(1+q^2+q^6+\cdots)}{2\ q_8^6(1+q^6+q^{18}+\cdots)}. \\ &\qquad (\text{ since } \theta_2(z) = 2\ q_8(1+q+q^3+\cdots)) \end{aligned}$$

Thus by Table 1, $j_{1,12}$ has a simple pole at $\frac{1}{6}$.
(viii) $s = \frac{1}{8}$: Because $\left(\begin{smallmatrix} -5 & 1 \\ -36 & 7 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ sends ∞ to $\frac{1}{8}$,

$$\begin{aligned} j_{1,12}\left(\frac{1}{8}\right) &= \lim_{z\to i\infty} \left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{\left(\begin{smallmatrix} -5 & 1 \\ -36 & 7 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)} \\ &= \lim_{z\to i\infty} -\left.\frac{\theta_3(2z)}{\theta_3(6z)}\right|_{\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)} \qquad \text{by (a)} \\ &= -\,j_{1,12}\left(\frac{1}{4}\right) = -\sqrt{3}i. \end{aligned}$$

(ix) $s=\frac{1}{9}$: Because $\begin{pmatrix} -5 & 2 \\ -48 & 19 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ sends ∞ to $\frac{1}{9}$,

$$\begin{aligned} j_{1,12}\left(\frac{1}{9}\right) &= \lim_{z\to i\infty} \frac{\theta_3(2z)}{\theta_3(6z)}\bigg|_{\begin{pmatrix} -5 & 2 \\ -48 & 19 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}} \\ &= \lim_{z\to i\infty} -\frac{\theta_3(2z)}{\theta_3(6z)}\bigg|_{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}} \qquad \text{by (a)} \\ &= -j_{1,12}\left(\frac{1}{3}\right) = -i. \end{aligned}$$

(x) $s=\frac{5}{12}$: $\begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}\cdot\infty = \frac{5}{12}$.

$$\begin{aligned} j_{1,12}\left(\frac{5}{12}\right) &= \lim_{z\to i\infty} \frac{\theta_3(2z)}{\theta_3(6z)}\bigg|_{\begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}} \\ &= \lim_{z\to i\infty} -\frac{\theta_3(2z)}{\theta_3(6z)} \qquad \text{by (a)} \\ &= -1. \end{aligned}$$

□

We will now construct the Hauptmodul $\mathcal{N}(j_{1,12})$ from the modular function $j_{1,12}$ mentioned in Theorem 4.

$$\begin{aligned} \frac{2}{j_{1,12}(z)-1} &= \frac{2\ \theta_3(6z)}{\theta_3(2z)-\theta_3(6z)} = \frac{2(1+2q^3+2q^{12}+2q^{27}+\cdots)}{2q-2q^3+2q^4+2q^9-2q^{12}+\cdots} \\ &= \frac{1}{q}+q+q^2+q^3-q^6-q^7-q^8-q^9+q^{11}+2q^{12}+\cdots, \end{aligned}$$

which is in $q^{-1}\mathbb{Z}[[q]]$. From the uniqueness of the normalized generator it follows that $\mathcal{N}(j_{1,12})=\frac{2}{j_{1,12}-1}$. By Theorem 4-(b) we have the following table:

Table 2. Cusp values of $j_{1,12}$ and $\mathcal{N}(j_{1,12})$

s	∞	0	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$1/8$	$1/9$	$5/12$
$j_{1,12}(s)$	1	$\sqrt{3}$	0	i	$\sqrt{3}i$	$-\sqrt{3}$	∞	$-\sqrt{3}i$	$-i$	-1
$\mathcal{N}(j_{1,12})(s)$	∞	$\sqrt{3}+1$	-2	$-1-i$	$\frac{-1-\sqrt{3}i}{2}$	$1-\sqrt{3}$	0	$\frac{-1+\sqrt{3}i}{2}$	$-1+i$	-1

Theorem 5. *Let d be a square free positive integer and $t=N(j_{1,N})$ be the normalized generator of $K(X_1(N))$. Let s be a cusp of $\Gamma_1(N)$ whose width is h_s. If $t\in q^{-1}\mathbb{Z}[[q]]$ and $\prod_{s\in S_{\Gamma_1(N)}\setminus\{\infty\}}(t(z)-t(s))^{h_s}$ is a polynomial in $\mathbb{Z}[t]$, then $t(\tau)$ is an algebraic integer for $\tau\in\mathbb{Q}(\sqrt{-d})\cap\mathfrak{H}$.*

Proof. Let $j(z)=\frac{1}{q}+744+196884q+\cdots$, the elliptic modular function. It is well-known that $j(\tau)$ is an algebraic integer for $\tau\in\mathbb{Q}(\sqrt{-d})\cap\mathfrak{H}$ ([23], [30]). For algebraic proofs, see [4], [26], [29] and [32]. Now, we view j as a function on the modular curve $X_1(N)$. Then j has a pole of order h_s at the cusp s. On the other hand, $t(z)-t(s)$ has a simple zero at s. Thus

$$j \times \prod_{s\in S_{\Gamma_1(N)}\setminus\{\infty\}} (t(z)-t(s))^{h_s}$$

has a pole only at ∞ whose degree is $\mu_N = [\overline{\Gamma}(1) : \overline{\Gamma}_1(N)]$, and so by the Riemann-Roch Theorem it is a monic polynomial in t of degree μ_N which we denote by $f(t)$. Since $\prod_{s\in S_{\Gamma_1(N)}\setminus\{\infty\}}(t(z)-t(s))^{h_s}$ is a polynomial in $\mathbb{Z}[t]$ and j, t have integer coefficients in the q-expansions, $f(t)$ is a monic polynomial in $\mathbb{Z}[t]$ of degree μ_N. This shows that $t(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$. Therefore $t(\tau)$ is integral over $\mathbb{Z}$ for $\tau \in \mathbb{Q}(\sqrt{-d})\cap\mathfrak{H}$. □

Corollary 6. *For $\tau \in \mathbb{Q}(\sqrt{-d})\cap\mathfrak{H}$, $\mathcal{N}(j_{1,12})(\tau)$ is an algebraic integer.*

Proof. $\mathcal{N}(j_{1,12})$ has integral Fourier coefficients. And by Table 1 and 2,

$$\prod_{s\in S_{\Gamma_1(12)}\setminus\{\infty\}} (t(z)-t(s))^{h_s}$$
$$= (t^2-2t-2)^{12}\ (t+2)^6\ (t^2+2t+2)^4\ (t^2+t+1)^3\ t^2\ (t+1)\ \in \mathbb{Z}[t].$$

Now the assertion is immediate from Theorem 5. □

3. Super-replication formulae

Let Δ^n be the set of 2×2 integral matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, where $a\in 1+N\mathbb{Z}$, $c\in N\mathbb{Z}$, and $ad-bc=n$. Then Δ^n has the following right coset decomposition: (See [21], [25], [30])

(7) $$\Delta^n = \bigcup_{\substack{a|n \\ (a,N)=1}} \bigcup_{i=0}^{\frac{n}{a}-1} \Gamma_1(N)\sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a}\end{pmatrix},$$

where $\sigma_a\in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a\end{smallmatrix}\right) \mod N$. Let $f(z)$ be a modular function with respect to $\Gamma_1(N)$. For brevity, let us call it *$f(z)$ is on $\Gamma_1(N)$*. For $f\in K(X_1(N))$ we define an operator U_n and T_n by

$$f|_{U_n} = n^{-1}\sum_{i=0}^{n-1} f|_{\left(\begin{smallmatrix} 1 & i \\ 0 & n\end{smallmatrix}\right)}$$

and

$$f|_{T_n} = n^{-1}\sum_{\substack{a|n \\ (a,N)=1}} \sum_{i=0}^{\frac{n}{a}-1} f|_{\sigma_a\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a}\end{smallmatrix}\right)}.$$

Lemma 7. *For $f\in K(X_1(N))$ and $\gamma_0\in\Gamma_0(N)$, $(f|_{T_n})|_{\gamma_0} = (f|_{\gamma_0})|_{T_n}$ for any positive integer n. In particular, $f|_{T_n}$ is again on $\Gamma_1(N)$.*

Proof. First we claim that

$$\Delta^n\gamma_0 = \gamma_0\Delta^n \quad \text{for } \gamma_0 = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \Gamma_0(N).$$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta^n$. Then

$$\gamma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\gamma_0 = \left(\begin{smallmatrix} w & -y \\ -z & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & d \end{smallmatrix}\right) \mod N.$$

Hence $\gamma_0^{-1}\Delta^n\gamma_0 \subset \Delta^n$ so that $\Delta^n\gamma_0 \subset \gamma_0\Delta^n$. By the same argument we can show the reverse inclusion. We note that

$$\Delta^n\gamma_0 = \bigcup_{\substack{a|n \\ (a,N)=1}} \bigcup_{i=0}^{\frac{n}{a}-1} \Gamma_1(N)\sigma_a\begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix}\gamma_0$$

and

$$\begin{aligned}\gamma_0\Delta^n &= \bigcup_{\substack{a|n \\ (a,N)=1}} \bigcup_{i=0}^{\frac{n}{a}-1} \gamma_0\Gamma_1(N)\sigma_a\begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \\ &= \bigcup_{\substack{a|n \\ (a,N)=1}} \bigcup_{i=0}^{\frac{n}{a}-1} \Gamma_1(N)\gamma_0\sigma_a\begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \quad \text{because } \Gamma_1(N) \lhd \Gamma_0(N).\end{aligned}$$

Here we note that $\sigma_a\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)\gamma_0$'s are the matrices appearing in the definition of $(f|_{T_n})|_{\gamma_0}$ and $\gamma_0\sigma_a\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)$'s are those appearing in the definition of $(f|_{\gamma_0})|_{T_n}$. Now the assertion follows. □

For a positive integer N with $g_{1,N} = 0$, we let t (resp. t_0) be the Hauptmodul of $\Gamma_1(N)$ (resp. $\Gamma_0(N)$). And, we write $X_n(t) = \frac{1}{n}q^{-n} + \sum_{m\geq 1} H_{m,n}q^m$ and $X_n(t_0) = \frac{1}{n}q^{-n} + \sum_{m\geq 1} h_{m,n}q^m$.

Lemma 8. *For positive integers m and n, $H_{m,n} = H_{n,m}$ and $h_{m,n} = h_{n,m}$.*

Proof. Let $p = e^{2\pi iy}$ and $q = e^{2\pi iz}$ with $y, z \in \mathfrak{H}$. Note that $X_n(t)$ can be viewed as the coefficient of p^n-term in $-\log p - \log(t(y) - t(z))$ ([27]). Thus $H_{m,n}$ becomes the coefficient of p^nq^m-term of

$$\begin{aligned}&-\log p - \log(t(y) - t(z)) \\ &= -\log(1 - p/q) + \log(p^{-1} - q^{-1}) - \log(t(y) - t(z)) \\ &= \sum_{i\geq 1} \frac{1}{i}(p/q)^i - F(p,q),\end{aligned}$$

where $F(p,q) = \log\left(\frac{p^{-1}-q^{-1}+\sum_{i\geq 1} H_i(p^i-q^i)}{p^{-1}-q^{-1}}\right)$. We then come up with $F(p,q)$ $= F(q,p)$, which implies that $H_{m,n} = H_{n,m}$. Similarly if we work with t_0 instead of t, the identity $h_{m,n} = h_{n,m}$ follows. □

Theorem 9. *For positive integers n and l such that $(n, N) = (l, n) = 1$,*

$$X_l(t)|_{T_n} = X_{ln}(t)|_{\sigma_n} + c,$$

where c is a constant. In particular,

$$t|_{T_n} = X_n(t)|_{\sigma_n} + c.$$

Proof. Since $X_l(t)$ has poles only at $\Gamma_1(N)\infty$, the poles of $X_l(t)|_{T_n}$ can occur only at $\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)^{-1} {\sigma_a}^{-1}\Gamma_1(N)\infty$, where a and i are the indices appearing in the definition of T_n. On the other hand, we have

$$\begin{aligned}\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)^{-1} {\sigma_a}^{-1}\Gamma_1(N)\infty &= n^{-1}\left(\begin{smallmatrix} \frac{n}{a} & -i \\ 0 & a \end{smallmatrix}\right){\sigma_a}^{-1}\Gamma_1(N)\infty \\ &= \left(\begin{smallmatrix} \frac{n}{a} & -i \\ 0 & a \end{smallmatrix}\right){\sigma_a}^{-1}\Gamma_1(N)\infty.\end{aligned}$$

Let γ be an element in $\Gamma_1(N)$. Then

$$\begin{aligned}\left(\begin{smallmatrix} \frac{n}{a} & -i \\ 0 & a \end{smallmatrix}\right){\sigma_a}^{-1}\gamma\infty &\equiv \left(\begin{smallmatrix} \frac{n}{a} & -i \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\infty \mod N \\ &\equiv \left(\begin{smallmatrix} n & * \\ 0 & 1 \end{smallmatrix}\right)\infty = \frac{n+Nm}{Nk}\end{aligned}$$

for some $k, m \in \mathbb{Z}$. If there exists an integer $x > 1$ with $x \mid (n+Nm, Nk)$, then x must divide $\det\left[\left(\begin{smallmatrix} \frac{n}{a} & -i \\ 0 & a \end{smallmatrix}\right){\sigma_a}^{-1}\gamma\right] = n$. In this case $x \nmid N$ because $(n, N) = 1$. Therefore $\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)^{-1}{\sigma_a}^{-1}\gamma\infty$ is of the form $\gamma_0\infty$ for some $\gamma_0 \in \Gamma_0(N)$. Now we conclude that $X_l(t)|_{T_n}$ can have poles only at $\gamma_0\infty$ for some $\gamma_0 \in \Gamma_0(N)$. By Lemma 7,

$$\begin{aligned}&n(X_l(t)|_{T_n})|_{\gamma_0} = n(X_l(t)|_{\gamma_0})|_{T_n} \\ &= \sum_{a|n}\sum_{i=0}^{\frac{n}{a}-1}(X_l(t)|_{\gamma_0})|_{\sigma_a\left(\begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix}\right)} = \sum_{a|n}\frac{n}{a}(X_l(t)|_{\gamma_0})|_{\sigma_a}|_{U_{\frac{n}{a}}}(az) \\ &= \sum_{\substack{a|n \\ a\neq n}}\frac{n}{a}(X_l(t)|_{\gamma_0})|_{\sigma_a}|_{U_{\frac{n}{a}}}(az) + X_l(t)|_{\gamma_0\sigma_n}(nz).\end{aligned}$$

Here we note that

$$\sum_{\substack{a|n \\ a\neq n}}\frac{n}{a}(X_l(t)|_{\gamma_0})|_{\sigma_a}|_{U_{\frac{n}{a}}}(az) = O(1).$$

In fact, if $\gamma_0\sigma_a \notin \pm\Gamma_1(N)$, it is clear that $(X_l(t)|_{\gamma_0})|_{\sigma_a}|_{U_{\frac{n}{a}}}$ has a holomorphic q-expansion. Otherwise

$$\begin{aligned}X_l(t)|_{\gamma_0\sigma_a U_{\frac{n}{a}}} &= X_l(t)|_{U_{\frac{n}{a}}} \\ &= (l^{-1}q^{-l} + \text{ terms of positive degree })|_{U_{\frac{n}{a}}} = O(1)\end{aligned}$$

because $(l,n)=1$. Now we have

$$n(X_l(t)|_{T_n})|_{\gamma_0} = X_l(t)|_{\gamma_0\sigma_n}(nz) + O(1). \tag{8}$$

This implies that $X_l(t)|_{T_n}$ has a pole at $\gamma_0\infty$ if and only if $\gamma_0\sigma_n \in \pm\Gamma_1(N)$, that is, $\gamma_0 \in \pm\Gamma_1(N){\sigma_n}^{-1}$. Hence $X_l(t)|_{T_n}$ has poles only at cusps $\Gamma_1(N){\sigma_n}^{-1}\infty$. In this case we derive from (8)

$$n(X_l(t)|_{T_n})|_{\sigma_n^{-1}} = X_l(t)|_{\sigma_n^{-1}\sigma_n}(nz) + O(1) = l^{-1}q^{-ln} + O(1).$$

Thus $(X_l(t)|_{T_n})|_{\sigma_n^{-1}} = (nl)^{-1}q^{-ln} + O(1)$ and $(X_l(t)|_{T_n})|_{\sigma_n^{-1}}$ has poles only at cusps $\sigma_n\Gamma_1(N){\sigma_n}^{-1}\infty = \Gamma_1(N)\infty$. On the other hand, $X_{ln}(t)$ has poles only at $\Gamma_1(N)\infty$ too and $X_{ln}(t) = (ln)^{-1}q^{-ln} + O(1)$. Therefore $(X_l(t)|_{T_n})|_{\sigma_n^{-1}} = X_{ln}(t) + c$ for some constant c. Then we have $X_l(t)|_{T_n} = X_{ln}(t)|_{\sigma_n} + c$, as desired. □

Corollary 10. *Let N be a positive integer such that the genus $g_{1,N}$ is zero and $[\overline{\Gamma}_0(N) : \overline{\Gamma}_1(N)] \le 2$. For positive integers n, l and m such that $(n,N) = (l,n) = 1$, we have*

$$\sum_{\substack{e|(m,n)\\ e>0}} e^{-1}\left\{\psi(e)\left(2H_{\frac{mn}{e^2},l} - h_{\frac{mn}{e^2},l}\right) + h_{\frac{mn}{e^2},l}\right\}$$
$$= \psi(n)(2H_{m,ln} - h_{m,ln}) + h_{m,ln},$$

where $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ is a character defined by

$$\psi(e) = \begin{cases} 1, & \text{if } e \equiv \pm 1 \mod N \\ -1, & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 9 that

$$X_l(t)|_{T_n} = \sum_{\substack{e|n\\ e>0}} e^{-1}\left(X_l(t)|_{\sigma_e}\right)|_{U_{\frac{n}{e}}}(ez) = X_{ln}(t)|_{\sigma_n} + \text{ constant}. \tag{9}$$

Note that for each positive integer r,

$$X_r(t)|_{\sigma_e} + X_r(t) = \begin{cases} 2X_r(t), & \text{if } e \equiv \pm 1 \mod N \\ X_r(t_0) + \text{ constant}, & \text{otherwise.} \end{cases}$$

In the above when e is not congruent to $\pm 1 \mod N$, $X_r(t)|_{\sigma_e} + X_r(t)$ is on $\Gamma_0(N)$ and has poles only at $\Gamma_0(N)\infty$ with $r^{-1}q^{-r}$ as its pole part, which guarantees the above equality. We then have

$$X_r(t)|_{\sigma_e} = \frac{1}{2}\{\psi(e)(2X_r(t) - X_r(t_0)) + X_r(t_0)\} + \text{ constant}.$$

Now (9) reads

$$\sum_{\substack{e|n\\ e>0}} e^{-1}\cdot \frac{1}{2}\{\psi(e)(2X_l(t)-X_l(t_0))+X_l(t_0)\}|_{U_{\frac{n}{e}}}(ez)$$
$$=\frac{1}{2}\{\psi(n)(2X_{ln}(t)-X_{ln}(t_0))+X_{ln}(t_0)\}+\text{constant}.$$

Comparing the coefficients of q^m-terms on both sides, we get the corollary. □

Corollary 11. *Let N be a positive integer such that the genus $g_{1,N}$ is zero and $[\overline{\Gamma}_0(N):\overline{\Gamma}_1(N)]\le 2$. For positive integers a,b,c,d with $ab=cd$, $(a,b)=(c,d)$ and $(b,N)=(d,N)=1$,*

$$H_{a,b}=\psi(bd)H_{c,d}+\frac{(1-\psi(bd))}{2}h_{c,d}. \tag{10}$$

Proof. In Corollary 10 we take $n=b$, $l=1$ and $m=a$. Then it follows from the conditions and the replicability of $h_{m,n}$ that

$$\psi(b)(2H_{a,b}-h_{a,b})=\psi(d)(2H_{c,d}-h_{c,d}).$$

Now the assertion follows. □

Corollary 12. *Let N be a positive integer with $g_{1,N}=0$ and $[\overline{\Gamma}_0(N):\overline{\Gamma}_1(N)]=2$. If $(mn,N)=1$ and $mn\not\equiv\pm 1 \mod N$, then $h_{m,n}=2H_{m,n}$.*

Proof. In Corollary 11 we take $a=m, b=n$ and $c=n, d=m$. The condition that $\psi(mn)=-1$ implies

$$\begin{aligned}H_{m,n}&=-H_{n,m}+h_{n,m}\\&=-H_{m,n}+h_{m,n}\quad\text{by Lemma 8.}\end{aligned}$$

This proves the corollary. □

4. Vanishing property in the Fourier coefficients of $\mathcal{N}(j_{1,12})$

As mentioned in the introduction, many of the Thompson series have periodically vanishing properties among the Fourier coefficients. Now we will give a more theoretical explanation for these phenomena.

Let T_g be the Thompson series of type g and Γ_g be its corresponding genus zero group. To describe Γ_g we are in need of some notations. Let N be a positive integer and Q be any Hall divisor of N, that is, Q be a positive divisor of N for which $(Q,N/Q)=1$. We denote by $W_{Q,N}$ a matrix $\begin{pmatrix} Qx & y\\ Nz & Qw\end{pmatrix}$ with $\det W_{Q,N}=Q$ and x,y,z and $w\in\mathbb{Z}$, and call it an Atkin-Lehner involution. Let S be a subset of Hall divisors of N and let $\Gamma=N+S$ be the subgroup of $PSL_2(\mathbb{R})$ generated by $\Gamma_0(N)$ $(=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\mid c\equiv 0 \mod N\})$ and all Atkin-Lehner involutions $W_{Q,N}$ for $Q\in S$. For a positive divisor h of 24, let n be a multiple of h and set $N=nh$. When S is a subset of Hall divisors of n/h, we denote by $\Gamma_0(n|h)+S$ the group generated by $\left(\begin{smallmatrix} h & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\circ\Gamma_0(n/h)\circ\left(\begin{smallmatrix} h & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} h & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\circ W_{Q,n/h}\circ\left(\begin{smallmatrix} h & 0\\ 0 & 1\end{smallmatrix}\right)$ for all $Q\in S$. If there exists a homomorphism λ of $\Gamma_0(n|h)+S$ into $\mathbb{C}^*$ such that

(11) $\lambda(\Gamma_0(N))=1,$

(12) $\lambda(\left(\begin{smallmatrix} 1 & 1/h\\ 0 & 1\end{smallmatrix}\right))=e^{-2\pi i/h},$

(13) $\lambda(\left(\begin{smallmatrix} 1 & 0\\ n & 1\end{smallmatrix}\right))=\begin{cases} e^{2\pi i/h} & \text{if } n/h\in S,\\ e^{-2\pi i/h} & \text{if } n/h\notin S,\end{cases}$

(14) λ is trivial on all Atkin-Lehner involutions of $\Gamma_0(N)$ in $\Gamma_0(n|h)+S$,

then we let $n|h+S$ be the kernel of λ which is a subgroup of $\Gamma_0(n|h)+S$ of index h. Ferenbaugh ([6]) found out a necessary and sufficient condition for the homomorphism λ to exist and calculated the genera of groups of type $n|h+S$. All the genus zero groups of type $n|h+S$ are listed in [6], Table 1.1 and 1.2. Now we have the following theorem.

Theorem 13. *Write $T_g(q)=q^{-1}+\sum_{m\geq 1}c_g(m)q^m$.*

(i) *If $\Gamma_g=\Gamma_0(N)$ and h is the largest integer such that $h\mid 24$ and $h^2\mid N$, then $c_g(m)=0$ unless $m\equiv -1 \mod h$.*

(ii) *If $\Gamma_g=N+S$ and there exists a prime p such that $p^2|N$ and $p\nmid Q$ for all $Q\in S$, then $c_g(m)=0$ whenever $m\equiv 0 \mod p$.*

(iii) *If $\Gamma_g=n|h+S$, then $c_g(m)=0$ unless $m\equiv -1 \mod h$.*

Proof. (i) Note that $\left(\begin{smallmatrix} 1 & h^{-1}\\ 0 & 1\end{smallmatrix}\right)$ belongs to the normalizer of $\Gamma_0(N)$. Thus $T_g|_{\left(\begin{smallmatrix} 1 & h^{-1}\\ 0 & 1\end{smallmatrix}\right)}$ has poles only at ∞, which is a simple pole. This enables us to write $T_g|_{\left(\begin{smallmatrix} 1 & h^{-1}\\ 0 & 1\end{smallmatrix}\right)}=c\cdot T_g$ for some constant c. By comparing the coefficients of q^m-terms on both sides, the assertion follows.

(ii) From Corollary 3.1 [22] it follows that $T_g|_{U_p}=0$. Thus (ii) is clear.

(iii) Considering the identity in [22], p.27 we have $T_g(z+1/h)=e^{-2\pi i/h}\cdot T_g(z)$. Then $T_g(q)=q^{-1}+\sum_{0<l\in\mathbb{Z}}c_g(lh-1)q^{lh-1}$, which implies (iii). □

Unlike the cases of Thompson series, when we handle the super-replicable function $\mathcal{N}(j_{1,12})$ we can not directly use the ingredients adopted in Theorem 13. Therefore we start with

Lemma 14. *For $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(12)$, $j_{1,12}|_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}=\left(\frac{3}{d}\right)j_{1,12}$. Here $\left(\frac{\cdot}{\cdot}\right)$ denotes the generalized quadratic residue symbol.*

Proof. Immediate from Lemma 3 and Theorem 4. □

We fix $N=12$ and let t denote the Hauptmodul $\mathcal{N}(j_{1,12})$ in what follows.

Lemma 15. *For $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(24)$,*

$$(t|_{U_2})_\chi|_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}=(-1)^{\frac{c}{24}}\cdot\left(\frac{3}{a+\frac{c}{4}}\right)\cdot(t|_{U_2})_\chi,$$

where $\chi = \left(\frac{-1}{\cdot}\right)$ *is the Jacobi symbol and* $(t|_{U_2})_\chi$ *is the twist of* $t|_{U_2}$ *by* χ *([21], p.127).*

Proof. From [21], p.128 we observe that

$$(15) \qquad (t|_{U_2})_\chi = \frac{1}{\sqrt{-4}}\left(t|_{U_2}\left(z+\frac{1}{4}\right) - t|_{U_2}\left(z+\frac{3}{4}\right)\right)$$
$$= \frac{1}{\sqrt{-4}} \sum_{i\in(\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right)\left(t|_{U_2}+\frac{1}{2}\right)|_{\left(\begin{smallmatrix}4 & i\\ 0 & 4\end{smallmatrix}\right)}.$$

It then follows from [19], Corollary 28 that

$$\left(t - t\left(\frac{1}{6}\right)\right) \times t|_{U_2} = H_2.$$

If we compare the coefficients of q-term on both sides, we get $H_4 - t\left(\frac{1}{6}\right)\cdot H_2 = 0$. And, substituting $H_2 = 1$ and $H_4 = 0$ we get $t\left(\frac{1}{6}\right) = 0$. Now

$$(16) \qquad t|_{U_2} = \frac{1}{t} = \frac{j_{1,12}-1}{2}.$$

Then for $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(12)$

$$(17) \qquad \left(t|_{U_2}+\frac{1}{2}\right)|_{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)} = \frac{1}{2}j_{1,12}|_{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)} = \frac{1}{2}\cdot\left(\frac{3}{d}\right)j_{1,12} \quad \text{by Lemma 14}$$
$$= \left(\frac{3}{d}\right)\left(t|_{U_2}+\frac{1}{2}\right) = \left(\frac{3}{a}\right)\left(t|_{U_2}+\frac{1}{2}\right)$$

since $ad \equiv 1 \mod 12$. Let $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(24)$. For $i = 1, 3$, we consider $\left(\begin{smallmatrix}4 & i\\ 0 & 4\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}4a+ic & 4b+id\\ 4c & 4d\end{smallmatrix}\right)$. Since $(4a+ic, 4c)$ divides det $\left(\begin{smallmatrix}4a+ic & 4b+id\\ 4c & 4d\end{smallmatrix}\right)$, we must have $(4a+ic, 4c) = 4$; hence $(a+ic/4, c) = 1$. Thus we can choose integers x_i and y_i such that $\gamma_i = \left(\begin{smallmatrix}a+ic/4 & x_i\\ c & y_i\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Write ${\gamma_i}^{-1}\left(\begin{smallmatrix}4 & i\\ 0 & 4\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}4 & z_i\\ 0 & 4\end{smallmatrix}\right)$ for some integer z_i. Then we have

$$\left(\begin{smallmatrix}4 & i\\ 0 & 4\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}a+ic/4 & x_i\\ c & y_i\end{smallmatrix}\right)\left(\begin{smallmatrix}4 & z_i\\ 0 & 4\end{smallmatrix}\right)$$

for some integer z_i. Comparing (1,2)-component on both sides we get

$$4b + id = (a+ic/4)z_i + 4x_i.$$

Thus $id \equiv (a+ic/4)z_i \mod 4$. Then

$$\begin{aligned} z_i &\equiv (a+ic/4)id \mod 4, \\ &\qquad \text{because } n^2 \equiv 1 \mod 4 \text{ for every odd integer } n \\ &\equiv aid + i^2\cdot\frac{c}{4}\cdot d \mod 4 \\ &\equiv i + \frac{c}{4}\cdot d \mod 4, \qquad \text{due to } ad \equiv 1 \mod 4 \\ &\equiv i + 6c_1 d \mod 4, \qquad \text{where we write } c = 24c_1. \end{aligned}$$

Therefore we derive that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(24)$,

$$\begin{aligned}
&\sqrt{-4} \cdot (t|_{U_2})_\chi |_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \\
&= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} 4 & i \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \quad \text{by (15)} \\
&= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} a+ic/4 & x_i \\ c & y_i \end{smallmatrix}\right)\left(\begin{smallmatrix} 4 & z_i \\ 0 & 4 \end{smallmatrix}\right)} \\
&= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(\frac{3}{a+ic/4}\right) \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} 4 & z_i \\ 0 & 4 \end{smallmatrix}\right)} \quad \text{by (17).}
\end{aligned}$$

Now if we set $c = 24c_1$ as before, then we have

$$\begin{aligned}
&\sqrt{-4} \cdot (t|_{U_2})_\chi |_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \\
&= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(\frac{3}{a+6c_1 i}\right) \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} 4 & i+6c_1 d \\ 0 & 4 \end{smallmatrix}\right)} \\
&= \left(\frac{3}{a+6c_1}\right) \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} 4 & i+6c_1 d \\ 0 & 4 \end{smallmatrix}\right)} \\
&\qquad \text{since } a + 6c_1 i \equiv a + 6c_1 \mod 12 \\
&= \left(\frac{3}{a+6c_1}\right) \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(\frac{-1}{i+6c_1 d}\right) \left(\frac{-1}{i+6c_1 d}\right) \\
&\qquad \times \left(t|_{U_2} + \frac{1}{2}\right) |_{\left(\begin{smallmatrix} 4 & i+6c_1 d \\ 0 & 4 \end{smallmatrix}\right)} \\
&= (-1)^{c_1} \cdot \left(\frac{3}{a+6c_1}\right) \cdot \sqrt{-4} \cdot (t|_{U_2})_\chi
\end{aligned}$$

because $\left(\frac{-1}{i}\right)\left(\frac{-1}{i+6c_1 d}\right) = \left(\frac{-1}{i^2+6c_1 id}\right) = \left(\frac{-1}{1+2c_1 id}\right) = (-1)^{c_1 id} = (-1)^{c_1}$ and $i + 6c_1 d$ runs over $(\mathbb{Z}/4\mathbb{Z})^\times$. This completes the lemma. □

Lemma 16. (i) *For each $k \geq 1$, we put*

$$g(z) = (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)(z) - X_{2^k}(t)\left(z + \frac{1}{2}\right)\right).$$

Then g belongs to $K(X_1(24))$.

(ii) *For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(24)$ and $k \geq 1$,*

$$\left(t|_{U_{2^k}}\right)_\chi |_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = (-1)^{\frac{c}{24}} \cdot \left(\frac{3}{a+\frac{c}{4}}\right) \cdot \left(t|_{U_{2^k}}\right)_\chi.$$

In particular, $\left(t|_{U_{2^k}}\right)_\chi$ lies in $K(X_1(24))$.

Proof. First we note that for $n \mid N^\infty$, $T_n = U_n$. Here, by $n \mid N^\infty$ we mean that n divides some power of N. To show $g \in K(X_1(24))$, we observe that

$$g = (-1)^{k-1}\left(X_{2^k}(t)(z) - X_{2^k}(t)|_{U_2}(2z)\right).$$

Then using Lemma 7 we obtain that $g \in K(X_1(24))$. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(12)$,

$$\begin{aligned}
\left(t|_{U_{2^k}} + \frac{1}{2}\right)|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} &= \left(t|_{U_2} + \frac{1}{2}\right)|_{U_{2^{k-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \\
&= \left(t|_{U_2} + \frac{1}{2}\right)|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)U_{2^{k-1}}} \quad \text{by Lemma 7} \\
&= \left(\frac{3}{a}\right)\left(t|_{U_2} + \frac{1}{2}\right)|_{U_{2^{k-1}}} \quad \text{by (17)} \\
&= \left(\frac{3}{a}\right)\left(t|_{U_{2^k}} + \frac{1}{2}\right).
\end{aligned} \tag{18}$$

Now we can proceed in the same manner as in the proof of Lemma 15. □

Lemma 17. *For $k \geq 1$,*

(i) $$\left(t|_{U_{2^k}}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)} = \begin{cases} \frac{1}{2}{q_2}^{-1} + O(1), & \text{if } k = 1 \\ O(1), & \text{otherwise.} \end{cases}$$

(ii) $$\left(t|_{U_{2^k}}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)} = \begin{cases} O(1), & \text{if } k = 1 \\ \frac{1}{4}{q_4}^{-1} + O(1), & \text{if } k = 2 \\ -\frac{i}{8}{q_2}^{-1} + O(1), & \text{if } k = 3 \\ (-1)^{k-1} \cdot \frac{i}{2^k} \cdot q^{-2^{k-4}} + O(1), & \text{if } k \geq 4. \end{cases}$$

Proof. (i) First, $t|_{\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)}$ is holomorphic at ∞ because t has poles only at the cusps $\Gamma_1(12)\infty$. Now for $k \geq 1$,

$$\begin{aligned}
\left(t|_{U_{2^k}}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)} &= \left(\left(t|_{U_{2^{k-1}}}\right)|_{U_2}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)} \\
&= \frac{1}{2}\left(t|_{U_{2^{k-1}}}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)} + \frac{1}{2}\left(t|_{U_{2^{k-1}}}\right)|_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)} \\
&= \frac{1}{2}\left(t|_{U_{2^{k-1}}}\right)|_{\left(\begin{smallmatrix} 1 & 0 \\ 12 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)} + \frac{1}{2}\left(t|_{U_{2^{k-1}}}\right)|_{\left(\begin{smallmatrix} 7 & 4 \\ 12 & 7 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 2 \end{smallmatrix}\right)} \\
&= \frac{1}{2}\left(t|_{U_{2^{k-1}}}\right)\left(\frac{z}{2}\right) + \frac{1}{2}\left(\left(t|_{\left(\begin{smallmatrix} 7 & 4 \\ 12 & 7 \end{smallmatrix}\right)}\right)|_{U_{2^{k-1}}}\right)|_{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 2 \end{smallmatrix}\right)} \\
&\qquad \text{by Lemma 7} \\
&= \begin{cases} \frac{1}{2}{q_2}^{-1} + O(1), & \text{if } k = 1 \\ O(1), & \text{otherwise.} \end{cases}
\end{aligned}$$

(ii) We observe that $t|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} \in O(1)$. And for $k \geq 1$,

$$(t|_{U_{2^k}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} = ((t|_{U_{2^{k-1}}})|_{U_2})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} \tag{19}$$
$$= \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} + \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}$$
$$= \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)} + \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)}$$

has a holomorphic Fourier expansion if $k = 1$. Thus we suppose $k \geq 2$. We then derive that

$$\begin{aligned}
&(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)} \\
=\ &((t|_{U_{2^{k-2}}})|_{U_2})|_{\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)} \\
=\ &\frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)} + \frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)} \\
=\ &\frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right)} + \frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}11&-1\\12&-1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)} \\
=\ &\frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right)} + \frac{1}{2}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)}.
\end{aligned}$$

If we substitute the above into (19), for $k \geq 2$,

$$(t|_{U_{2^k}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} \tag{20}$$
$$= \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)} + \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)} + \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)} \tag{21}$$
$$= \frac{1}{2}(t|_{U_{2^{k-1}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)} + \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&1\\0&2\end{smallmatrix}\right)} + \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)\left(\begin{smallmatrix}4&1\\0&1\end{smallmatrix}\right)}. \tag{22}$$

When $k = 2$,

$$\begin{aligned}
(t|_{U_4})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} &= \frac{1}{2}(t|_{U_2})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)}\left(\frac{z}{2}\right) + \frac{1}{4}t|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)}\left(z + \frac{1}{2}\right) \\
&\quad + \frac{1}{4}t|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z + 1) \\
&= \frac{1}{4}q_4{}^{-1} + O(1) \quad \text{by (i)}.
\end{aligned}$$

If $k=3$,

$$\begin{aligned}(t|_{U_8})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} &= \frac{1}{2}(t|_{U_4})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)}\left(\frac{z}{2}\right)+\frac{1}{4}(t|_{U_2})|_{\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)}\left(z+\frac{1}{2}\right)\\ &\quad+\frac{1}{4}(t|_{U_2})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z+1)\\ &= \frac{1}{8}e^{-\pi i(z+\frac{1}{2})}+O(1) = -\frac{i}{8}{q_2}^{-1}+O(1)\\ &\quad\text{by (i) and the case } k=1 \text{ in (ii)}.\end{aligned}$$

For $k\geq 4$, we will show by induction on k that

$$(t|_{U_{2^k}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} = (-1)^{k-1}\cdot\frac{i}{2^k}\cdot q^{-2^{k-4}}+O(1). \tag{23}$$

First we note that by (i) and (20)

$$(t|_{U_{2^k}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} = \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z+1)+O(1).$$

If $k=4$,

$$\begin{aligned}(t|_{U_{2^4}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} &= \frac{1}{4}(t|_{U_{2^{4-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z+1)+O(1)\\ &= \frac{1}{16}e^{-\frac{\pi i}{2}(4z+1)}+O(1) \quad \text{by the case } k=2\\ &= (-1)^{4-1}\cdot\frac{i}{2^4}\cdot q^{-2^{4-4}}+O(1).\end{aligned}$$

Thus when $k=4$, (23) holds. Meanwhile, if $k=5$ then we get that

$$\begin{aligned}(t|_{U_{2^5}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} &= \frac{1}{4}(t|_{U_{2^{5-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z+1)+O(1)\\ &= -\frac{i}{32}e^{-\pi i(4z+1)}+O(1) \quad \text{by the case } k=3\\ &= (-1)^{5-1}\cdot\frac{i}{2^5}\cdot q^{-2^{5-4}}+O(1).\end{aligned}$$

Therefore in this case (23) is also valid. Now for $k\geq 6$,

$$\begin{aligned}(t|_{U_{2^k}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)} &= \frac{1}{4}(t|_{U_{2^{k-2}}})|_{\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)}(4z+1)+O(1)\\ &= \frac{1}{4}\cdot(-1)^{k-2-1}\cdot\frac{i}{2^{k-2}}\cdot e^{-2\pi i(4z+1)\cdot 2^{k-2-4}}+O(1)\\ &\quad\text{by induction hypothesis for } k-2\\ &= (-1)^{k-1}\cdot\frac{i}{2^k}\cdot q^{-2^{k-4}}+O(1).\end{aligned}$$

This proves the lemma. □

Theorem 18. *For $k\geq 1$,*

$$(t|_{U_{2^k}})_\chi(z) = (-1)^{k-1}\cdot\frac{1}{2}\cdot\left(X_{2^k}(t)(z)-X_{2^k}(t)\left(z+\frac{1}{2}\right)\right). \tag{24}$$

This identity twisted by a character χ is analogous to the "2^k-plication formula" ([7], [22]) satisfied by the Hauptmodul of $\Gamma_0(N)$.

Proof. Put $g(z) = (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)(z) - X_{2^k}(t)\left(z + \frac{1}{2}\right)\right)$ as before. We see by Lemma 16 that both $\left(t|_{U_{2^k}}\right)_\chi$ and g sit in $K(X_1(24))$. For $\left(t|_{U_{2^k}}\right)_\chi = g$, we will show that $\left(t|_{U_{2^k}}\right)_\chi - g$ has no poles in $\mathfrak{H}^*$. Recall that

$$\begin{aligned}\left(t|_{U_{2^k}}\right)_\chi &= \frac{1}{\sqrt{-4}} \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \chi(i) \left(t|_{U_{2^k}}\right)|_{\left(\begin{smallmatrix} 4 & i \\ 0 & 4 \end{smallmatrix}\right)} \\ &= \frac{1}{\sqrt{-4}} \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \sum_{j=0}^{2^k-1} \chi(i) t|_{\left(\begin{smallmatrix} 1 & j \\ 0 & 2^k \end{smallmatrix}\right)\left(\begin{smallmatrix} 4 & i \\ 0 & 4 \end{smallmatrix}\right)}.\end{aligned}$$

Since t has poles only at $\Gamma_1(12)\infty$, $\left(t|_{U_{2^k}}\right)_\chi$ can also have poles only at

$$\begin{aligned}\left(\begin{smallmatrix} 4 & i \\ 0 & 4 \end{smallmatrix}\right)^{-1} \begin{pmatrix} 1 & j \\ 0 & 2^k \end{pmatrix}^{-1} \Gamma_1(12)\infty &= 16^{-1} \cdot 2^{-k} \cdot \left(\begin{smallmatrix} 4 & -i \\ 0 & 4 \end{smallmatrix}\right) \begin{pmatrix} 2^k & -j \\ 0 & 1 \end{pmatrix} \Gamma_1(12)\infty \\ &= \begin{pmatrix} 2^{k+2} & -4j-i \\ 0 & 4 \end{pmatrix} \Gamma_1(12)\infty.\end{aligned}$$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(12)$. Then

$$\begin{pmatrix} 2^{k+2} & -4j-i \\ 0 & 4 \end{pmatrix} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \infty = \begin{pmatrix} 2^{k+2}a-4jc-ic & * \\ 4c & * \end{pmatrix} \infty = \frac{2^{k+2}a - 4jc - ic}{4c}.$$

Observe that $(2^{k+2}a-4jc-ic, 4c) \mid 2^k \cdot 16 = 2^{k+4}$. Write $(2^{k+2}a-4jc-ic, 4c) = 2^l$ for some integer $l \geq 0$. Then

$$s = \frac{(2^{k+2}a - 4jc - ic)/2^l}{4c/2^l}$$

is in lowest terms. Since $12 \mid c$, s is of the form $s = \frac{n}{3m}$ for some integers m and n. We assume $(3m, n) = 1$. Here we consider two cases.

(i) $2^2 \nmid m$:

Choose integers x and y such that $\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and consider

$$\begin{aligned}\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)} &= \frac{1}{\sqrt{-4}} \left(t|_{U_{2^k}\left(\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)} - t|_{U_{2^k}\left(\begin{smallmatrix} 4 & 3 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)}\right) \\ &= \frac{1}{\sqrt{-4}} \left(t|_{U_{2^k}\left(\begin{smallmatrix} 4n+3m & * \\ 12m & * \end{smallmatrix}\right)} - t|_{U_{2^k}\left(\begin{smallmatrix} 4n+9m & * \\ 12m & * \end{smallmatrix}\right)}\right).\end{aligned}$$

Since $(3m, n) = 1$ and $2^2 \nmid m$, we can write $\left(\begin{smallmatrix} 4n+3m & * \\ 12m & * \end{smallmatrix}\right) = \gamma_0 U_1$ and $\left(\begin{smallmatrix} 4n+9m & * \\ 12m & * \end{smallmatrix}\right) = \gamma_0' U_2$, where both γ_0 and γ_0' are in $\Gamma_0(12)$ and U_1, U_2 are upper triangular matrices. Then by (18), $\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)} \in O(1)$. Hence, if $2^2 \nmid m$ then $\left(t|_{U_{2^k}}\right)_\chi$ is holomorphic at the cusp $s = \frac{n}{3m}$.

(ii) $2^2 \mid m$:

If $2^3 \mid m$, then s is of the form $s = \left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)\infty$ with $\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right) \in \Gamma_0(24)$. Thus by Lemma 16,

$$\left(t|_{U_{2^k}}\right)_\chi\Big|_{\left(\begin{smallmatrix} n & x \\ 3m & y \end{smallmatrix}\right)} = (-1)^{\frac{3m}{24}} \cdot \left(\frac{3}{n+3m/4}\right) \cdot \left(t|_{U_{2^k}}\right)_\chi \in O(1).$$

As for the other cases, if we use Lemma 1, it is easy to see that s is equivalent to $\frac{1}{12}$ or $\frac{5}{12}$ under $\Gamma_1(24)$.

Thus we conclude that $\left(t|_{U_{2^k}}\right)_\chi$ can have poles only at $\frac{1}{12}$, $\frac{5}{12}$ under $\Gamma_1(24)$-equivalence. Next, let us investigate the poles of g. Recall that $X_{2^k}(t)$ has poles only at $\Gamma_1(12)\infty$. Therefore g can have poles only at $\left(\begin{smallmatrix} 2 & i \\ 0 & 2 \end{smallmatrix}\right)^{-1}\Gamma_1(12)\infty$ for $i = 0, 1$. And, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(12)$,

$$\begin{aligned}\left(\begin{smallmatrix} 2 & i \\ 0 & 2 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\infty &= \left(\begin{smallmatrix} 2 & -i \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\infty = \frac{2a-ic}{2c} \\ &= \frac{a-ic/2}{c} \quad \text{in lowest terms.}\end{aligned}$$

Hence by Lemma 1, g can have poles only at $\frac{1}{24}$, $\frac{5}{24}$, $\frac{7}{24}$, $\frac{11}{24}$, $\frac{1}{12}$, $\frac{5}{12}$ under $\Gamma_1(24)$-equivalence. At $\frac{1}{24}$, $\frac{5}{24}$, $\frac{7}{24}$ and $\frac{11}{24}$, it is easy to check that g is holomorphic. For example, at $\frac{5}{24}$,

$$\begin{aligned} g|_{\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)} &= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)|_{\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)}\right) \\ &= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)|_{\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix} 34 & 7 \\ 48 & 10 \end{smallmatrix}\right)}\right) \\ &= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)|_{\left(\begin{smallmatrix} 5 & 1 \\ 24 & 5 \end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix} 17 & 12 \\ 24 & 17 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & -1 \\ 0 & 2 \end{smallmatrix}\right)}\right) \\ &\in O(1) \quad \text{since } \frac{5}{24}, \frac{17}{24} \notin \Gamma_1(12)\infty. \end{aligned}$$

Now it remains to show that $\left(t|_{U_{2^k}}\right)_\chi - g$ has no poles at the cusps equivalent to $\frac{1}{12}$, $\frac{5}{12}$ under $\Gamma_1(24)$. At $\frac{1}{12}$,

$$\begin{aligned} &\left(t|_{U_{2^k}}\right)_\chi\Big|_{\left(\begin{smallmatrix} 1 & 0 \\ 12 & 1 \end{smallmatrix}\right)} \\ &= \frac{1}{\sqrt{-4}}\left(t|_{U_{2^k}\left(\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 12 & 1 \end{smallmatrix}\right)} - t|_{U_{2^k}\left(\begin{smallmatrix} 4 & 3 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 12 & 1 \end{smallmatrix}\right)}\right) \\ &= \frac{1}{\sqrt{-4}}\left(t|_{U_{2^k}\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 16 & 1 \\ 0 & 1 \end{smallmatrix}\right)} - t|_{U_{2^k}\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 8 & 1 \\ 0 & 2 \end{smallmatrix}\right)}\right) \\ &= \frac{1}{\sqrt{-4}}\left(t|_{U_{2^k}\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 16 & 1 \\ 0 & 1 \end{smallmatrix}\right)} - t|_{U_{2^k}\left(\begin{smallmatrix} 11 & -1 \\ 12 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 8 & 1 \\ 0 & 2 \end{smallmatrix}\right)}\right) \\ &= \frac{1}{\sqrt{-4}}\left(t|_{U_{2^k}\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)}(16z+1) - t|_{U_{2^k}\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)}\left(4z+\frac{1}{2}\right)\right). \end{aligned} \tag{25}$$

By (25) and Lemma 17, we have the following:

If $k=1$,
$$\left(t|_{U_2}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} = \frac{1}{\sqrt{-4}}\cdot\left(-\frac{1}{2}e^{-\pi i\left(4z+\frac{1}{2}\right)}+O(1)\right) = \frac{1}{4}q^{-2}+O(1).$$

If $k=2$,
$$\left(t|_{U_{2^2}}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} = \frac{1}{\sqrt{-4}}\cdot\left(\frac{1}{4}e^{-\frac{\pi i}{2}(16z+1)}+O(1)\right) = -\frac{1}{8}q^{-4}+O(1).$$

If $k=3$,
$$\left(t|_{U_{2^3}}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} = \frac{1}{\sqrt{-4}}\cdot\left(-\frac{i}{8}e^{-\pi i(16z+1)}+O(1)\right) = \frac{1}{16}q^{-8}+O(1).$$

If $k\geq 4$,
$$\begin{aligned}&\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)}\\ &= \frac{1}{\sqrt{-4}}\cdot\left((-1)^{k-1}\cdot\frac{i}{2^k}\cdot e^{-2\pi i\cdot 2^{k-4}\cdot(16z+1)}+O(1)\right)\\ &= (-1)^{k-1}\cdot\frac{1}{2^{k+1}}\cdot q^{-2^k}+O(1).\end{aligned}$$

Observe that the identities for $k=1,2$ and 3 are the same as the last one when $k\geq 4$. Hence we conclude that for all $k\geq 1$,
$$\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} = (-1)^{k-1}\cdot\frac{1}{2^{k+1}}\cdot q^{-2^k}+O(1).$$

On the other hand,
$$\begin{aligned} g|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} &= (-1)^{k-1}\cdot\frac{1}{2}\cdot\left(X_{2^k}(t)|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix}2&1\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)}\right)\\ &= (-1)^{k-1}\cdot\frac{1}{2}\cdot\left(X_{2^k}(t) - X_{2^k}(t)|_{\left(\begin{smallmatrix}7&4\\12&7\end{smallmatrix}\right)\left(\begin{smallmatrix}2&-1\\0&2\end{smallmatrix}\right)}\right)\\ &= (-1)^{k-1}\cdot\frac{1}{2^{k+1}}\cdot q^{-2^k}+O(1).\end{aligned}$$

Thus
$$\left(\left(t|_{U_{2^k}}\right)_\chi - g\right)|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} \in O(1).$$

At $\frac{5}{12}$, we see that $\left(\begin{smallmatrix}5&2\\12&5\end{smallmatrix}\right) = \left(\begin{smallmatrix}-19&2\\-48&5\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)$ sends ∞ to $\frac{5}{12}$. Then
$$\begin{aligned}&\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix}5&2\\12&5\end{smallmatrix}\right)}\\ &= \left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix}-19&2\\-48&5\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)}\\ &= (-1)^{\frac{-48}{24}}\cdot\left(\frac{3}{-19+\frac{-48}{4}}\right)\cdot\left(t|_{U_{2^k}}\right)_\chi|_{\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)} \text{ by Lemma 16}\end{aligned}$$

$$= (-1) \cdot \left(t|_{U_{2^k}}\right)_\chi |_{\left(\begin{smallmatrix} 1 & 0 \\ 12 & 1 \end{smallmatrix}\right)}$$
$$= (-1)^k \cdot \frac{1}{2^{k+1}} \cdot q^{-2^k} + O(1).$$

And

$$\begin{aligned} g|_{\left(\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}\right)} &= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)|_{\left(\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}\right)} \right) \\ &= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left(X_{2^k}(t)|_{\left(\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}\right)} - X_{2^k}(t)|_{\left(\begin{smallmatrix} 11 & -1 \\ 12 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)} \right) \\ &= (-1)^k \cdot \frac{1}{2} \cdot X_{2^k}(t)|_{\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)} + O(1) \\ &= (-1)^k \cdot \frac{1}{2^{k+1}} \cdot q^{-2^k} + O(1). \end{aligned}$$

This implies that

$$\left(\left(t|_{U_{2^k}}\right)_\chi - g\right)|_{\left(\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}\right)} \in O(1),$$

from which the theorem follows. □

Now, we are ready to show periodically vanishing property of $\mathcal{N}(j_{1,12})$.

Corollary 19. *As before we let t be the Hauptmodul of $\Gamma_1(12)$ and write $X_n(t) = \frac{1}{n}q^{-n} + \sum_{m\geq 1} H_{m,n}q^m$. Then we have*
(i) $H_{m,2^k} = (-1)^{k-1}\left(\frac{-1}{m}\right) H_{2^k m,1}$ *for odd m.*
(ii) $H_m = 0$ *whenever $m \equiv 4 \mod 6$, and $m = 5$.*

Proof. First if we compare the coefficients of q^m-terms on both sides of the identity in Theorem 18, we get (i). We see from the Appendix, Table 4 that $H_5 = 0$. On the other hand, by the super-replication formula (Corollary 11) it follows that for m relatively prime to 12,

$$H_{m,2^k} = H_{2^k,m} = \begin{cases} H_{2^k m,1}, & \text{if } m \equiv \pm 1 \mod 12 \\ -H_{2^k m,1}, & \text{if } m \equiv \pm 5 \mod 12 \end{cases}$$

because $h_{2^k m,1} = 0$ in this case ([22], Corollary 3.1). Then $H_{2^k m,1} = 0$ when k is odd and $m \equiv 5 \mod 6$, or k is even and $m \equiv 1 \mod 6$. It is easy to see that

$$\{2^k m \mid k, m \geq 1,\ k \text{ odd},\ m \equiv 5 \mod 6\} \cup \{2^k m \mid k, m \geq 1,\ k \text{ even},$$
$$m \equiv 1 \mod 6\} = \{l \in \mathbb{Z} \mid l \equiv 4 \mod 6\}.$$

This proves (ii). □

Appendix. Fourier coefficients of the Hauptmodul $\mathcal{N}(j_{1,N})$

We shall make use of the following modular forms to construct $j_{1,N}$. For $z \in \mathfrak{H}$,

- $\eta(z)$ the Dedekind eta function
- $G_2(z)$ Eisenstein series of weight 2

- $G_2^{(p)}(z) = G_2(z) - pG_2(pz)$ for each prime p
- $E_2(z) = G_2(z)/(2\zeta(2))$ normalized Eisenstein series of weight 2
- $E_2^{(p)}(z) = E_2(z) - pE_2(pz)$ for each prime p
- $\mathcal{P}_{N,\mathbf{a}}(z) = \mathcal{P}_{L_z}\left(\frac{a_1z+a_2}{N}\right)$ N-th division value of $\mathcal{P}$, where $\mathbf{a} = (a_1, a_2)$, $L_z = \mathbb{Z}z + \mathbb{Z}$ and $\mathcal{P}_L(\tau)$ is the Weierstrass $\mathcal{P}$-function (relative to a lattice L)

Now we get the following tables due to [16]-[18]:

Table 3. Hauptmoduln $\mathcal{N}(j_{1,N})$

N	$j_{1,N}$	$\mathcal{N}(j_{1,N})$
1	$j(z)$	$j(z) - 744$
2	$\theta_2(z)^8/\theta_4(2z)^8$	$256/j_{1,2} + 24$
3	$E_4(z)/E_4(3z)$	$240/(j_{1,3} - 1) + 9$
4	$\theta_2(2z)^4/\theta_3(2z)^4$	$16/j_{1,4} - 8$
5	$\frac{4\eta(z)^5/\eta(5z)+E_2^{(5)}(z)}{\eta(5z)^5/\eta(z)}$	$-8/(j_{1,5} + 44) - 5$
6	$\frac{G_2^{(2)}(z)-G_2^{(2)}(3z)}{2G_2^{(2)}(z)-G_2^{(3)}(z)}$	$2/(j_{1,6} - 1) - 1$
7	$\frac{\mathcal{P}_{7,(1,0)}(7z)-\mathcal{P}_{7,(2,0)}(7z)}{\mathcal{P}_{7,(1,0)}(7z)-\mathcal{P}_{7,(4,0)}(7z)}$	$-1/(j_{1,7} - 1) - 3$
8	$\theta_3(2z)/\theta_3(4z)$	$2/(j_{1,8} - 1) - 1$
9	$\frac{\mathcal{P}_{9,(1,0)}(9z)-\mathcal{P}_{9,(2,0)}(9z)}{\mathcal{P}_{9,(1,0)}(9z)-\mathcal{P}_{9,(4,0)}(9z)}$	$-1/(j_{1,9} - 1) - 2$
10	$\frac{\mathcal{P}_{10,(1,0)}(10z)-\mathcal{P}_{10,(2,0)}(10z)}{\mathcal{P}_{10,(1,0)}(10z)-\mathcal{P}_{10,(4,0)}(10z)}$	$-1/(j_{1,10} - 1) - 2$
12	$\theta_3(2z)/\theta_3(6z)$	$2/(j_{1,12} - 1)$

When $N = 1, 2, 3, 4, 6$, $\mathcal{N}(j_{1,N})$ becomes a Thompson series T_g with $\Gamma_g = \Gamma_0(N)$. Hence, if $N = 4$, $\mathcal{N}(j_{1,4})$ has periodically vanishing property by Theorem 13-(i). Otherwise, the Fourier coefficients of $\mathcal{N}(j_{1,N})$ do not vanish (see [24], Table 1). Therefore, we consider only the following cases N for which $\overline{\Gamma}_1(N) \neq \overline{\Gamma}_0(N)$.

Table 4. Fourier coefficients H_m of $\mathcal{N}(j_{1,N})$ for $1 \leq m \leq 60$

	$\mathcal{N}(j_{1,5})$	$\mathcal{N}(j_{1,7})$	$\mathcal{N}(j_{1,8})$	$\mathcal{N}(j_{1,9})$	$\mathcal{N}(j_{1,10})$	$\mathcal{N}(j_{1,12})$
H_1	10	4	3	2	2	1
H_2	5	3	2	2	1	1
H_3	-15	0	1	1	1	1
H_4	-24	-5	-2	-1	0	0
H_5	15	-7	-4	-2	-1	0
H_6	70	-2	-4	-3	-2	-1
H_7	30	8	0	-2	-2	-1
H_8	-125	16	6	1	-1	-1
H_9	-175	12	9	4	1	-1
H_{10}	95	-7	8	6	3	0

	$\mathcal{N}(j_{1,5})$	$\mathcal{N}(j_{1,7})$	$\mathcal{N}(j_{1,8})$	$\mathcal{N}(j_{1,9})$	$\mathcal{N}(j_{1,10})$	$\mathcal{N}(j_{1,12})$
H_{11}	420	-29	-1	5	4	1
H_{12}	180	-35	-12	1	4	2
H_{13}	-615	-10	-20	-5	1	2
H_{14}	-826	37	-16	-11	-2	2
H_{15}	410	70	1	-12	-6	1
H_{16}	1760	53	22	-7	-8	0
H_{17}	705	-21	38	3	-7	-2
H_{18}	-2415	-106	30	15	-3	-3
H_{19}	-3100	-126	1	22	4	-4
H_{20}	1530	-38	-40	19	10	-4
H_{21}	6270	119	-64	5	14	-2
H_{22}	2460	226	-52	-15	12	0
H_{23}	-8090	164	-2	-32	6	3
H_{24}	-10174	-70	68	-36	-6	5
H_{25}	4840	-326	107	-22	-16	7
H_{26}	19570	-378	88	8	-22	6
H_{27}	7500	-106	-2	40	-20	4
H_{28}	-24360	353	-112	58	-8	0
H_{29}	-30024	652	-180	50	8	-4
H_{30}	14130	469	-144	12	26	-8
H_{31}	55970	-189	3	-41	34	-10
H_{32}	21155	-885	182	-84	31	-9
H_{33}	-67380	-1015	292	-93	12	-6
H_{34}	-81926	-290	228	-54	-14	0
H_{35}	37895	910	4	22	-41	6
H_{36}	148410	1664	-286	103	-54	12
H_{37}	55305	1179	-452	148	-47	14
H_{38}	-174500	-483	-356	124	-20	14
H_{39}	-209577	-2205	-4	32	23	8
H_{40}	96025	-2492	440	-96	61	0
H_{41}	371620	-692	686	-200	84	-10
H_{42}	137160	2212	544	-219	72	-18
H_{43}	-427665	3998	-5	-128	31	-22
H_{44}	-508800	2809	-668	46	-32	-20
H_{45}	230670	-1120	-1044	231	-90	-12
H_{46}	885070	-5119	-816	330	-122	0
H_{47}	323605	-5754	5	275	-107	15
H_{48}	-1001340	-1598	996	67	-44	26
H_{49}	-1181123	4992	1563	-216	45	33
H_{50}	531545	8968	1210	-439	133	29

	$\mathcal{N}(j_{1,5})$	$\mathcal{N}(j_{1,7})$	$\mathcal{N}(j_{1,8})$	$\mathcal{N}(j_{1,9})$	$\mathcal{N}(j_{1,10})$	$\mathcal{N}(j_{1,12})$
H_{51}	2022670	6251	6	-477	174	19
H_{52}	734130	-2506	-1464	-275	154	0
H_{53}	-2253515	-11285	-2276	107	61	-20
H_{54}	-2639348	-12579	-1768	501	-68	-37
H_{55}	1178880	-3455	-8	708	-192	-45
H_{56}	4456650	10812	2128	590	-254	-42
H_{57}	1606500	19278	3284	146	-220	-26
H_{58}	-4901250	13362	2552	-447	-90	0
H_{59}	-5703676	-5278	-9	-911	100	27
H_{60}	2532720	-23765	-3056	-987	272	52

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