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EXTINCTION AND PERMANENCE OF A KIND OF PEST-PREDATOR MODELS WITH IMPULSIVE EFFECT AND INFINITE DELAY

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Abstract. In this paper, a kind of pest-predator model with impulsive effect and infinite delay is considered by the method of chain transform. By using Floquet's theorem, it is shown that there exists a globally asymptotically stable periodic pest eradication solution when the impulsive period is less than or equal to some critical value which is a directly proportional function with respect to the population of release. Furthermore, it is proved that the system is permanent if the impulsive period is larger than some critical value. Finally, the results of the corresponding systems are compared, those results obtained in this paper are confirmed by numerical simulation.

1. Introduction

Biological control and chemical control are two methods of pest control. There are many articles on the pest-control system (see [3]$\sim$[7]) which have obtained many excellent results. In biological control, harvesting pest and releasing predator (such as natural enemies) are usually used in the biological pest control. Xianning Liu and Lansun Chen (see [3]) develop the Voltka-Volterra predator-prey system

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[-r_2 + a_{21}x_1(t)], \end{cases} \tag{1.1}$$

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by introducing an impulsive constant planting effect periodically for the predator, that is,

$$
(1.2) \qquad \begin{cases} \left.\begin{array}{l} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[-r_2 + a_{21}x_1(t)], \end{array}\right\} & t \neq n\tau, \\ \left.\begin{array}{l} \Delta x_1(t) = 0, \\ \Delta x_2(t) = b, \end{array}\right\} & t = n\tau, \end{cases}
$$

where $x_1(t)$, $x_2(t)$ denote the densities of the pest and predator (natural enemies) at the time t respectively, $r_1 > 0$ is the intrinsic growth rate of pest, $r_2 > 0$ is the death rate of the predator, a_{11} is the coefficient of intraspecific competition or density dependence, $a_{12} > 0$ is the per-capita rate of predation of the predator, a_{21} is the product of the per-capita rate of predation and the rate of the converting pest into predator, τ is the period of the impulsive effect, b is the population of the predator released.

The dynamics of system (1.1) are very simple, $x_2(t)$ goes extinct and $x_1(t)$ tends to r_1/a_{11}, the capacity of the prey, if there is no positive equilibrium ($r_1/a_{11} \le r_2/a_{21}$) or there is a positive equilibrium which is global asymptotically stable ($r_1/a_{11} > r_2/a_{21}$). In each case the prey cannot be extinct. That is why the classical approach of this kind in pest control is not so effective. Usually biological pest control requires the introduction of a predator decreasing the pest population to an acceptable level. It provides only a short term results as after some time this kind of predator-prey system will reach its coexisting equilibrium no matter how large the initial density of the predator is. System (1.2) suggests an approach in pest control by adding some suitable amount of predator after some fixed period of time which proves more effective.

Xianning Liu and Lansun Chen study the extinction of the prey in the impulsive system (1.2) and the permanent property of (1.2) in case that the impulsive invasion of predator be disaster to the prey which need protection. The conditions of the extinction and permanence of (1.2) are given in [3]. The solution of (1.2) $x_1(t) \to 0$, $x_2(t) \to x_2^*(t)$ as $t \to \infty$ provided one of the following conditions (A1) and (A2) is satisfied: (A1) $b > r_1 r_2 \tau / a_{12}$, (A2) $b = r_1 r_2 \tau / a_{12}$ and $x_{02} \ge \dfrac{b}{1 - \exp(-r_2\tau)}$, where $x_{02} = \dfrac{b\exp(-r_2(t - n\tau))}{1 - \exp(-r_2\tau)}$, $t \in (n\tau, (n+1)\tau]$, $n \in N$, $x_2^*(0^+) = \dfrac{b}{1 - \exp(-r_2\tau)}$. System (1.2) is permanent if $b < r_1 r_2 \tau / a_{12}$.

Another important method for pest control is chemical control (such as pesticide spray). Pesticides are useful because they quickly kill a significant portion of a pest population and they sometimes provide the only feasible method for preventing economic loss. During investigating the pest-predator models under pulse use of insecticide, Zhonghua Lu and Lansun Chen (see [4]) develop (1.1) by harvesting the pest, then (1.1) becomes

$$
(1.3) \qquad \begin{cases} \left.\begin{array}{l} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[-r_2 + a_{21}x_1(t)], \end{array}\right\} & t \neq n\tau, \\ \left.\begin{array}{l} \Delta x_1(t) = -E_1 x_1(t), \\ \Delta x_2(t) = 0, \end{array}\right\} & t = n\tau, \end{cases}
$$

where E_1 represents the fraction of population that dies due to the pesticide. If $E_1 \geq E_1^* = 1 - e^{-r_1\tau}$, then (0,0) equilibrium of system (1.3) is stable and there does not exist boundary nontrivial periodic solution. If $r_1/a_{11} \leq r_2/a_{21}$ and $E_1 < E_1^*$, or $r_1/a_{11} > r_2/a_{21}$ and $E_1^{**} < E_1 < E_1^*$, $E_1^{**} = 1 - \exp(\frac{a_{11}r_2 - a_{21}r_1}{a_{21}}\tau)$, then there exists a unique periodic predator-free solution $(\widetilde{x}_1(t), 0)$, which is asymptotically stable.

However, pesticide pollution is not only recognized as major health hazard to human beings and to natural enemies but also is resited by pest. And the latter results in the use of high rates and more toxic materials to combat pests. Beneficial insects are often susceptible to chemical insecticides applied for the target pest. One of the side effects of the high rates of pesticide use is that natural enemies and other small animals that might otherwise feed on pests are killed and pests population burst out once again after beneficial insects being killed (see [5]).

The main purpose of this paper is to construct a simple mathematical model of a system of biological control by periodic release of natural enemies and investigate the dynamics of this system. For simplicity, we denote $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$, $\Delta x_1 = x_1(n\tau^+) - x_1(n\tau)$, $\Delta x_2 = x_2(n\tau^+) - x_2(n\tau)$, $\Delta x_3 = x_3(n\tau^+) - x_3(n\tau)$. We suggest an impulsive system including one prey and one predator to model the process of periodic release of natural enemies at fixed moments and consider a kind of pest-predator models with impulsive effect and infinite delay described by the system as follows

$$
(1.4) \qquad \begin{cases} \left.\begin{array}{l} \dot{x}_1 = x_1[r_1 - a_{11}x_1 - a_{12}x_2], \\ \dot{x}_2 = x_2[-r_2 + a_{21}\int_{-\infty}^{t} F(t-s)x_1(s)ds], \end{array}\right\} & t \neq n\tau, \\ \left.\begin{array}{l} \Delta x_1 = 0, \\ \Delta x_2 = p \end{array}\right\} & t = n\tau, \end{cases}
$$

where τ is the period of the impulsive release, $p > 0$ is the amount of the predator released, $a > 0$. The exponential weight function satisfies

$$
\int_{-\infty}^{t} F(t-s)ds = \lim_{A \to -\infty} \int_{A}^{t} ae^{-a(t-s)}ds = 1,
$$

Let $F(t) = ae^{-at}$, $x_3(t) = \int_{-\infty}^{t} F(t-s)x_1(s)ds$, then (1.4) becomes

$$
\begin{cases}
\left.\begin{array}{l}
\dot{x}_1 = x_1[r_1 - a_{11}x_1 - a_{12}x_2], \\
\dot{x}_2 = x_2[-r_2 + a_{21}x_3], \\
\dot{x}_3 = a(x_1 - x_3),
\end{array}\right\} & t \neq n\tau, \\
\left.\begin{array}{l}
\Delta x_1 = x_1(n\tau^+) - x_1(n\tau) = 0, \\
\Delta x_2 = x_2(n\tau^+) - x_2(n\tau) = p, \\
\Delta x_3 = x_3(n\tau^+) - x_3(n\tau) = 0,
\end{array}\right\} & t = n\tau.
\end{cases}
\tag{1.5}
$$

Because

$$\int_{-\infty}^{t} F(t-s)ds = \lim_{A \to -\infty} \int_{A}^{t} ae^{-a(t-s)}ds = 1,$$

$\int_{-\infty}^{t} F(t-s)x_1(s)ds$ is convergent,

$$
\begin{aligned}
\Delta x_3(t) &= \int_{-\infty}^{n\tau^+} F(t-s)x_1(s)ds - \int_{-\infty}^{n\tau} F(t-s)x_1(s)ds \\
&= ae^{-an\tau} \int_{n\tau}^{n\tau^+} F(t-s)x_1(s)ds = 0.
\end{aligned}
$$

From discussion above, the property of (1.4) can be obtained by investigating (1.5), so in the following we will consider (1.5) mainly.

2. Periodic pest-free solution and its stability property

In the absence of the pest x_1, that is $x_1 = 0, x_3 = 0$, system (1.5) reduces to

$$
\begin{cases}
\dot{x}_2 = -r_2 x_2, & t \neq n\tau, \\
\Delta x_2 = p, & t = n\tau.
\end{cases}
\tag{2.1}
$$

Integrating and solving the first equation of the system (2.1) at the interval $(n\tau, (n+1)\tau]$, we have

$$x_2(t) = x_2(n\tau^+)e^{-r_2(t-n\tau)}.$$

By means of the stroboscopic mapping:

$$x_2(n\tau^+)e^{-r_2\tau} + p = x_2(n\tau^+) = x_2((n+1)\tau^+),$$

we can get the fixed point of the mapping:

$$x_2 e^{-r_2\tau} + p = x_2 \Rightarrow \widetilde{x_2}(0^+) = \frac{p}{1 - e^{-r_2\tau}}.$$

So the system (2.1) has a periodic solution which eradicate pest at the interval $(n\tau, (n+1)\tau]$:

$$\widetilde{x_2}(t) = \frac{p}{1 - e^{-r_2\tau}} e^{-r_2(t-n\tau)}.$$

The general solution of the system (1.5) can be obtained by the iterative method from $(n\tau, (n+1)\tau]$ to $(0, \tau]$

$$\begin{aligned} x_2(t) &= x_2(n\tau^+)e^{-r_2(t-n\tau)} = (x_2(n\tau)+p)e^{-r_2(t-n\tau)} \\ &= [x_2(0^+)e^{-r_2 n\tau} + p(e^{-r_2 n\tau} + \cdots + e^{-r_2\tau} + 1)]e^{-r_2(t-n\tau)} \\ &= x_2(0^+)e^{-r_2 t} - \frac{pe^{-r_2 t}}{1-e^{-r_2\tau}} + \frac{p}{1-e^{-r_2\tau}}e^{-r_2(t-n\tau)} \\ &= (x_2(0^+) - \frac{p}{1-e^{-r_2\tau}})e^{-r_2 t} + \widetilde{x_2}(t). \end{aligned}$$

From discussion above, we have

Theorem 2.1. *The system* (2.1) *has a positive periodic solution* $\widetilde{x_2}(t)$*, and* $x_2(t) \to \widetilde{x_2}(t), t \to \infty$ *for all solution which satisfy the initiate condition* $x_2(0^+) = \frac{p}{1-e^{-r_2\tau}} \geq 0$. *The system* (1.5) *has a periodic solution which eradicate pest* $(0, \widetilde{x_2}(t), 0) = (0, \frac{pe^{-r_2(t-n\tau)}}{1-e^{-r_2\tau}}, 0)$.

Lemma 2.1. [2] *Assume that* $m \in PC[R_+, R]$ *with points of discontinuity at* $t = t_k$ *and is left continuous at* $t = t_k$, $k = 1, 2, \ldots,$ *and*

$$\begin{cases} D_- m(t) \leq g(t, m(t)), & t \neq t_k, \ k = 1, 2, \ldots, \\ m(t_k^+) \leq \psi_k(m(t_k)), & t = t_k, \ k = 1, 2, \ldots, \end{cases}$$

where $g \in C[R_+ \times R_+, R]$, $\psi_k \in C[R, R]$ *and* $\psi_k(u)$ *is nondecreasing in* u *for each* $k = 1, 2, \ldots$. *Let* $r(t)$ *be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} \dot{u} = g(t, u), & t \neq t_k, \ k = 1, 2, \ldots, \\ u(t_k^+) = \psi_k(u(t_k)) \geq 0, & t = t_k, \ t_k > t_0, \ k = 1, 2, \ldots, \\ u(t_0^+) = u_0, \end{cases}$$

existing on $[t_0, \infty)$. *Then* $m(t_0^+) \leq u_0$ *implies* $m(t) \leq r(t)$, $t \geq t_0$.

Theorem 2.2. *Let* (x_1, x_2, x_3) *be any solution of system* (1.5). *Then* $(0, \widetilde{x_2}(t), 0)$ *is globally asymptotically stable* $\tau \leq \frac{pa_{12}}{r_1 r_2}$, *where* $\widetilde{x_2}(t) = \frac{p\exp(-r_2(t-n\tau))}{1-\exp(-r_2\tau)}$, $t \in (n\tau, (n+1)\tau]$, $n \in N$, $\widetilde{x_2}(0^+) = \frac{p}{1-\exp(-r_2\tau)}$.

Proof. By means of the theory of Floquet and $x_1 = y_1(t)$, $x_2 = y_2(t) + \widetilde{x_2}(t)$, $x_3 = y_3(t)$, we linearize the system (1.5) and obtain

$$(2.2) \qquad \begin{cases} \left.\begin{aligned} &\dot{y}_1(t) = (r_1 - a_{12}\widetilde{x_2}(t))y_1, \\ &\dot{y}_2(t) = -r_2 y_2 + a_{21}\widetilde{x_2}(t)y_3(t), \\ &\dot{y}_3(t) = a(y_1(t) - y_3(t)), \end{aligned}\right\} & t \neq n\tau, \\ \left.\begin{aligned} &\Delta y_1 = 0, \\ &\Delta y_2 = 0, \\ &\Delta y_3 = 0, \end{aligned}\right\} & t = n\tau. \end{cases}$$

Let $\Phi(t)$ be the fundamental matrix of (2.2), then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t),$$

where

$$A(t) = \begin{pmatrix} r_1 - a_{12}\widetilde{x_2}(t) & 0 & 0 \\ 0 & -r_2 & a_{21}\widetilde{x_2}(t) \\ a & 0 & -a \end{pmatrix}.$$

From the fourth to the sixth equations, we can obtain

$$\begin{pmatrix} y_1(n\tau^+) \\ y_2(n\tau^+) \\ y_3(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(n\tau) \\ y_2(n\tau) \\ y_3(n\tau) \end{pmatrix}.$$

The monodromy matrix of system (2.2) is:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(\tau),$$

$$\Phi(\tau) = \Phi(0)\exp(\int_0^\tau A(t)dt) = \Phi(0)\exp(\overline{A}),$$

where $\Phi(0) = E$, let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of matric $\overline{A}$,

$$\overline{\lambda_1} = \int_0^\tau (r_1 - a_{12}\widetilde{x_2}(t))dt, \quad \overline{\lambda_2} = -r_2\tau, \quad \overline{\lambda_3} = -a\tau,$$

Therefore, the eigenvalues of M are $\lambda_1 = e^{\overline{\lambda_1}}, \lambda_2 = e^{\overline{\lambda_2}}, \lambda_3 = e^{\overline{\lambda_3}}$ respectively.

It is clear that $\lambda_2,\ \lambda_3 < 1$. Our following purpose is to prove $\lambda_1 \leq 1$, that is, $\int_0^\tau (r_1 - a_{12}\widetilde{x_2}(t))dt \leq 0$.

$$\begin{aligned} \textstyle\int_0^\tau (r_1 - a_{12}\widetilde{x_2}(t))dt &= r_1\tau - a_{12}\textstyle\int_0^\tau \widetilde{x_2}(t)dt \\ &= r_1\tau - a_{12}\frac{p}{1-e^{-r_2\tau}}\int_0^\tau e^{-r_2 t}dt \\ &= r_1\tau - a_{12}\frac{p}{1-e^{-r_2\tau}}(-\frac{1}{r_2})(e^{-r_2\tau}-1) \\ &= r_1\tau - \frac{a_{12}p}{r_2}. \end{aligned}$$

If $r_1\tau - \frac{a_{12}p}{r_2} \leq 0,\ \tau \leq \frac{a_{12}p}{r_1 r_2}$, $\lambda_1 \leq 1$, the periodic solution of system (1.5) is locally stable.

In the following, we prove the global attractivity. Noting that $\dot{x}_2(t) \geq -r_2 x_2(t)$, we consider the impulse differential equation

$$\begin{cases} \dot{y}(t) = -r_2 y(t), & t \neq n\tau, \\ \Delta y(t) = y(n\tau^+) - y(n\tau) = p, & t = n\tau, \\ y(0^+) = x_2(0^+) \geq 0. \end{cases}$$

From Lemma 2.1, we have $x_2(t) \geq y(t),\ y(t) \to \widetilde{x_2}(t),\ t \to \infty$. Hence

$$x_2(t) \geq y(t) > \widetilde{x_2}(t) - \varepsilon, \tag{2.3}$$

(2.3) holds for all t large enough. For simplification we may assume (2.3) holds for all $t \geq 0$.

From the first equation of system (1.5), we have

$$\dot{x}_1(t) \leq x_1(t)(r_1 - a_{12}(\widetilde{x_2}(t) - \varepsilon)). \tag{2.4}$$

Integrate (2.4) in $(n\tau, (n+1)\tau]$

$$x_1((n+1)\tau) \leq x_1(n\tau)\exp((r_1 + a_{12}\varepsilon)\tau - \frac{a_{12}p}{r_2}).$$

Let $\sigma = (r_1 + a_{12}\varepsilon)\tau - \dfrac{a_{12}p}{r_2} < 0$, we can choose $\varepsilon > 0$ such that $\sigma < 0$, which yields

$$x_1(n\tau) \leq x_1(0^+)\exp(n\sigma),$$

Thus $x_1(n\tau) \to 0$ as $n \to \infty$, therefore $x_1(t) \to 0,\ t \to \infty$ since $0 < x_1(t) \leq x_1(n\tau)\exp(r_1\tau)$.

For x_3: Since $x_1(t) \leq x_1(0^+)\exp(n\sigma) = g(n)$, and $g(n) \to 0$ when $n \to \infty$, then $\dot{x}_3 \leq a(g(n) - x_3)$,

$$\begin{array}{rcl} x_3(t) & \leq & g(n) - (g(n) - x_3(n\tau))e^{-a(t-n\tau)} \\ & = & g(n)(1 - e^{-a(t-n\tau)}) + x_3(n\tau)e^{-a(t-n\tau)}, \\ x_3((n+1)\tau) & \leq & g(n)(1 - e^{-a\tau}) + x_3(n\tau)e^{-a\tau} \\ & \leq & x_1(0^+)\exp(n\sigma)(1 - e^{-a\tau})(1 + e^{-a\tau} + e^{-a2\tau} \\ & & + \cdots + e^{-an\tau}) + x_3(0^+)e^{-na\tau} \\ & \leq & x_1(0^+)\exp(n\sigma)(1 - e^{-an\tau}) + x_3(0^+)e^{-na\tau}. \end{array} \tag{2.5}$$

The right side of inequality (2.5) tends to zero as $n \to \infty$, so $x_3(n\tau) \to 0$ as $n \to \infty$. In addition, since $x_3(t) \leq g(n)(1 - e^{-a(t-n\tau)}) + x_3(n\tau)e^{-a(t-n\tau)}$, then $x_3(t) \to 0,\ n \to \infty$.

Next, we prove that

$$x_2(t) \to \widetilde{x_2}(t),\ t \to \infty$$

if $\lim\limits_{t\to\infty} x_1(t) = 0,\ \lim\limits_{t\to\infty} x_3(t) = 0$.

For $0 < \varepsilon_1 < \dfrac{r_2}{a_{21}}$, there exists a $\tau > 0$ such that $0 < x_1(t) < \varepsilon_1,\ 0 < x_3(t) < \varepsilon_1$, when $t > T$

$$-r_2 x_2(t) \leq \dot{x}_2(t) \leq x_2(t)(-r_2 + a_{21}\varepsilon_1),$$

$$y_1(t) \leq x_2(t) \leq y_2(t),\ \ y_1(t) \to \widetilde{x_2}(t),\ \ y_2(t) \to \widetilde{y_2}(t),$$

where $y_1(t),\ y_2(t),\ \widetilde{y_2}(t)$ is the solution and periodic solution of the following equation

$$\left\{\begin{array}{ll} \dot{y}_1(t) = -r_2 y_1(t), & t \neq n\tau, \\ \Delta y_1(t) = p, & t = n\tau, \\ y_1(0^+) = x_2(0^+) \geq 0, & \end{array}\right. \text{ and } \left\{\begin{array}{ll} \dot{y}_2(t) = y_2(t)(-r_2 + a_{21}\varepsilon_1), & t \neq n\tau, \\ \Delta y_2(t) = p, & t = n\tau, \\ y_2(0^+) = x_2(0^+) \geq 0, & \end{array}\right.$$

$$\widetilde{y_2}(t) = \frac{p\exp((-r_2 + a_{21}\varepsilon_1)(t - n\tau)}{1 - \exp((-r_2 + a_{21}\varepsilon_1)\tau)}, n\tau < t \leq (n+1)\tau.$$

From $y_1(t) \leq x_2(t) \leq y_2(t),\ y_1(t) \to \widetilde{x_2}(t),\ y_2(t) \to \widetilde{y_2}(t)$, we have

$$\widetilde{x_2}(t) - \varepsilon_2 < x_2(t) \leq \widetilde{y_2}(t) + \varepsilon_2,$$

then $\widetilde{y_2}(t) \to \widetilde{x_2}(t)$, as $\varepsilon_1 \to 0$, therefore $x_2(t) \to \widetilde{x_2}(t)$, $t \to \infty$. The proof is completed. □

3. Permanence of the system (1.5)

Theorem 3.1. *If $r_2 a_{11} - a_{21} r_1 > 0$ holds, there exist constants $M_1, M_2, M_3 > 0$ such that $x_1 \le M_1, x_2 \le M_2, x_3 \le M_3$ for each positive solution $x(t) = (x_1(t), x_2(t), x_3(t))$ of (1.5) with t large enough.*

Proof. From the first equation of system (1.5), we have $\dot{x}_1(t) \le x_1(t)(r_1 - a_{11}x_1(t))$. Considering the system

$$\begin{cases} \dot{y}_3(t) = y_3(t)(r_1 - a_{11}y_3(t)), & t \ne n\tau, \\ \Delta y_3(t) = 0, & t = n\tau, \\ y_3(0^+) = x_2(0^+) \ge 0, \end{cases} \tag{3.1}$$

the solution of (3.1) is

$$y_3(t) = \frac{r_1 y(0^+)}{a_{11} y_3(0^+) + [r_1 - a_{11} y_3(0^+)]e^{-r_1 t}},$$

$y_3(t) \to \dfrac{r_1}{a_{11}} = M_1$ as $t \to \infty$, therefore there exists a $T_1 > 0$, $x_1(t) \le y_3(t) \le M_1$ as $t > T_1$. Furthermore, from

$$x_3(t) = \int_{-\infty}^{t} ae^{-a(t-s)} x_1(s) ds \le M_1 \int_{-\infty}^{t} ae^{-a(t-s)} ds = M_1,$$

we can get $x_3(t) \le M_1 = M_3$.

From the second equation of system (1.5), we have

$$\dot{x}_2(t) \le x_2(t)[-r_2 + a_{21}M_1].$$

If $r_2 a_{11} - a_{21} r_1 > 0$ holds, then $\delta := -r_2 + a_{21}M_1 < 0$, so we have

$$x_2(t) \le (x_2(0^+) - \frac{p}{1 - e^{\delta\tau}})e^{\delta t} + \widetilde{x_2}(t).$$

there exists a T_2, when $t > T_2$,

$$x_2(t) \le \widetilde{x_2}(t) + \varepsilon \le \frac{p}{1 - \exp(-r_2\tau)} + \varepsilon = M_2$$

for any $\varepsilon > 0$, that is $x_2(t) \le M_2$, therefore $T_3 = \max\{T_1, T_2\}$, when $t > T_3$, $x_1(t) \le M_1$, $x_3(t) \le M_1$, $x_2(t) \le M_2$. The proof is completed. □

Theorem 3.2. *The system* (1.5) *is permanent if* $\tau > \dfrac{a_{12}p}{r_1 r_2}$.

Proof. Suppose $x(t)$ is a solution of (1.5) with $x(0) > 0$. From Theorem 3.1 we may assume $\dot{x}_2(t) \le M_2$. Noting that $\dot{x}_1(t) \le x_1(t)(r_1 - a_{11}x_1(t))$, Considering (3.1) and the comparison theorem, we have $x_1(t) \le y_3(t) \le M_1 = \dfrac{r_1}{a_{11}}$, thus $x_1(t) < \dfrac{r_1}{a_{11}} + \varepsilon_3$, $\varepsilon_3 > 0$. Without loss of generality, we may assume $x_1(t) <$

$\frac{r_1}{a_{11}} + \varepsilon_3$ for $t > 0$. Similarly, we can assume $x_3(t) < M_1 + \varepsilon_3 = M_3$. Let $m_2 = \frac{pe^{-r_2\tau}}{1-e^{-r_2\tau}} - \varepsilon > 0$, $\varepsilon > 0$. Since $\widetilde{x_2}(t) - \varepsilon \le x_2(t) \le \widetilde{x_2}(t) + \varepsilon$, and $\widetilde{x_2}(t) \ge \frac{pe^{-r_2\tau}}{1-e^{-r_2\tau}}$, $x_2(t) > m_2$.

In the following, we first prove that there exists $\overline{m}_1 > 0$ such that $x_1(t) \ge \overline{m}_1$ for t large enough.

Let $0 < m_1 < \frac{r_2}{a_{21}}$. We will prove that , $x_1(t) < m_1$ doesn't hold for all $t \ge 0$.

Step 1. Choose ε_4 large enough such that $\sigma_1 := (r_1 - a_{11}m_1 - a_{12}\varepsilon_4)\tau - \frac{a_{12}p}{r_2 - a_{21}M_3} > 0$. According to the assumption and the second equation of system (1.5), we get

$$\dot{x}_2(t) \le x_2(t)(-r_2 + a_{21}M_3),$$
$$x_2(t) \le y_4(t), y_4(t) \to \widetilde{y_4}(t), t \to \infty.$$

$y_4(t)$ is the solution of the following equation

$$\begin{cases} \dot{y}_4(t) = y_4(t)(-r_2 + a_{21}M_3), & t \ne n\tau, \\ \Delta y_4(t) = p, & t = n\tau, \\ y_4(0^+) = x_2(0^+) \ge 0. \end{cases} \tag{3.2}$$

The periodic solution of (3.2) is

$$\widetilde{y_4}(t) = \frac{p\exp((-r_2 + a_{21}M_3)(t - n\tau))}{1 - \exp((-r_2 + a_{21}M_3)\tau)}, \quad n\tau < t \le (n+1)\tau.$$

Furthermore, there exist $T_4 > 0$ and $\varepsilon_5 > 0$ such that $x_2(t) \le y_4(t) < \widetilde{y_4}(t) + \varepsilon_5$ for $t \ge T_4$,

$$\dot{x}_1(t) \ge x_1(t)[r_1 - a_{11}m_1 - a_{12}(\widetilde{y_4}(t) + \varepsilon_5)], \quad t \ge \tau_1, \tag{3.3}$$

there exists $N_1 \in Z_+$, $N_1 T \ge T_4$, integrating (3.3) on $(n\tau, (n+1)\tau]$, $n \ge N_1$, we have

$$\begin{aligned} x_1((n+1)\tau) &\ge x_1(n\tau)\exp(\textstyle\int_{n\tau}^{(n+1)\tau}(r_1 - a_{11}m_1 - a_{12}(\widetilde{y_4}(t) + \varepsilon_5)dt) \\ &= x_1(n\tau)\exp(\sigma_1), \end{aligned}$$

therefore

$$x_1((N_1+1)\tau) \ge x_1(N_1\tau)\exp(k\sigma_1) \to \infty.$$

It is contradiction to the boundedness of $x_1(t)$.

Step 2. If $x_1(t) \ge m_1$ for all $t \ge t_1$, then our aim is obtained. Otherwise, $x_1(t) < m_1$ for some $t \ge t_1$. Setting $t^* = \inf_{t \ge t_1}\{x_1(t) < m_1\}$, we have $x_1(t) \ge m_1$, $t \in [t_1, t^*)$, $t^* \in (n_1\tau, (n_1+1)\tau]$, $n_1 \in Z_+$. Since $x_1(t)$ is continuous,it is clear that $x_1(t^*) = m_1$.

Choose $n_2, n_3 \in Z_+$ such that

$$n_2\tau > \frac{1}{-r_2 + a_{21}M_3}\ln\frac{\varepsilon_3}{M_1 + p},$$

$$\exp(\delta(n_2+1)\tau)\exp(n_3\sigma_1) > 1, \quad \delta = r_1 - a_{11}m_1 - a_{12}M_2 > 0.$$

Set $\tau' = n_2\tau + n_3\tau$, we claim that there must exists a $t' \in ((n_1+1)\tau,\ (n_1+1)\tau + \tau']$ such that $x_1(t') \geq m_1$. Otherwise $x_1(t) < m_1,\ t \in ((n_1+1)\tau,\ (n_1+1)\tau+\tau']$. Considering (3.2) with $y_4((n_1+1)\tau^+) = x_2((n_1+1)\tau^+)$, we have

$$\begin{aligned} y_4(t) &= \left(y_4((n_1+1)\tau^+) - \frac{p}{1-\exp((-r_2+a_{21}M_3)\tau)} \right) \\ &\quad \times \exp[(-r_2+a_{21}M_3)(t-(n_1+1)\tau)] + \widetilde{y_4}(t) \end{aligned}$$

for $t \in (n\tau, (n+1)\tau],\ n_1+1 \leq n \leq n_1+1+n_2+n_3$. Then

$$|y_4(t) - \widetilde{y_4}(t)| < (M_2+p)\exp(-r_2+a_{21}M_3)(t-(n_1+1)\tau) < \varepsilon_5$$

and $x_2(t) \leq y_4(t) < \widetilde{y_4}(t) + \varepsilon_5,\ (n_1+1+n_2)\tau \leq t \leq (n_1+1)\tau+\tau'$, which implies that $\dot{x}_1(t) \geq x_1(t)[r_1 - a_{11}m_1 - a_{12}(\widetilde{y_4}(t)+\varepsilon_5)],\ (n_1+1+n_2)\tau \leq t \leq (n_1+1)\tau+\tau'$, that is to say, (3.3) holds.

Similarly, there are two possible cases for

$$x_1((n_1+1+n_2+n_3)\tau) \geq x_1((n_1+1+n_2)\tau)\exp(n_3\sigma_1)$$

on $t \in (t^*, (n_1+1)\tau]$.

Case (a). If $x_1(t) < m_1, t \in (t^*, (n_1+1)\tau]$, then $x_1(t) < m_1$ holds for all $t \in (t^*, (n_1+1+n_2)\tau]$, system (1.5) gives

$$\dot{x}_1(t) \geq x_1(t)(r_1 - a_{11}m_1 - a_{12}M_3) = \delta x_1(t), \tag{3.4}$$

$\delta := r_1 - a_{11}m_1 - a_{12}M_2,\ t \in (t^*, (n_1+1+n_2)\tau]$. Integrating (3.4) on $(t^*, (n_1+1+n_2)\tau]$, we have

$$x_1(n_1+1+n_2)\tau) \geq m_1\exp(\delta(n_2+1)\tau),$$

thus

$$x_1(n_1+1+n_2+n_3)\tau) \geq m_1\exp(\delta(n_2+1)\tau)\exp(n_3\sigma_1) > m_1,$$

which is a contradiction to $x_1(t) < m_1$.

Further let $\bar{t} = \inf_{t>t^*}\{x_1(t) \geq m_1\}$, then $x_1(\bar{t}) = m_1$ and (3.4) holds for $t \in [t^*, \bar{t})$. Integrating (3.4) on $[t^*, \bar{t})$ yields

$$x_1(t) \geq x_1(t^*)\exp(\delta(t-t^*) \geq m_1\exp(\delta(1+n_2+n_3)\tau) = \overline{m}_1.$$

For $t > \bar{t}$, the same argument can be continued since $x_1(\bar{t}) \geq m_1$. Hence $x_1(\bar{t}) \geq \overline{m}_1$ for all $t > t_1$.

Case (b). There exists a $t'' \in (t^*, (n_1+1)\tau]$ such that $x_1(t'') \geq m_1$. Let $\widehat{t} = \inf_{t>t^*}\{x_1(t) \geq m_1\}$, then $x_1(t) < m_1,\ t \in [t^*, \widehat{t})$ and $x_1(\widehat{t}) = m_1$. (3.4) holds for $t \in [t^*, \widehat{t})$.

Integrating (3.4) on $t \in [t^*, \widehat{t})$, we have

$$x_1(t) \geq x_1(t^*)\exp(\delta(t-t^*) \geq m_1\exp(\delta(\tau) > \overline{m}_1.$$

This process can be continued since $x_1(\widehat{t}) \geq m_1$ and we have $x_1(t) \geq \overline{m}_1,\ t \geq t_1$.

Thus in both cases, we conclude $x_1(t) \geq \overline{m}_1$ for all $t \geq t_1$. From $x_3(t) = \int_{-\infty}^{t} ae^{-a(t-s)}x_1(s)ds$ and $x_1(t) \geq \overline{m}_1$ for all $t \geq t_1$ we can get $x_3(t) \leq \overline{m}_1 \int_{-\infty}^{t} ae^{-a(t-s)}ds = \overline{m}_1$. The proof is completed. □

4. Discussion

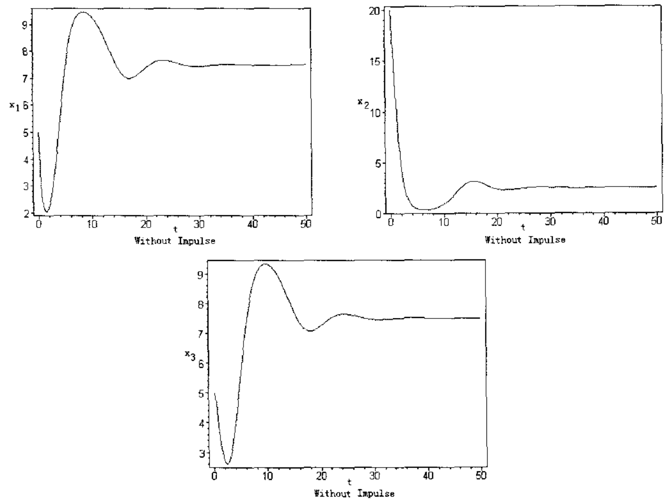

Fig.1 The numerical graph of system (4.1)(without Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 1; \tau > \widehat{T}_1 = 0.666666666667.$

In this paper, we investigate a kind of pest-predator system with infinite delay and periodic constant impulsive effect on predator at fixed moment. We have shown that there exists a globally asymptotically stable pest periodic eradication solution.

If system (1.5) has neither impulsive release nor continuous release, it becomes

$$(4.1) \qquad \begin{cases} \dot{x}_1 = x_1[r_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2 = x_2[-r_2 + a_{21}x_3], \\ \dot{x}_3 = a(x_1 - x_3), \end{cases}$$

which have two equilibria $(0,0,0)$ and $(\frac{r_1}{a_{11}}, 0, \frac{r_1}{a_{11}})$. If $r_2a_{11} - r_2a_{21} > 0$, $(0,0,0)$ is unstable and $(\frac{r_1}{a_{11}}, 0, \frac{r_1}{a_{11}})$ is stable equilibrium, (4.1) also has a unstable equilibrium $(\frac{r_2}{a_{21}}, \frac{r_2a_{11} - r_1a_{21}}{a_{12}a_{21}}, \frac{r_2}{a_{21}})$.

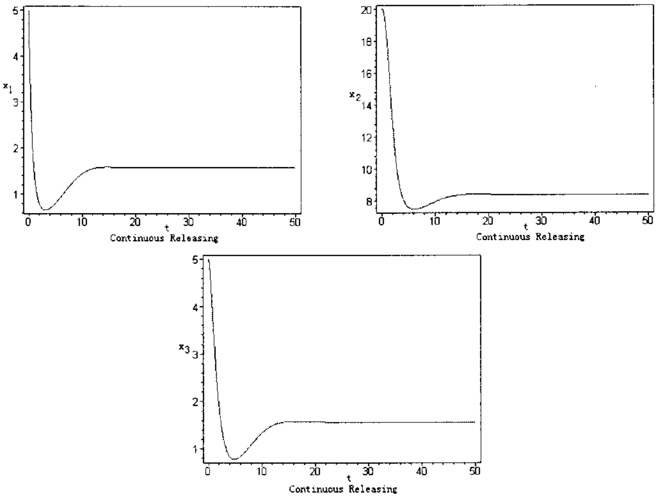

Fig.2 The numerical graph of system (4.2)(Continuous Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 1; \tau > \widehat{T}_1 = 0.666666666667.$

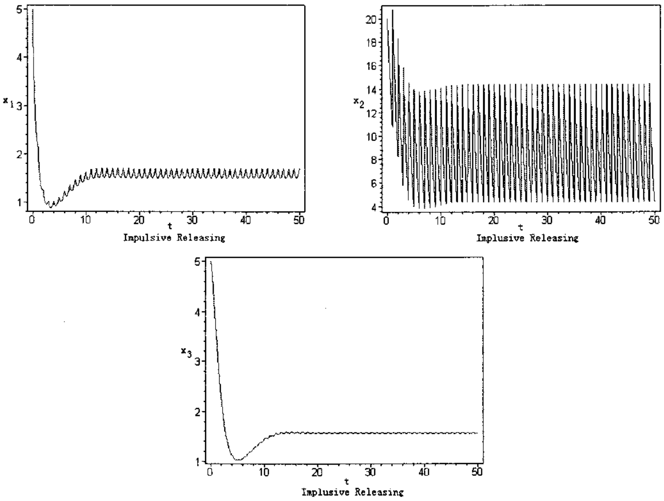

Fig.3 The numerical graph of system (1.5)(Impulse Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 1; \tau > \widehat{T}_1 = 0.666666666667.$

If we replace the pulse release predator (nature enemy) in system with continuous release, the system (1.5) becomes

$$\begin{cases} \dot{x}_1 = x_1[r_1 - a_{11}x_1(t) - a_{12}x_2(t)] \\ \dot{x}_2 = x_2[-r_2 + a_{21}x_3] + \dfrac{p}{\tau} \\ \dot{x}_3 = a(x_1 - x_3) \end{cases} \tag{4.2}$$

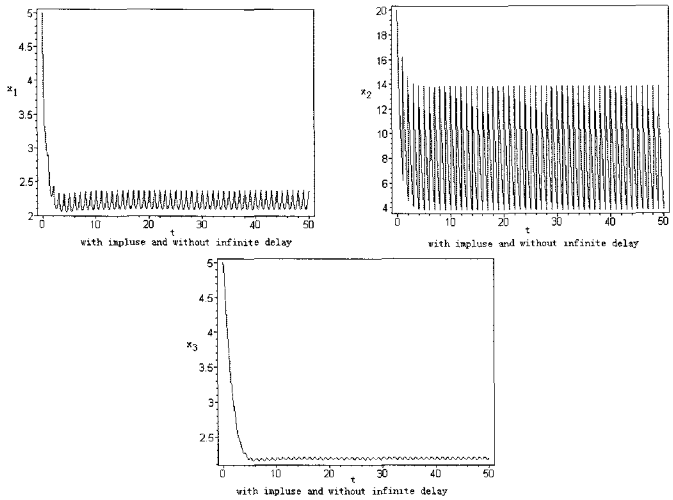

Fig.4 The numerical graph of system (1.2)(Impulse Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 1; \tau > \widehat{T_1} = 0.666666666667.$

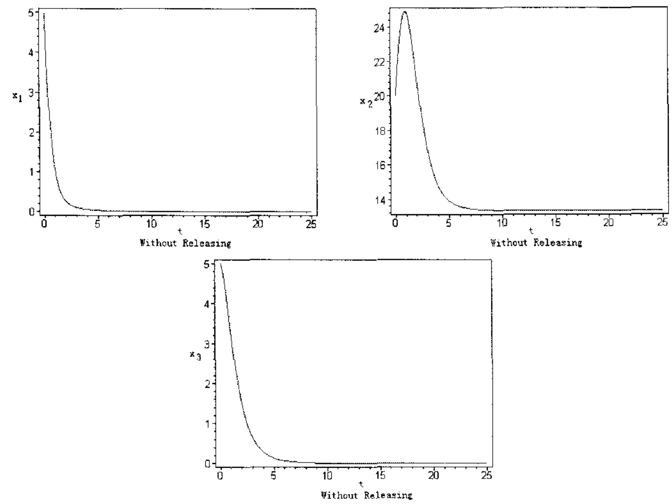

Fig.5 The numerical graph of system (4.1)(without Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 0.5; \tau < \widehat{T_1} = 0.666666666667.$

There also exists a pest eradication equilibrium for system (4.2), that is $E(0, \frac{p}{r_2\tau}, 0)$ which is globally stable if the condition $r_1r_2\tau - pa_{12} < 0$ is satisfied,

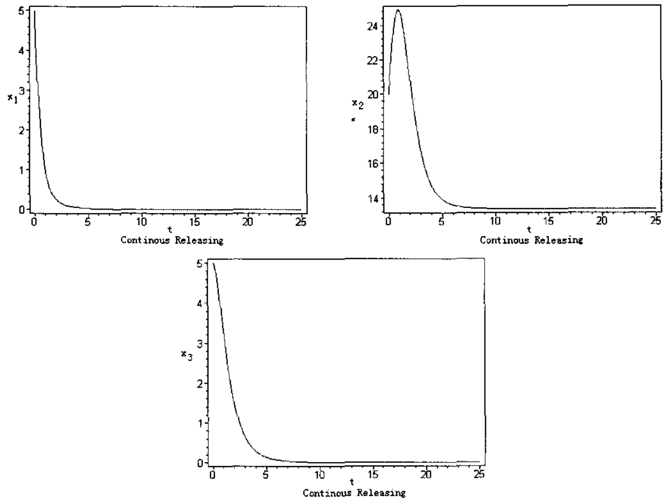

Fig.6 The numerical graph of system (4.2)(Continuous Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 0.5; \tau < \widehat{T_1} = 0.666666666667.$

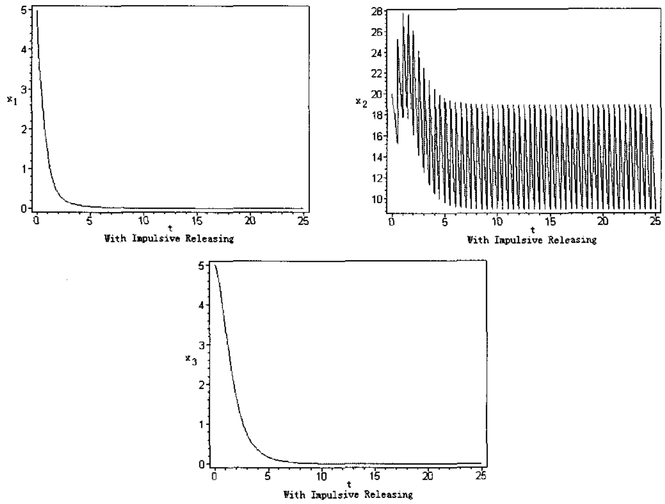

Fig.7 The numerical graph of system (1.5)(Impulse Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 0.5; \tau < \widehat{T_1} = 0.666666666667.$

that is, $\tau < \dfrac{pa_{12}}{r_1 r_2}$. The results is the same as our system (1.5). Although pests can both be eradicated by controlling τ in (1.5) and (4.2), it is impossible to release a natural enemy continuously in pest control. If predators are released

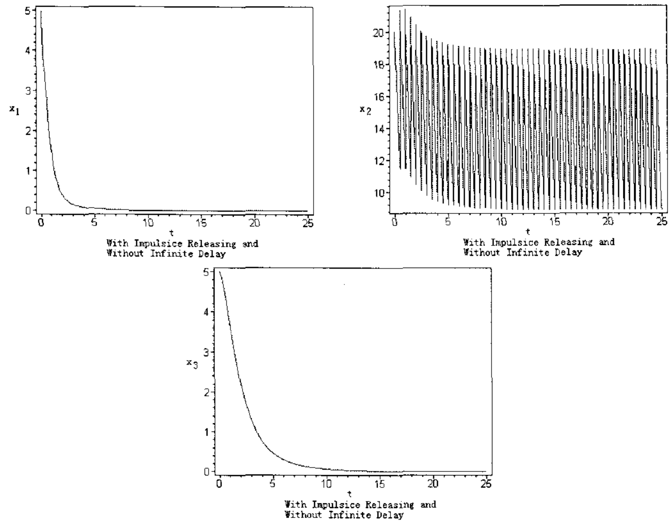

Fig.8 The numerical graph of system (1.2)(Impulse Releasing)
$r_1 = 1, r_2 = 1.5, a = 1, a_{11} = 0.1, a_{12} = 0.1, a_{21} = 0.2, p = 10, x_0 = 5;$
$y_0 = 20, z_0 = 5; \tau = 0.5; \tau < \widehat{T_1} = 0.666666666667.$

continuously, predators will be another disaster after the extinction of pests. So it is more reliable that predators are released impulsively.

From Theorem 2.2 we know the periodic pest eradication solution $(0,\widetilde{x_2}(t),0)$ is globally asymptotically stable provided $\tau \le \dfrac{pa_{12}}{r_1 r_2} = \widehat{T_1}$.

If the system (1.4) has no infinite delay, the system (1.5) becomes (1.2). The dynamics behaviors of (1.2) had been investigated by [3]. Our results in this paper are the same as the results in [3], which show that there is no effect for infinite delay on the system.

From the discussion above, we can see that $\widehat{T_1}$ is the critical value of the release period which directly propositional function with respect to p. Therefore, we can determine the parameter p on the cost of releasing natural enemies such that $\tau < \widehat{T_1}$. In Fig.7 $\widehat{T_1} = 0.666666666667$, that is to say, we can make the impulsive period smaller than 0.666666666667 in order to drive a pest population to extinction.

Fig.1~Fig.4 ($\tau > \widehat{T_1}$) show that the system (1.5) and (1.2) is permanent. Fig.5~Fig.8 ($\tau < \widehat{T_1}$) show there exists a globally asymptotically stable pest eradication periodic solution of system (1.5) and (1.2).

The use of natural predators for pest control aims to inhibit any large pest increase by a corresponding increase in the predator population. The aim is to keep pests at acceptably low levels: not to eradicate them, only to control their populations. To choose the policy of release, there are two ways: The first is if the period of release τ is given,we will choose the appropriate population of release p. The second is if the population of release is given, we will choose the appropriate period of release. The larger the period is, the more the pests are, so we can control the pest population below some given levels by choosing an appropriate impulsive period τ. Therefore, the periodic release of natural enemies at fixed moment changes the properties of the system without impulse, and it is a highly effective method in pest control which the classical method cannot emulate.

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