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REAL HYPERSURFACES IN COMPLEX SPACE FORMS WITH ξ-PARALLEL RICCI TENSOR AND STRUCTURE JACOBI OPERATOR

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Abstract. We know that there are no real hypersurfaces with parallel Ricci tensor or parallel structure Jacobi operator in a nonflat complex space form (See [4], [6], [10] and [11]). In this paper we investigate real hypersurfaces M in a nonflat complex space form $M_n(c)$ under the condition that $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi = 0$, where S and R_ξ respectively denote the Ricci tensor and the structure Jacobi operator of M in $M_n(c)$.

0. Introduction

A Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. It is well known that complete and simply connected complex space forms are isometric to a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC according as $c > 0$, $c = 0$ and $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kaehler metric of $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A denotes the shape operator of M and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a *Hopf hypersurface* if the structure vector field ξ of M is principal.

In the study of real hypersurfaces in P_nC, Takagi [12] classified all homogeneous real hypersurfaces and Cecil and Ryan [2] showed that they can be regarded as the tubes of constant radius over Kaehler submanifolds when the structure vector field ξ is principal. Such tubes can be divided into six kinds of type A_1, A_2, B, C, D and E.

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On the other hand, real hypersurfaces in H_nC have been investigated by Berndt [1], Montiel and Romero [7] and so on. Berndt [1] classified all homogeneous real hypersurfaces in H_nC and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Now, let M be a real hypersurface in $M_n(c), c \neq 0$. Then we introduce the following theorems due to Okumura [9] for $c > 0$ and Montiel and Romero [7] for $c < 0$ respectively.

Theorem A. *Let M be a real hypersurface of $P_nC, n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{0.1}$$

for any vector fields X and Y, then M is locally a tube of radius r over one of the following Kaehler submanifolds:

(A_1) *a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$,*
(A_2) *a totally geodesic P_kC $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$.*

Theorem B. *Let M be a real hypersurface of $H_nC, n \geq 2$. If it satisfies* (0.1), *then M is locally one of the following hypersurfaces :*

(A_0) *a horosphere in H_nC, i.e., a Montiel tube,*
(A_1) *a geodesic hypersphere and a tube over a hyperplane $H_{n-1}C$,*
(A_2) *a tube over a totally geodesic H_kC $(1 \leq k \leq n-2)$.*

On the other hand, it is well known that there are no real hypersurfaces with parallel Ricci tensor in $M_n(c)$, $n \geq 3$, $c \neq 0$ (see [4]). Recently, Kim [6] proved that this is also true when $n = 2$. So it should be natural to investigate real hypersurfaces M in $M_n(c)$ by using some conditions about covariant derivative of S which are weaker than $\nabla S = 0$, where ∇ and S denotes the Levi-Civita connection and the Ricci tensor of M in $M_n(c)$ respectively. Along this direction we introduce a theorem due to [3] as follows:

Theorem C. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ satisfying $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ for some constant σ. Then M is a Hopf hypersurface.*

A Jacobi field along geodesics of a given Riemannian manifold (M, g) is an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. The Jacobi operator is defined by $(R_X(Y))(p) = (R(Y,X)X)(p)$, where R denotes the curvature tensor of M and X, Y denote tangent vector fields on M. Then we see that R_X is a self-adjoint endomorphism on the tangent space of M and is related to the differential equation, so called Jacobi equation, which is given by $\nabla_{\gamma'}(\nabla_{\gamma'}Y) + R(Y,\gamma')\gamma' = 0$ along a geodesic γ on M, where γ' denotes the velocity vector along γ on M.

When we study a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$, we will call the Jacobi operator on M with respect to the structure vector ξ the *structure Jacobi operator* on M and will denote it by R_ξ, where R_ξ is defined by $R_\xi(X) = R(\xi, X)X$ for the curvature tensor R and any tangent vector field X on M. But, recently it is known that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M in $M_n(c)$, $c\neq 0$ (see [10] and [11]).

Motivated by Theorem C and such a view point of the parallel structure Jacobi operator we are able to consider a covariant derivative or a Lie derivative for the Ricci tensor S and the structure Jacobi operator R_ξ along the direction of ξ. This condition $\nabla_\xi S = 0$ (resp. $\nabla_\xi R_\xi = 0$ or $\mathcal{L}_\xi R_\xi = 0$ in [11]) are weaker than the notion of $\nabla S = 0$ (resp. $\nabla R_\xi = 0$ or $\mathcal{L} R_\xi = 0$), respectively.

Now in this paper we prove the following:

Theorem 0.1. *Let M be a real hypersurface in a complex space form $M_n(c)$ satisfying $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi = 0$. Then M becomes a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces:*

(1) *In cases P_nC*

(A_1) *a tube of radius r over a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $P_kC (1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$,*

(T) *a tube of radius $\frac{\pi}{4}$ over a certain complex submanifold in P_nC,*

(2) *In cases H_nC*

(A_0) *a horosphere in H_nC, i.e., a Montiel tube,*

(A_1) *a geodesic hypersphere and a tube over a hyperplane $H_{n-1}C$,*

(A_2) *a tube over a totally geodesic H_kC $(1 \le k \le n-2)$.*

1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$ with parallel almost complex structure J and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where ∇ and g denote the Riemannian connection and the Riemannian metric induced from $\tilde{g}$ respectively, and A denotes the shape operator in the direction of N.

For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we may see that (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. From the fact $\tilde{\nabla} J = 0$ and by using the Gauss and Weingarten formulas, we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.1}$$

$$\nabla_X \xi = \phi AX. \tag{1.2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{1.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Now let us denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), \gamma = \eta(A^3\xi)$ and $h = TrA$, and ∇f the gradient vector field of the function f defined on M.

Now let us denote by S the Ricci tensor of M in $M_n(c)$. Then we have from (1.3)

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \tag{1.5}$$

which together with (1.2) implies that

$$\begin{aligned} (\nabla_X S)Y =& -\frac{3}{4}c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY \\ & + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY, \end{aligned} \tag{1.6}$$

where I denotes the identity map on the tangent space T_pM, $p \in M$.

We put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector field ξ. Thus it is, using (1.2), seen that

$$\phi U = -A\xi + \alpha\xi, \tag{1.7}$$

which shows that $g(U,U) = \beta - \alpha^2$. We easily see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

If $A\xi - g(A\xi, \xi)\xi \neq 0$, then we can put

$$A\xi = \alpha\xi + \mu W, \tag{1.8}$$

where W is a unit vector field orthogonal to ξ. Then by (1.2) we see that $U = \mu\phi W$ and hence $g(U,U) = \mu^2$. So we have

$$\mu^2 = \beta - \alpha^2. \tag{1.9}$$

Further, W is also orthogonal to U.

Using (1.2) and (1.8), it is seen that

$$\mu g(\nabla_X W, \xi) = g(AU, X), \tag{1.10}$$

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{1.11}$$

Now, differentiating (1.7) covariantly along M and making use of (1.1) and (1.2), we find

$$\begin{aligned} &g(\phi X, \nabla_Y U) + \eta(X) g(AU + \nabla\alpha, Y) \\ &= g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.12}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.13}$$

because of (1.4).

By the definition of U, (1.1) and (1.13), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{1.14}$$

From the Gauss equation (1.3) the structure Jacobi operator R_ξ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{1.15}$$

for any vector field X on M.

We set $\Omega = \{p \in M | \mu(p) \neq 0\}$, and suppose that $\Omega \neq \emptyset$, that is, ξ is not a principal curvature vector on M. Hereafter, unless otherwise stated, we continue our discussions on the open set Ω of M.

2. Real hypersurfaces in $M_n(c)$ satisfying $\nabla_\xi R_\xi = 0$

Differentiating (1.15) covariantly, we find

$$\begin{aligned} &g((\nabla_X R_\xi)Y, Z) \\ &= -\frac{c}{4}\{\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)\} + (X\alpha)g(AY, Z) \\ &\quad + \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &\quad - g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\}, \end{aligned}$$

which together with (1.2) and (1.13) implies that

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = &-\frac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_\xi A)Y, Z) - g(A\xi, Z)\{3g(AU, Y) + Y\alpha\} \\ &- g(A\xi, Y)\{3g(AU, Z) + Z\alpha\}, \end{aligned} \tag{2.1}$$

where u is a 1-form defined by $u(X) = g(U, X)$ for any vector field X and $U = \mu\phi W$ defined in (1.8).

Moreover, assume that $\nabla_\xi R_\xi = 0$. Then we have from (2.1)

$$\begin{aligned} \alpha(\nabla_\xi A)X + (\xi\alpha)AX = &\frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)\{3AU + \nabla\alpha\} \\ &+ \{3g(AU, X) + X\alpha\}A\xi. \end{aligned} \tag{2.2}$$

Putting $X = \xi$ in this and making use of (1.13), we find

$$\alpha AU + \frac{c}{4}U = 0, \tag{2.3}$$

which shows that $\alpha \neq 0$ on Ω.

Putting $X = \alpha U$ in (2.2) and using (2.3), we obtain

$$\alpha^2(\nabla_\xi A)U - \frac{c}{4}(\xi\alpha)U = \frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}A\xi. \tag{2.4}$$

Because of (2.3), the equation (1.14) turns out to be

$$\alpha\nabla_\xi U = \frac{3}{4}c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha.$$

Differentiating (2.3) covariantly along Ω, we find

$$(X\alpha)AU + \alpha(\nabla_X A)U + \alpha A(\nabla_X U) + \frac{c}{4}\nabla_X U = 0.$$

If we replace $X = \alpha\xi$ in this and take account of (2.3) and (2.4), then we obtain

$$\frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}A\xi + \alpha^2 A(\nabla_\xi U) + \frac{c}{4}\alpha\nabla_\xi U = 0, \tag{2.5}$$

which together with (1.14) gives

$$\alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \mu(\alpha^2 + \frac{3}{4}c)\{AW - \mu\xi - \frac{1}{\alpha}(\mu^2 - \frac{c}{4})W\} = 0, \tag{2.6}$$

where we have used (1.8).

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $\nabla_\xi S = 0$

In this section, we will continue our discussions on a real hypersurface M in $M_n(c)$ satisfying $\nabla_\xi R_\xi = 0$ and $\nabla_\xi S = 0$. Then replacing X by ξ in (1.6) and using the Codazzi equation (1.4), we obtain

$$\begin{aligned} &\frac{3}{4}c\{u(X)\eta(Y) + u(Y)\eta(X)\} + \frac{c}{4}\{g(AY, \phi X) + g(AX, \phi Y)\} \\ &= (\xi h)g(AX, Y) + hg((\nabla_Y A)X, \xi) - \frac{c}{4}hg(\phi X, Y) \\ &\quad - g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi). \end{aligned} \tag{3.1}$$

On the other hand, $\nabla_\xi R_\xi = 0$ implies

$$\begin{aligned} \alpha^2(\nabla_\xi A)X = &- \alpha(\xi\alpha)AX + \frac{c}{4}\alpha\{u(X)\xi + \eta(X)U\} + \{\alpha(X\alpha) \\ &- \frac{3}{4}cu(X)\}A\xi + (\alpha\nabla\alpha - \frac{3}{4}cU)g(A\xi, X) \end{aligned} \tag{3.2}$$

because of (2.2) and (2.3). Combining above two equations, we get

$$\begin{aligned}&\frac{c}{4}(3\alpha^2-h\alpha-\frac{c}{4})(u(X)\xi+\eta(X)U)-2\alpha(\xi\alpha)A^2X\\&+\alpha\{h(\xi\alpha)-\alpha(\xi h)\}AX\\&=g(A\xi,X)\{h\alpha\nabla\alpha-\alpha A\nabla\alpha+\frac{3}{4}c(AU-hU)-\frac{c}{4}\alpha U\}\\&+\big[h\alpha(X\alpha)-\alpha g(A\nabla\alpha,X)+\frac{3}{4}c\{g(AU,X)-hu(X)\}-\frac{c}{4}\alpha u(X)\big]A\xi\\&+\{\frac{3}{4}cu(X)-\alpha(X\alpha)\}A^2\xi+\{\frac{3}{4}cU-\alpha\nabla\alpha\}g(A^2\xi,X).\end{aligned}\tag{3.3}$$

Since U is orthogonal to ξ, we see, using (1.8) and (2.3), that $g(A^2\xi,U)=0$. Thus, replacing X by U in (3.3) and taking account of (2.3), we find

$$\begin{aligned}&\frac{c}{4}(3\alpha^2-h\alpha-\frac{c}{4})\mu^2\xi-\frac{c}{4}\{(\frac{c}{2\alpha}+h)\xi\alpha-\alpha(\xi h)\}U\\&=\{(h\alpha+\frac{c}{4})U\alpha-\frac{3}{4}c(\frac{c}{4\alpha}+h)\mu^2-\frac{c}{4}\alpha\mu^2\}A\xi+\{\frac{3}{4}c\mu^2-\alpha(U\alpha)\}A^2\xi,\end{aligned}\tag{3.4}$$

which enables us to obtain

$$(\frac{c}{2}+h\alpha)\xi\alpha-\alpha^2(\xi h)=0.\tag{3.5}$$

We notice here that $\alpha(U\alpha)\neq\frac{3}{4}c\mu^2$ on Ω. In fact, if not, then we have $\alpha A\xi=(h\alpha+\frac{c}{4}-3\alpha^2)\xi$ by virtue of (3.4) and (3.5). From this, by taking an inner product with ξ, we have $4\alpha^2=h\alpha+\frac{c}{4}$ on this subset. So we have $A\xi=\alpha\xi$, a contradiction. Therefore $\alpha(U\alpha)-\frac{3}{4}c\mu^2\neq0$ is satisfied everywhere. Consequently we have from (3.4) and (3.5) the following:

$$A^2\xi=\rho A\xi+(\beta-\rho\alpha)\xi,\tag{3.6}$$

where the function ρ (resp. β) is defined by $\mu\rho=g(A^2\xi,W)$ (resp. $\mu^2=\beta-\alpha^2$ in (1.9)).

Combining (1.8) to (3.6), we see that

$$AW=\mu\xi+(\rho-\alpha)W\tag{3.7}$$

because of $\mu\neq0$.

Differentiating (3.7) covariantly, we find

$$(\nabla_XA)W+A\nabla_XW=(X\mu)\xi+\mu\nabla_X\xi+X(\rho-\alpha)W+(\rho-\alpha)\nabla_XW,$$

which shows that

$$g((\nabla_XA)W,W)=-2g(AU,X)+X\rho-X\alpha,\tag{3.8}$$

$$\mu(\nabla_WA)\xi=(\rho-2\alpha)AU-\frac{c}{2}U+\mu\nabla\mu,\tag{3.9}$$

where we have used (1.4), (1.8) and (1.10). From the last two equations, it follows that

$$W\mu=\xi\rho-\xi\alpha.\tag{3.10}$$

Taking an inner product (2.6) with W and using (3.7), we have

$$-\{\alpha(\rho-\alpha)+\frac{c}{4}\}\phi W\alpha+\mu U\alpha+\mu(\alpha^2+\frac{3}{4}c)\{(\rho-\alpha)-\frac{1}{\alpha}(\mu^2-\frac{c}{4})\}=0.$$

On the other hand, by applying ξ to (2.6) and using (1.8) we have

$$\alpha U\alpha=\alpha\mu\phi W\alpha.$$

Substituting this into the above equation and using (1.9), we have

$$(\beta-\rho\alpha-\frac{c}{4})\{\alpha(U\alpha)-(\alpha^2+\frac{3}{4}c)\mu^2\}=0. \tag{3.11}$$

We are now going to prove $\alpha(U\alpha)=(\alpha^2+\frac{3}{4}c)\mu^2$ on Ω. For this purpose we prepare the following facts.

Lemma 3.1. $\Omega=\emptyset$ *provided that*

$$\beta-\rho\alpha-\frac{c}{4}=0. \tag{3.12}$$

Proof. From our assumption we have

$$A^2\xi=\rho A\xi+\frac{c}{4}\xi \tag{3.13}$$

by virtue of (3.6). Differentiating (3.13) covariantly and using (1.2), we find

$$\begin{aligned} &g((\nabla_X A)A\xi,Y)+g(A(\nabla_X A)\xi,Y)+g(A^2\phi AX,Y)-\rho g(A\phi AX,Y)\\ &=(X\rho)g(A\xi,Y)+\rho g((\nabla_X A)\xi,Y)+\frac{c}{4}g(\phi AX,Y), \end{aligned} \tag{3.14}$$

which together with (1.4) and (1.13) yields

$$(\nabla_\xi A)A\xi=\rho AU-\frac{c}{4}U+\frac{1}{2}\nabla\beta. \tag{3.15}$$

If we put $X=\xi$ in (3.14) and use (1.13) and the last equation, then we get

$$3A^2U-2\rho AU-\frac{c}{2}U=(\xi\rho)A\xi-A\nabla\alpha+\rho\nabla\alpha-\frac{1}{2}\nabla\beta. \tag{3.16}$$

If we replace X by $A\xi$ in (3.1) and make use of (1.13), (3.13) and (3.15), we obtain

$$\begin{aligned} &\rho A^2U+(\rho^2-h\rho+\frac{c}{4})AU+\frac{c}{4}(3\alpha+h-\rho)U\\ &=(\xi h)A^2\xi-\frac{c}{4}\nabla\alpha+\frac{1}{2}(h-\rho)\nabla\beta-\frac{1}{2}A\nabla\beta. \end{aligned} \tag{3.17}$$

On the other hand, we have from (1.5) and (3.13)

$$S\xi=\frac{c}{4}(2n-3)\xi+(h-\rho)A\xi. \tag{3.18}$$

Differentiating this covariantly, we find

$$\begin{aligned} (\nabla_X S)\xi+S\nabla_X\xi=&\frac{c}{4}(2n-3)\nabla_X\xi+X(h-\rho)A\xi\\ &+(h-\rho)(\nabla_X A)\xi+(h-\rho)A\nabla_X\xi. \end{aligned}$$

Putting $X = \xi$ in the last equation and taking account of $\nabla_\xi S = 0$, (1.5) and (1.13), we obtain

$$A^2U + (2h - 3\rho)AU - cU = \xi(\rho - h)A\xi + (\rho - h)\nabla\alpha. \tag{3.19}$$

We here note that $\rho - h \neq 0$ on Ω. In fact, if not, then by (3.18) we obtain $S\xi = g(S\xi, \xi)\xi$, where $g(S\xi, \xi) = \frac{c}{4}(2n-3)$. Then by virtue of Theorem C in [3], we are able to assert that M is a Hopf real hypersurface. So we have $\Omega = \emptyset$, a contradiction. Thus, $\rho - h \neq 0$ is proved everywhere.

Taking an inner product (3.19) with ξ or W, we obtain respectively

$$\begin{aligned}(h - \rho)\xi\alpha + \alpha(\xi h - \xi\rho) = 0,\\ (h - \rho)W\alpha + \mu(\xi h - \xi\rho) = 0,\end{aligned} \tag{3.20}$$

which enables us to obtain

$$\mu(\xi\alpha) = \alpha(W\alpha). \tag{3.21}$$

From this, together with (1.9) and (3.10), we get

$$W\beta = 2\mu(\xi\rho). \tag{3.22}$$

Taking an inner product (3.15) with ξ and using (1.9), (3.12) and (3.21), we obtain

$$\alpha^2(\xi\rho) = (\rho\alpha + \frac{c}{2})\xi\alpha. \tag{3.23}$$

By the way, if we apply (3.17) by W and make use of (3.7) and (3.22), then we get

$$\mu\rho(\xi h) - \frac{c}{4}W\alpha + \mu(h - 2\rho + \alpha)\xi\rho - \frac{1}{2}\mu(\xi\beta) = 0,$$

which together with (3.12), (3.20) and (3.21) implies that

$$\alpha(h - \rho + \frac{1}{2}\alpha)\xi\rho + (\rho^2 - \rho h - \frac{1}{2}\rho\alpha - \frac{c}{4})\xi\alpha = 0.$$

From this and (3.23) we see that $(h - \rho)\xi\alpha = 0$ and hence $\xi\alpha = 0$, because $\rho - h \neq 0$ on Ω. So we have $W\alpha = 0$, $\xi h = 0$ and $\xi\rho = 0$ because of (2.4), (3.5), (3.20) and (3.21). Further, we have $W\rho = 0$ by virtue of (3.10) and (3.12).

By putting $X = \mu W$ in (3.1) and using (1.4), (1.13), (3.7), (3.9) and (3.12), we find

$$\begin{aligned}\mu A\nabla\mu =&(2\alpha - \rho)A^2U - (2h\alpha - h\rho + \rho^2 - \rho\alpha + \frac{c}{4})AU\\ &- \frac{c}{4}(h - \rho + \alpha)U + (h - \rho + \alpha)\mu\nabla\mu - \mu^2\nabla\alpha,\end{aligned} \tag{3.24}$$

where we have used (1.9).

Now, differentiating (3.6) covariantly and using (1.2), we have

$$\begin{aligned}&g((\nabla_X A)\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y)\\ &= (X\rho)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) + X(\beta - \rho\alpha)\eta(Y)\\ &\quad + (\beta - \rho\alpha)g(\phi AX, Y).\end{aligned}$$

Then by the equation of Codazzi (1.4), the above equation becomes

$$\begin{aligned}&\frac{c}{4}\{u(Y)\eta(X)-u(X)\eta(Y)\}+\frac{c}{2}(\rho-\alpha)g(\phi Y,X)-g(A^2\phi AX,Y)\\&+g(A^2\phi AY,X)+2\rho g(\phi AX,AY)-(\beta-\rho\alpha)\{g(\phi AY,X)-g(\phi AX,Y)\}\\=\;&g(AY,(\nabla_X A)\xi)-g(AY,(\nabla_Y A)\xi)+(Y\rho)g(A\xi,X)-(X\rho)g(A\xi,Y)\\&+Y(\beta-\rho\alpha)\eta(X)-X(\beta-\rho\alpha)\eta(Y).\end{aligned}$$

Here, we replace X by μW to both sides and take account of (1.4), (3.7) and (3.9), we have

$$\begin{aligned}&(3\alpha-2\rho)A^2U+(2\rho^2-2\rho\alpha+c)AU+\frac{c}{4}(\alpha-\rho)U\\&=\mu A\nabla\mu+(\alpha-\rho)\mu\nabla\mu+\mu^2(\nabla\rho-\nabla\alpha)-\mu(W\rho)A\xi-\mu W(\beta-\rho\alpha)\xi.\end{aligned}\tag{3.25}$$

Substituting (3.24) into (3.25) and using (3.12), we obtain

$$\begin{aligned}&(\alpha-\rho)A^2U+(3\rho^2-3\rho\alpha+2h\alpha-h\rho+\frac{5}{4}c)AU+\frac{c}{4}(h-2\rho+2\alpha)U\\&=\frac{1}{2}(h-2\rho+2\alpha)\nabla\beta-(\alpha h+\frac{c}{2}\nabla\alpha)+\mu^2\{\nabla\rho-(W\rho)W\}.\end{aligned}\tag{3.26}$$

Then from (3.19) and the fact that $\xi h=\xi\rho=W\rho=0$, the last equation becomes

$$(2h\rho+\frac{5}{2}c)AU=\frac{c}{2}(6\rho-6\alpha-h)U+(h\alpha+\frac{c}{2})\nabla\rho-(h\rho+c)\nabla\alpha.\tag{3.27}$$

Combining (3.16) to (3.17), we also obtain

$$2\alpha A\nabla\rho=\alpha(2h-\rho)\nabla\rho-(\rho^2+c)\nabla\alpha-(2\rho^2+c)AU+c(2\rho-h-3\alpha)U,\tag{3.28}$$

where we have used (3.12) and the fact that $\xi\rho=\xi h=0$.

If we take account of (2.4), then (3.27) turns out to be

$$f_1U=\alpha(h\alpha+\frac{c}{2})\nabla\rho-\alpha(h\rho+c)\nabla\alpha,\tag{3.29}$$

where we have put

$$f_1=-\frac{c}{4}(2h\rho+\frac{5}{2}c)+\frac{c}{2}(6\rho\alpha-6\alpha^2-h\alpha).\tag{3.30}$$

Differentiating (3.29) covariantly and taking the skew-symmetric part, we get

$$f_1\{g(\nabla_\xi U,X)+g(\nabla_X\xi,U)\}=0$$

for any vector X, where we have used $\xi h=\xi\rho=\xi\alpha=0$. This, together with (1.2), (1.11), (1.14) and (3.7) implies that $f_1\{\phi(3AU+\nabla\alpha)+\mu\rho W\}=0$ and hence $f_1(\nabla\alpha-\rho U+3AU)=0$. Therefore, it follows that $f_1=0$ on Ω.

In fact, if not, then we have

$$\nabla\alpha=\rho U-3AU.\tag{3.31}$$

From this, combining with (3.12) and (3.16), and using $\xi\rho=0$, we have

$$\alpha\nabla\rho=-\rho AU+(\rho^2+c)U\tag{3.32}$$

on the subset Ω.

Using (3.31), the equation (3.19) can be written as

$$A^2U = hAU + (\rho^2 - \rho h + c)U. \tag{3.33}$$

Substituting (3.31) and (3.32) into (3.28) and making use of the last equation, we obtain

$$h = 3\alpha - 2\rho. \tag{3.34}$$

Using (3.31), (3.32) and (3.34), we have from (3.27)

$$AU = hU - \nabla\rho$$

on the subset, which together with (3.32) yields

$$(\rho - \alpha)AU = (\rho^2 - 3\alpha^2 + 2\rho\alpha + c)U.$$

Comparing this with (2.4), we obtain $(\rho + 3\alpha)(\rho\alpha - \alpha^2 + \frac{c}{4}) = 0$, which together with (1.9) and (3.12) implies that $\rho + 3\alpha = 0$ on this subset. By the way, we have from (2.4) and (3.33) the following

$$3\rho\alpha^2(\rho - \alpha) + \frac{c}{4}\alpha(\alpha + 2\rho) = (\frac{c}{4})^2.$$

Therefore α is constant and hence ρ does so, a contradiction. Thus, $f_1 = 0$ is proved everywhere on Ω.

Accordingly we have

$$(h\rho + c)\nabla\alpha = (h\alpha + \frac{c}{2})\nabla\rho \tag{3.35}$$

by virtue of (3.29).

From (2.4) and (3.19) we have

$$(\rho - h)\alpha^2\nabla\alpha = \frac{c}{4}(\frac{c}{4} - 2h\alpha + 3\rho\alpha - 4\alpha^2)U.$$

In the same way as above, we verify from this that

$$2h\alpha + 4\alpha^2 - 3\rho\alpha - \frac{c}{4} = 0 \tag{3.36}$$

and hence $\nabla\alpha = 0$ by virtue of (2.4) and Theorem C in the introduction. So we have $\nabla\rho = 0$ because of (3.35) and (3.36), which together with (3.12) yields that $\nabla\beta = 0$. Thus, by using (2.4) the equations (3.16) and (3.28) imply respectively to

$$\rho\alpha - \alpha^2 + \frac{3}{8}c = 0, \ 2\rho^2 + c = 8\rho\alpha - 4h\alpha - 12\alpha^2.$$

From the last three equations, it follows that $\rho^2 + \alpha^2 = -\frac{9}{8}c$ and hence $\rho^2 - 3\rho\alpha + 4\alpha^2 = 0$, a contradiction. Thus, $\beta - \rho\alpha - \frac{c}{4} \neq 0$ on Ω is proved. □

4. Proof of a key lemma

In this section we give another important lemma which will be useful in the proof of our Main Theorem stated in the introduction.

In this section we also assume that a real hypersurface M in $M_n(c)$ has ξ-parallel Ricci tensor and structure Jacobi operator. Now from (3.4) we obtain

$$A^2\xi = hA\xi + \frac{c}{4\alpha^2}(h\alpha + \frac{c}{4} - 3\alpha^2)\xi,$$

which implies that

$$\alpha^2(\beta - h\alpha + \frac{3}{4}c) = \frac{c}{4}(h\alpha + \frac{c}{4}). \tag{4.1}$$

Further, from (3.6), we get $h = \rho$ and hence (3.6) becomes

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi. \tag{4.2}$$

Putting $X = \xi$ in (1.5) and using (4.2), we find

$$S\xi = \{\frac{c}{2}(n-1) + h\alpha - \beta\}\xi. \tag{4.3}$$

Differentiating this covariantly, we obtain

$$(\nabla_X S)\xi + S\nabla_X\xi = X(h\alpha - \beta)\xi + \{\frac{c}{2}(n-1) + h\alpha - \beta\}\nabla_X\xi.$$

If we replace X by ξ in this, and take account of (1.5) and $\nabla_\xi S = 0$, then we have

$$A^2U - hAU - (\beta - h\alpha + \frac{3}{4}c)U + \xi(h\alpha - \beta)\xi = 0.$$

Since $g(A^2U, \xi) = 0$ because of (2.3), it follows from the last equation

$$\xi(h\alpha - \beta) = 0 \tag{4.4}$$

and hence

$$A^2U = hAU + (\beta - h\alpha + \frac{3}{4}c)U. \tag{4.5}$$

Combining (3.5) to (4.4), we obtain

$$\frac{1}{2}\alpha(\xi\beta) = (h\alpha + \frac{c}{4})\xi\alpha. \tag{4.6}$$

Differentiating (4.2) covariantly and making use of (1.2), we find

$$\begin{aligned} &g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - hg(A\phi AX, Y) \\ &= (Xh)g(A\xi, Y) + hg((\nabla_X A)\xi, Y) + X(\beta - h\alpha)\eta(Y) \\ &\quad + (\beta - h\alpha)g(\phi AX, Y), \end{aligned} \tag{4.7}$$

which together with the equation of Codazzi(1.4), (1.13) and (3.7) implies that

$$(\nabla_\xi A)A\xi = hAU - \frac{c}{4}U + \frac{1}{2}\nabla\beta, \tag{4.8}$$

where we have used that $A\phi A^2\xi = \mu\rho A\phi W$, $hAU = h\mu A\phi W$ and $h = \rho$.

If we replace X by ξ in (4.7) and use (1.3), (1.4), (3.11) and the last equation, then we get

$$3A^2U - 2hAU - (\beta - h\alpha + \frac{c}{4})U = (\xi h)A\xi - A\nabla\alpha + h\nabla\alpha - \frac{1}{2}\nabla\beta,$$

which, together with (2.3) and (4.5) gives

$$hAU + 2(\beta - h\alpha + c)U = (\xi h)A\xi - A\nabla\alpha + h\nabla\alpha - \frac{1}{2}\nabla\beta. \tag{4.9}$$

Now by applying A to both sides of (4.9), and using (4.2) and (4.5), we have

$$\begin{aligned}(h^2 + 2\beta - 2h\alpha + 2c)AU + h(\beta - h\alpha + \frac{3}{4}c)U \\ = h(\xi h)A\xi + (\beta - h\alpha)(\xi h)\xi - A^2\nabla\alpha - \frac{1}{2}A\nabla\beta + hA\nabla\alpha.\end{aligned} \tag{4.10}$$

On the other hand, by applying h to both sides of (4.9), we have

$$h^2AU + 2h(\beta - h\alpha + c)U = h(\xi h)A\xi - hA\nabla\alpha + h^2\nabla\alpha - \frac{1}{2}h\nabla\beta. \tag{4.11}$$

If we replace Y by $A\xi$ in (3.1) and make use of (1.4), (1.13), (4.2) and (4.8), then we get

$$\frac{3}{4}c\alpha U + \frac{c}{4}AU = (\xi h)A^2\xi - (\beta - h\alpha)(2AU + \nabla\alpha) - hA^2U + \frac{c}{2}AU - \frac{1}{2}A\nabla\beta.$$

This together with (4.5) yields

$$\begin{aligned}(h^2 + 2\beta - 2h\alpha - \frac{c}{4})AU + \{h\beta - h^2\alpha + \frac{3}{4}c(h + \alpha)\}U \\ = (\xi h)A^2\xi - \frac{1}{2}A\nabla\beta - (\beta - h\alpha)\nabla\alpha.\end{aligned}$$

Applying (4.9) by A and using the above formula, we have

$$\frac{3}{4}c(3AU - \alpha U) + A^2\nabla\alpha - hA\nabla\alpha - (\beta - h\alpha)\nabla\alpha = 0.$$

Then substituting this formula into the right side of (4.10), we have

$$\begin{aligned}(h^2 + 2\beta - 2h\alpha + 2c)AU + h(\beta - h\alpha + \frac{3}{4}c)U \\ = h(\xi h)A\xi + (\beta - h\alpha)(\xi h)\xi - (\beta - h\alpha)\nabla\alpha \\ - \frac{3}{4}c(\alpha U - 3AU) - \frac{1}{2}A\nabla\beta.\end{aligned} \tag{4.12}$$

Substracting (4.12) from (4.11) gives the following

$$\begin{aligned}\frac{1}{2}(A\nabla\beta - h\nabla\beta) - h(A\nabla\alpha - h\nabla\alpha) + (\beta - h\alpha)\nabla\alpha \\ = \{2h\alpha - 2\beta + \frac{c}{4}\}AU + \{h\beta - h^2\alpha + \frac{5}{4}ch - \frac{3}{4}c\alpha\}U + (\beta - h\alpha)(\xi h)\xi.\end{aligned}$$

This implies again

$$\begin{aligned} &A^2\nabla\beta - hA\nabla\beta \\ &= 2h\{(\beta - h\alpha)\nabla\alpha + \frac{3}{4}c(\alpha U - 3AU)\} \\ &- 2(\beta - h\alpha)A\nabla\alpha + 2\{2h\alpha - 2\beta + \frac{c}{4}\}A^2U \\ &+ 2\{h\beta - h^2\alpha + \frac{5}{4}ch - \frac{3}{4}c\alpha\}AU + 2(\beta - h\alpha)(\xi h)A\xi. \end{aligned}$$

On the other hand, by (4.9) we see that

$$\begin{aligned} &2(\beta - h\alpha)\{h\nabla\alpha - A\nabla\alpha\} \\ &= 2(\beta - h\alpha)\{hAU + 2(\beta - h\alpha + c)U - (\xi h)A\xi + \frac{1}{2}\nabla\beta\}, \end{aligned}$$

from this, together with the above equation, it follows that

$$\frac{1}{2}\{A^2\nabla\beta - hA\nabla\beta - (\beta - h\alpha)\nabla\beta\} + \frac{3}{4}c\{(h+\alpha)AU - (\beta + \frac{c}{4})U\} = 0.$$

Then by using (4.5), we have

$$\frac{1}{2}U\beta = -(h+\alpha)g(AU, U) + (\beta + \frac{c}{4})\mu^2.$$

From this and (2.4) it follows that

$$\frac{1}{2}\alpha(U\beta) = \{\alpha\beta + \frac{c}{4}(h + 2\alpha)\}\mu^2. \tag{4.13}$$

On the other hand, taking several choices of X and Y in (3.1) and (4.7), and using (1.4), (1.13), (3.7), (4.2), (4.5) and (4.8), we obtain (for detail, see [3]).

$$\begin{aligned} &(4\beta - 4h\alpha + h^2 + \frac{c}{4})AU + \frac{c}{4}(6\alpha - 5h)U \\ &= \mu(\xi h)AW - \mu(Wh)A\xi - \mu\{W\beta - h(W\alpha) - \alpha(Wh)\}\xi \\ &\quad + \frac{1}{2}(2\alpha - h)\nabla\beta + (h\alpha - 2\beta)\nabla\alpha + \mu^2\nabla h, \end{aligned} \tag{4.14}$$

where we have used the formula (4.3) and the assumption $\nabla_\xi S = 0$. Applying (4.14) by U and using (3.11), Lemma 3.1 and (4.13), we get

$$\mu^2(Uh) = (\frac{c}{4} - 2\mu^2)g(AU, U) + (h\mu^2 + c(\alpha - h))\mu^2,$$

which together with (2.3) gives

$$\alpha(Uh) = (h\alpha + \frac{c}{2})\mu^2 + c(\alpha^2 - \alpha h) - (\frac{c}{4})^2. \tag{4.15}$$

Now, we prepare the following:

Lemma 4.1. *$\xi\alpha = 0$, $\xi h = 0$, $\xi\beta = 0$ $W\alpha = 0$ and $W\beta = 0$ on Ω.*

Proof. Taking an inner product (4.12) with ξ and using (1.8) and (4.4), we find

$$\mu(W\beta) = (2\beta - \alpha^2)\xi h + (h\alpha - 2\beta)\xi\alpha,$$

which connected with (3.5) yields

$$\mu\alpha^2(W\beta) = (2\beta h\alpha - 2\beta\alpha^2 + c\beta - \frac{c}{2}\alpha^2)\xi\alpha. \tag{4.16}$$

If we take an inner product (4.9) with ξ and make use of (1.8) and (4.4), then we also obtain

$$\frac{1}{2}\xi\beta = \alpha(\xi\alpha) + \mu(W\alpha),$$

which together with (1.9) implies that

$$\xi\mu = W\alpha. \tag{4.17}$$

Combining above two equations, it follows that

$$\mu\alpha(\xi\mu) = (h\alpha - \alpha^2 + \frac{c}{4})\xi\alpha, \tag{4.18}$$

where we have used (3.5), (4.4) and (4.6).

Because of (3.5), (4.6), (4.16) and (4.18) and the fact $\alpha\neq 0$ on the set Ω by Lemma 3.1, it suffices to show that $\xi\alpha = 0$ on Ω.

In order to do this, we differentiate (4.1) as follows:

$$2\alpha(\beta - h\alpha + \frac{3}{4}c)\nabla\alpha + \alpha^2\nabla\beta = (\alpha^2 + \frac{c}{4})(h\nabla\alpha + \alpha\nabla h).$$

From this, together with (4.1), we see that

$$\alpha^3\nabla\beta - \alpha^2(\alpha^2 + \frac{c}{4})\nabla h = (h\alpha^3 - \frac{c}{4}h\alpha - \frac{c^2}{8})\nabla\alpha. \tag{4.19}$$

Then by using (4.13), (4.15) and Lemma 3.1, we verify that

$$\begin{aligned}&\{2\alpha^3(\beta - h\alpha) + \frac{c}{2}\alpha^3 - \frac{c}{4}h\alpha^2 + \frac{3}{16}c^2h + \frac{3c^3}{32\alpha}\}\mu^2\\ &= c\alpha(\alpha^2 + \frac{c}{4})(\alpha^2 - h\alpha - \frac{c}{16}),\end{aligned}$$

or, using (1.9) and (4.1)

$$\begin{aligned}&(ch\alpha^3 - 4c\alpha^4 + \frac{c^2}{2}\alpha^2 + \frac{3}{4}c^2h\alpha + \frac{3}{8}c^3)(\beta - \alpha^2)\\ &= c(4\alpha^4 + c\alpha^2)(\alpha^2 - h\alpha - \frac{c}{16}).\end{aligned} \tag{4.20}$$

Now, differentiating (4.20) gives

$$\begin{aligned}
&2(h\alpha^3 - 4\alpha^4 + \frac{c}{2}\alpha^2 + \frac{3}{4}ch\alpha + \frac{3}{8}c^2)\mu(X\mu) \\
&+ (3\alpha^2 h - 16\alpha^3 + c\alpha + \frac{3}{4}ch)\mu^2(X\alpha) \\
&+ \{\alpha\mu^2(\alpha^2 + \frac{3}{4}c) + \alpha^3(4\alpha^2 + c)\}(Xh) \\
&= \alpha\{(8\alpha^2 + c)(2\alpha^2 - 2h\alpha - \frac{c}{8}) + \alpha(2\alpha - h)(4\alpha^2 + c)\}(X\alpha).
\end{aligned}$$

From this, putting $X = \xi$ and using (1.9), (3.5), (4.6) and (4.18),we obtain

$$\begin{aligned}
&(\frac{c}{2} + h\alpha)\{c\alpha^2(c + 4\alpha^2) + (\beta - \alpha^2)(\frac{3}{4}c^2 + c\alpha^2)\} \\
&+ 2(\frac{c}{4} + h\alpha - \alpha^2)(\frac{3}{8}c^3 - 4c\alpha^4 + \frac{3}{4}c^2 h\alpha + ch\alpha^3 + \frac{1}{2}c^2\alpha^2) \\
&= c\alpha^3(2\alpha - h)(c + 4\alpha^2) + c\alpha^2(2c + 16\alpha^2)(\alpha^2 - h\alpha - \frac{c}{16}) \\
&- \alpha(\frac{3}{4}c^2 h + c^2\alpha - 16\alpha^3 + 3ch\alpha^2)
\end{aligned} \tag{4.21}$$

provided that $\xi\alpha \neq 0$.

If we eliminate β from (4.1) and (4.20), then together with (4.21) we are able to assert that α is a root of an algebraic equation with constant coefficients. Hence, α should be constant, which makes a contradiction. Therefore, $\xi\alpha = 0$ on Ω. This gives the complete proof of our Lemma 4.1 □

From (2.3) and (4.9) we have

$$\alpha A\nabla\alpha = h\alpha\nabla\alpha - \frac{1}{2}\alpha\nabla\beta + \{\frac{c}{4}h - 2\alpha(\beta - h\alpha + c)\}U, \tag{4.22}$$

by virtue of $\xi h = 0$.

Putting $X = \xi$ in (3.3) and using (2.3), (4.1) and (4.22), we obtain

$$\beta\alpha\nabla\alpha - \frac{1}{2}\alpha^2\nabla\beta = \{(\beta - h\alpha)(2\alpha^2 + \frac{3}{4}c) + c(\alpha^2 - \frac{c}{8})\}U, \tag{4.23}$$

where we have used Lemma 4.1. By virtue of Lemma 4.1, equations (4.9), (4.12) and (4.14) mentioned above turn out respectively to be

$$hAU + 2(\beta - h\alpha + c)U = h\nabla\alpha - A\nabla\alpha - \frac{1}{2}\nabla\beta, \tag{4.24}$$

(4.25)

$$(h^2 + 2\beta - 2h\alpha - \frac{c}{4})AU + \{h\beta - h^2\alpha + \frac{3}{4}c(h + \alpha)\}U = -\frac{1}{2}A\nabla\beta + (h\alpha - \beta)\nabla\alpha,$$

and

$$\begin{aligned}
&(4\beta - 4h\alpha + h^2 + \frac{c}{4})AU + \frac{c}{4}(6\alpha - 5h)U \\
&= \frac{1}{2}(2\alpha - h)\nabla\beta + (h\alpha - 2\beta)\nabla\alpha + \mu^2\nabla h.
\end{aligned} \tag{4.26}$$

5. Proof of main theorem

We will continue our discussions for real hypersurfaces M in $M_n(c)$ with the assumption $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi = 0$ as in section 4. Now let us use the formulas (4.23)∼(4.26). From (4.23) we have

$$\beta\alpha(X\alpha) - \frac{1}{2}\alpha^2(X\beta) = fu(X), \tag{5.1}$$

where we have put

$$f = (\beta - h\alpha)(2\alpha^2 + \frac{3}{4}c) + c(\alpha^2 - \frac{c}{8}).$$

Differentiating (5.1) covariantly and taking the skew-symmetric parts obtained, we find

$$\begin{aligned}&(Yf)u(X) - (Xf)u(Y) + 2fdu(Y, X)\\ &= Y(\beta\alpha)(X\alpha) - X(\beta\alpha)(Y\alpha) + \frac{1}{2}\alpha\{(X\alpha)(Y\beta) - (Y\alpha)(X\beta)\},\end{aligned} \tag{5.2}$$

where the exterior derivative du of a 1-form u is given by

$$du(Y, X) = \frac{1}{2}\{Yu(X) - Xu(Y) - u([Y, X])\}.$$

If we replace Y by ξ in (5.2) and take account of Lemma 4.1, we get

$$fdu(\xi, X) = 0.$$

Now let us denote by Ω_0 be the set of points in Ω such that $du(\xi, X)_p \neq 0$ at $p \in \Omega$ and suppose that the set Ω_0 is nonempty. Then we have $f = 0$, that is,

$$2\alpha^2(\beta - h\alpha) + \frac{3}{4}c(\beta - h\alpha) + c(\alpha^2 - \frac{c}{8}) = 0.$$

This, together with (4.1) implies that

$$3\beta = h\alpha + 2\alpha^2 \tag{5.3}$$

on Ω_0. Further, we have by (4.23)

$$\frac{1}{2}\alpha\nabla\beta = \beta\nabla\alpha. \tag{5.4}$$

By using the similar method to (4.26) we have

$$7h\alpha - 10\alpha^2 + 3h^2 + \frac{3}{4}c = 0 \tag{5.5}$$

and

$$\frac{1}{2}(2\alpha - h)\nabla\beta + (h\alpha - 2\beta)\nabla\alpha + (\beta - \alpha^2)\nabla h = 0, \tag{5.6}$$

which together with (5.4) yields

$$h\nabla\alpha = \alpha\nabla h \tag{5.7}$$

on Ω_0. Differentiating (5.5) covariantly and using (5.7), we find $(7h\alpha - 10\alpha^2 + 3h^2)\nabla\alpha = 0$, which shows that $\nabla\alpha = 0$ and hence $\nabla\beta = 0$ and $\nabla h = 0$ on Ω_0 by virtue of (5.4) and (5.7). Thus, it follows from (2.3) and (4.24) that

$$\frac{c}{4}(h - 8\alpha) = 2\alpha(\beta - h\alpha), \tag{5.8}$$

which connected with (4.1) gives

$$2\alpha^2 + h\alpha + \frac{c}{2} = 0. \tag{5.9}$$

From this and (5.3) we see that

$$6\beta + c = 0. \tag{5.10}$$

However, it is verified, using (4.25) and (5.8), that

$$4\beta + h^2 - 2h\alpha - 6\alpha^2 - \frac{c}{2} = 0.$$

This, together with (2.3) and (4.26) implies that $h\alpha + \frac{c}{4} = 0$. Thus, (4.1) becomes $\beta - h\alpha + \frac{3}{4}c = 0$. So we have $\beta + c = 0$, which together with (5.10) will produce a contradiction. Hence we have $\Omega = \emptyset$. Thus, $du(\xi, X) = 0$ for any vector X. That is, $g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0$. This, together with (1.11), (1.14) and (3.7) with $h = \rho$, implies that

$$\phi(3AU + \nabla\alpha) + \mu h W = 0.$$

Accordingly, it follows that

$$\nabla\alpha = hU - 3AU.$$

Combining to (2.3), we have

$$\alpha\nabla\alpha = (h\alpha + \frac{3}{4}c)U.$$

From this, together with Lemma 3.1 and (3.11) we see that

$$h = \alpha \tag{5.11}$$

and hence

$$\alpha\nabla\alpha = (\alpha^2 + \frac{3}{4}c)U. \tag{5.12}$$

Further, (4.1) becomes

$$\alpha^2(\beta - \alpha^2) = (\frac{c}{4})^2 - \frac{c}{2}\alpha^2. \tag{5.13}$$

Now using (5.3), (5.4), (5.11) and (5.12), we have

$$\begin{aligned} \frac{1}{2}\alpha^2\nabla\beta &= \alpha\beta\nabla\alpha \\ &= \frac{1}{3}(h\alpha + 2\alpha^2)(\alpha^2 + \frac{3}{4}c)U \\ &= \alpha^2(\alpha^2 + \frac{3}{4}c)U, \end{aligned} \tag{5.14}$$

where in the first equality we have used (5.4) and (5.12), and in the second equality used (5.3) and finally in the third equality used (5.11) respectively.

Differentiating (5.13) covariantly and using (5.11) gives

$$\frac{1}{2}\alpha^2\nabla\beta = (2\alpha^2 - \beta - \frac{c}{2})(\alpha^2 + \frac{3}{4}c)U. \tag{5.15}$$

Then by comparing (5.14) and (5.15) we have

$$(\alpha^2 - \beta - \frac{c}{2})(\alpha^2 + \frac{3}{4}c) = 0. \tag{5.16}$$

On the other hand, substituting (5.11) into (5.9) gives $\alpha^2 = -\frac{c}{6}$. From this, together with (5.10) and (5.16), we have a contradiction. Thus, the set Ω should be empty. Therefore we see that the subset Ω in M on which $A\xi - g(A\xi,\xi)\xi \neq 0$ is an empty set. Namely, in $M_n(c)$, $c \neq 0$, every real hypersurface satisfying $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi$ is a Hopf hypersurface. So, we have $U = 0$ and moreover, the function α should be constant. Thus, (3.2) implies $\alpha\nabla_\xi A = 0$, which together with (1.4) and (1.12) yields

$$\alpha(A\phi - \phi A) = 0.$$

When the constant α identically vanishes, by Cecil and Ryan [2] we assert that M is a tube of radius $\frac{\pi}{4}$ over certain Kaehler submanifold in P_nC. But we here note that any Hopf hypersurfaces in H_nC the function α never vanishing(see [1], [7] and [8]). For the non-vanishing constant α, by virtue of Theorems A and B due to Okumura [9] for $c > 0$ and Montiel and Romero [7] for $c < 0$ respectively we complete the proof of our Main Theorem.

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