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GENERALIZED MULTIVALUED QUASIVARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

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ABSTRACT. In this paper, we introduce and study a class of generalized multivalued quasivariational inclusions for fuzzy mappings, and establish its equivalence with a class of fuzzy fixed-point problems by using the resolvent operator technique. We suggest a new iterative algorithm for the generalized multivalued quasivariational inclusions. Further, we establish a few existence results of solutions for the generalized multivalued quasivariational inclusions involving F_r-relaxed Lipschitz and F_r-strongly monotone mappings, and discuss the convergence criteria for the algorithm.

1. INTRODUCTION

Variational inequality theory plays an important and fundamental role in pure and applied science. In recent years, classical variational inequality problem has been extended to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity and applied sciences, etc., see [1], [3], [6]-[10] and the references therein. In 1998, Verma [9]considered the existence of solutions for a class of generalized variational inequalities involving relaxed Lipschitz mappings.

In 1989, Chang and Zhu [3] first introduced a class of variational inequalities for fuzzy mappings. In 1993, by using the projection technique, Noor [6] suggested an iterative scheme for finding the approximate solutions for a variational inequality for fuzzy mappings and proved that this approximate solution converges strongly to the exact solution for his problem. Afterwards, Al-Said [1] and Noor [7], [8]

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and others have introduced and studied a few classes of variational inequalities and quasi-variational inequalities for fuzzy mappings.

In this paper, we introduce and study a class of generalized multivalued quasivariational inclusion for fuzzy mappings, which include the variational inequalities and quasi-variational inequalities for fuzzy mappings in [1], [6]-[10] as special cases. Using the resolvent operator technique for maximal monotone mapping, we prove that the generalized multivalued quasivariational inclusions for fuzzy mappings are equivalent to the fuzzy fixed-point problems. We suggest a general and unified algorithm for the generalized multivalued quasivariational inclusions. Further we establish a few existence results of solutions for the generalized multivalued quasivariational inclusions involving F_r-relaxed Lipschitz and F_r-strongly monotone mappings, and discuss the convergence criteria for the algorithm. Our results are the extension and improvements of the earlier and recent results in this field.

2. PRELIMINARIES

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. We denote the collection of all fuzzy sets on H by $F(H) = \{a : H \to [0,1]\}$. Let I be the identity mapping on H, $A \in F(H)$ and $r \in (0,1]$. The r-level set of A, denoted $(A)_r$, is defined by $(A)_r = \{x \in H : Ax \geq r\}$. Let $F_r(H) = \{A \in F(H) : (A)_r$ is a bounded closed subset of $H\}$. A mapping T from H into $F(H)$ is called a fuzzy mapping.

Definition 2.1 ([2]). If $A : H \to 2^H$ is a maximal monotone mapping, then the *resolvent operator* associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u) \quad \text{for all } u \in H,$$

where $\rho > 0$ is a constant.

It is well known that the subdifferential of a proper lower semicontinuous convex function is maximal monotone.

Definition 2.2. A fuzzy mapping $T : H \to F_r(H)$ is said to be *F_r-Lipschitz continuous* if there exists a constant $t > 0$ such that for each $x, y \in H$,

$$H((Tx)_r, (Ty)_r) \leq t\|x - y\|,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on the family of nonempty bounded closed subsets of H.

Definition 2.3. A mapping $g : H \to H$ is said to be *Lipschitz continuous and strongly monotone* if there exist constants $t > 0$ and $s > 0$ such that for all $x, y \in H$,

$$\|gx - gy\| \le t\|x - y\| \text{ and } \langle gx - gy,\, x - y\rangle \ge s\|x - y\|^2,$$

respectively.

Definition 2.4. A mapping $N : H \times H \to H$ is said to be *Lipschitz continuous* with respect to the first argument if there exists a constant $t > 0$ such that for all $x, y, z \in H$,

$$\|N(x,z) - N(y,z)\| \le t\|x - y\|.$$

Similarly we can define the Lipschitz continuity of the mapping N with respect to the second argument.

Definition 2.5. A fuzzy mapping $T : H \to F(H)$ is said to be

(i) *F_r-relaxed Lipschitz* with respect to the first argument of $N : H \times H \to H$, if there exists a constant $t > 0$ such that for all $u, v, w \in H$, $x \in (Tu)_r$, $y \in (Tv)_r$,

$$\langle N(x,w) - N(y,w), u - v\rangle \le -t\|u - v\|^2;$$

(ii) *F_r-strongly monotone* with respect to the second argument of $N : H \times H \to H$, if there exists a constant $t > 0$ such that for all $u, v, w \in H$, $x \in (Tu)_r$, $y \in (Tv)_r$,

$$\langle N(w,x) - N(w,y), u - v\rangle \ge t\|u - v\|^2.$$

Lemma 2.1 ([4]). *Let $A : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator J_A is singlevalued and nonexpansive.*

Let $M : H \times H \to H$, $g, h : H \to H$ be mappings, $A, B, C, D : H \to F(H)$ be fuzzy mappings and $W : H \times H \to 2^H$ be a multivalued mapping such that for each $y \in H$, $W(\cdot, y) : H \to 2^H$ is a maximal monotone mapping. For given $f \in H$, we consider the following problem:

Find $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ such that $gu - hw \in dom(W(\cdot, z))$ and

$$f \in gu - M(x,y) + W(gu - hw, z), \tag{2.1}$$

which is called the generalized multivalued quasivariational inclusion for fuzzy mappings.

If $C = I$, $W(x,y) = W(x)$ for all $x, y \in H$, where $W : H \to 2^H$ is a maximal monotone mapping, then problem (2.1) is equivalent to finding $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $w \in (Du)_r$ such that $gu - hw \in dom(W)$ and

$$f \in gu - M(x,y) + W(gu - hw), \tag{2.2}$$

which appears to be a new one.

Let $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each $y \in H$, $\phi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper lower semicontinuous convex function on H and $\partial\phi(\cdot, y)$ denote the subdifferential of function $\phi(\cdot, y)$. Let $K(u)$ be a closed convex subset of H, $I_{K(u)}$ denote the indicator function of $K(u)$, $P_{K(u)}$ be the projection of H onto $K(u)$ for any $u \in H$.

In case $f = h = 0$, $C = D = I$, $W(x,y) = \partial\phi(x,y)$, $M(x,y) = gx - y$ for all $x, y \in H$, where $\phi(x,y) = I_{K(y)}(x)$, then problem (2,1) collapses to finding $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$ such that $gu \in K(u)$ and

$$\langle gu - (gx - y), v - gu \rangle \geq 0 \quad \text{for all } v \in K(u), \tag{2.3}$$

which is called the generalized quasi-variational inequality for fuzzy mappings studied by Al-Said [1].

If $f = h = 0$, $A = D = g = I$, $W = \partial\phi$, $M(x,y) = x - y$ for all $x, y \in H$, where $\phi : H \to R \cup \{+\infty\}$ is a proper lower semicontinuous convex function on H and $\partial\phi$ denotes the subdifferential of function ϕ, then problem (2.2) is equivalent to finding $u \in H$, $y \in (Bu)_r$ such that

$$\langle y, v - u \rangle \geq \phi(u) - \phi(v) \quad \text{for all } v \in H, \tag{2.4}$$

which is called the mixed variational inequality for fuzzy mappings introduced and studied by Noor [8].

If $f = h = 0$, $B = D = g = I$, $W = \partial\phi$, $\phi(x) = I_K(x)$, $M(x,y) = x - y$ for all $x, y \in H$, where K is a closed convex subset of H, then problem (2.2) is equivalent to finding $u \in K$, $x \in (Au)_r$ such that

$$\langle x, v - u \rangle \geq 0 \quad \text{for all } v \in K, \tag{2.5}$$

which is known as the variational inequality for fuzzy mappings introduced and studied by Noor [6], [7].

For appropriate and suitable choice of the mappings g, h, A, B, C, D, M, W and the element f, one can obtain a few new and known classes of variational inequalities and quasi-variational inequalities for fuzzy mappings from problem (2.1) as special cases.

3. Main Results

Lemma 3.1. *Let t and ρ be positive parameters. Then the following conditions are equivalent.*

(a) *the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ with $gu - hw \in dom(W(\cdot, z))$;*

(b) *there exist $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ satisfying*

$$gu = hw + J_{W(\cdot,z)}((1-\rho)gu - hw + \rho M(x,y) + \rho f);$$

(c) *the mapping $F : H \to 2^H$ defined by*

$$Fs = \cup_{x\in(As)_r, y\in(Bs)_r, z\in(Cs)_r, w\in(Ds)_r}\{(1-t)s + t[s - gs + hw + J_{W(\cdot,z)}((1-\rho)gs - hw + \rho M(x,y) + \rho f)]\} \quad \text{for all } s \in H$$

has a fixed point $u \in H$.

Proof. (a)⇔(b). It follows from Definition 2.1 and Lemma 2.1 that

$$\begin{aligned} & f \in gu - M(x,y) + W(gu - hw, z) \\ & \Leftrightarrow (1-\rho)gu - hw + \rho M(x,y) + \rho f \in gu - hw + \rho W(gu - hw, z) \\ & \Leftrightarrow gu = hw + J_{W(\cdot,z)}((1-\rho)gu - hw + \rho M(x,y) + \rho f). \end{aligned}$$

(b)⇔(c). It is easy to see that $u \in H$ is a fixed point of F if and only if there exist $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ satisfying

$$u = (1-t)u + t[u - gu + hw + J_{W(\cdot,z)}((1-\rho)gu - hw + \rho M(x,y) + \rho f)].$$

Notice that $t > 0$. Hence the above equation is equivalent to $gu = hw + J_{W(\cdot,z)}((1-\rho)gu - hw + \rho M(x,y) + \rho f)$. This completes the proof. □

Remark 3.1. Lemma 3.1 extends Lemma 3.1 in [1], [7], [8] and Lemma 3.2 in [6], [9]. Lemma 3.1 is very important from the numerical and approximation point of views. Based on Lemma 3.1 and Nadler's result, we suggest the following general and unified algorithm for generalized multivalued quasivariational inclusion for fuzzy mappings (2.1).

Algorithm 3.1. Let $g, h : H \to H$, $M : H \times H \to H$, $A, B, C, D : H \to F_r(H)$ and $f \in H$. For given $u_0 \in H$, $x_0 \in (Au_0)_r$, $y_0 \in (Bu_0)_r$, $z_0 \in (Cu_0)_r$, $w_0 \in (Du_0)_r$, compute $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ from the iterative schemes

$$\begin{aligned} u_{n+1} = (1-t)u_n + t[u_n - gu_n + hw_n + J_{W(\cdot, z_n)}((1-\rho)gu_n - hw_n \\ +\rho M(x_n, y_n) + \rho f)], \end{aligned} \tag{3.1}$$

$$\begin{aligned} x_n &\in (Au_n)_r, \|x_n - x_{n+1}\| \leq (1+(1+n)^{-1})H((Au_n)_r, (Au_{n+1})_r), \\ y_n &\in (Bu_n)_r, \|y_n - y_{n+1}\| \leq (1+(1+n)^{-1})H((Bu_n)_r, (Bu_{n+1})_r), \\ z_n &\in (Cu_n)_r, \|z_n - z_{n+1}\| \leq (1+(1+n)^{-1})H((Cu_n)_r, (Cu_{n+1})_r), \\ w_n &\in (Du_n)_r, \|w_n - w_{n+1}\| \leq (1+(1+n)^{-1})H((Du_n)_r, (Du_{n+1})_r) \end{aligned} \tag{3.2}$$

for all $n \geq 0$, where t and ρ are positive parameters with $t \leq 1$.

Remark 3.2. Algorithm 1 in [1] and Algorithm 3.1 in [6]-[9] are special cases of Algorithm 3.1.

Theorem 3.1. *Let $g, h : H \to H$ be Lipschitz continuous with constants m and l, respectively, g be strongly monotone with constants α, and $A, B, C, D : H \to F_r(H)$ be F_r-Lipschitz continuous with constants a, b, c, d, respectively. Let $M : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with constants η, σ, respectively, and A be F_r-relaxed Lipschitz with respect to the first argument of M with constant s. Suppose that $W : H \times H \to 2^H$ satisfy for each $x \in H$, $W(\cdot, x) : H \to 2^H$ is a maximal monotone mapping and*

$$\|J_{W(u,x)}(z) - J_{W(u,y)}(z)\| \leq \mu\|x - y\| \quad \text{for all } u, x, y, z \in H, \tag{3.3}$$

where $\mu > 0$ is a constant. Let $k = 2\sqrt{1 - 2\alpha + m^2} + 2ld + \mu c$, $j = \sigma b - \sqrt{1 - 2\alpha + m^2} \geq 0$, $L = 1 + 2s + \eta^2 a^2 - j^2$, $T = 1 + s - (1-k)j$ and $S = k^2 - 2K$. If there exists a constant $\rho \in (0, 1]$ satisfying

$$k + \rho j < 1, \tag{3.4}$$

and one of the following conditions

$$L > 0,\ T^2 + SL > 0,\ |\rho - TL^{-1}| < L^{-1}\sqrt{T^2 + SL}; \tag{3.5}$$

$$L = 0,\ T > 0,\ \rho > -(2T)^{-1}S; \tag{3.6}$$

$$L < 0,\ |\rho - TL^{-1}| > -L^{-1}\sqrt{T^2 + SL}, \tag{3.7}$$

then for each $f \in H$, the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ with $gu - hw \in dom(W(\cdot, z))$ and the sequences $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$,

$\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$, *generated by Algorithm 3.1 converge strongly to* u, x, y, z *and* w, *respectively.*

Proof. Since g is Lipschitz continuous and strongly monotone with constants m and α, respectively, we have

$$\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \leq \sqrt{1 - 2\alpha + m^2}\|u_n - u_{n-1}\|. \tag{3.8}$$

Because A is F_r-Lipschitz continuous with constant a and F_r-relaxed Lipschitz with respect to the first argument of M with constant s, and M is Lipschitz continuous with respect to the first argument with constants η, by (3.2), we conclude that

$$\begin{aligned}
&\|(1-\rho)(u_n - u_{n-1}) + \rho(M(x_n, y_n) - M(x_{n-1}, y_n))\|^2 \\
&= (1-\rho)^2\|u_n - u_{n-1}\|^2 + 2(1-\rho)\rho\langle M(x_n, y_n) - M(x_{n-1}, y_n), u_n - u_{n-1}\rangle \\
&\quad + \rho^2\|M(x_n, y_n) - M(x_{n-1}, y_n)\|^2 \\
&\leq ((1-\rho)^2 - 2(1-\rho)\rho s)\|u_n - u_{n-1}\|^2 \\
&\quad + \rho^2\eta^2(1 + n^{-1})^2)H^2((Au_n)_r, (Au_{n-1})_r) \\
&\leq ((1-\rho)^2 - 2(1-\rho)\rho s + \rho^2\eta^2 a^2(1 + n^{-1})^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{3.9}$$

Using (3.2), (3.3), (3.8), (3.9), Lemma 2.1 and the Lipschitz continuity of M with respect to the first and second arguments, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq (1-t)\|u_n - u_{n-1}\| + t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|hw_n - hw_{n-1}\| \\
&\quad + t\|J_{W(\cdot, z_n)}((1-\rho)gu_n - hw_n + \rho M(x_n, y_n) + \rho f) \\
&\quad - J_{W(\cdot, z_n)}((1-\rho)gu_{n-1} - hw_{n-1} + \rho M(x_{n-1}, y_{n-1}) + \rho f)\| \\
&\quad + t\|J_{W(\cdot, z_n)}((1-\rho)gu_{n-1} - hw_{n-1} + \rho M(x_{n-1}, y_{n-1}) + \rho f) \\
&\quad - J_{W(\cdot, z_{n-1})}((1-\rho)gu_{n-1} - hw_{n-1} + \rho M(x_{n-1}, y_{n-1}) + \rho f)\| \\
&\leq (1-t)\|u_n - u_{n-1}\| + t\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\
&\quad + t\|hw_n - hw_{n-1}\| \\
&\quad + t\|(1-\rho)(gu_n - gu_{n-1}) - (hw_n - hw_{n-1}) \\
&\quad + \rho(M(x_n, y_n) - \rho M(x_{n-1}, y_{n-1}))\| + t\mu\|z_n - z_{n-1}\| \\
&\leq (1-t)\|u_n - u_{n-1}\| + t(2-\rho)\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\
&\quad + 2t\|hw_n - hw_{n-1}\| \\
&\quad + t\|(1-\rho)(u_n - u_{n-1}) + \rho(M(x_n, y_n) - \rho M(x_{n-1}, y_n))\| \\
&\quad + t\rho\|M(x_{n-1}, y_n) - M(x_{n-1}, y_{n-1})\| + t\mu\|z_n - z_{n-1}\| \\
&\leq (1 - t + t(2-\rho)\sqrt{1 - 2\alpha + m^2})\|u_n - u_{n-1}\| + 2tl\|w_n - w_{n-1}\|
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&+t\sqrt{(1-\rho)^2-2(1-\rho)\rho s+\rho^2\eta^2a^2(1+n^{-1})^2}\|u_n-u_{n-1}\| \\
&+t\rho\sigma\|y_n-y_{n-1}\|+t\mu(1+n^{-1})H((Cu_n)_r,(Cu_{n-1})_r) \\
&\le (1-t(1-\theta_n))\|u_n-u_{n-1}\|,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_n = & \ (2-\rho)\sqrt{1-2\alpha+m^2}+(2ld+\mu c)(1+n^{-1}) \\
& +\sqrt{(1-\rho)^2-2(1-\rho)\rho s+\rho^2\eta^2a^2(1+n^{-1})^2}+\rho\sigma b(1+n^{-1}) \\
\to & \ \ \theta = k+\sqrt{(1-\rho)^2-2(1-\rho)\rho s+\rho^2\eta^2a^2}+\rho j
\end{aligned}
\tag{3.11}
$$

as $n\to\infty$. It is easy to verify that (3.4) means that

$$\theta<1 \Leftrightarrow \sqrt{(1-\rho)^2-2(1-\rho)\rho s+\rho^2\eta^2a^2}<1-k-\rho j \Leftrightarrow L\rho^2-2T\rho<S.$$

Thus one of (3.5)-(3.7) yields that $\theta<1$. Let $Q\in(\theta,1)$ be a fixed number. It follows from (3.11) that there exists a positive integer P satisfying $\theta_n<Q$ for all $n\ge P$. In view of (3.10) and $t\in(0,1]$, we easily conclude that $\{u_n\}_{n\ge 0}$ is a Cauchy sequence in H. Hence there exists $u\in H$ such that $u_n\to u$ as $n\to\infty$. From (3.2) and F_r-Lipschitz continuity of A,B,C,D, we know that $x_n\to x\in H$, $y_n\to y\in H$, $z_n\to z\in H$, $w_n\to w\in H$ as $n\to\infty$.

By virtue of the continuity of $g,h,J_{W(\cdot,z)}$, and the Lipschitz continuity of M with respect to the first and second arguments, respectively, by (3.1) we have

$$u=(1-t)u+t[u-gu+hw+J_{W(\cdot,z)}((1-\rho)gu-hw+\rho M(x,y)+\rho f)],$$

which yields that

$$gu=hw+J_{W(\cdot,z)}((1-\rho)gu-hw+\rho M(x,y)+\rho f).$$

Note that

$$
\begin{aligned}
d(x,(Au)_r) &= \inf\{\|x-p\|:p\in(Au)_r\}\le\|x-x_n\|+d(x_n,(Au)_r) \\
&\le \|x-x_n\|+H((Au_n)_r,(Au)_r))\le\|x-x_n\|+a\|u-u_n\|\to 0
\end{aligned}
$$

as $n\to\infty$. That is, $d(x,(Au)_r)=0$. Hence $x\in(Au)_r$ because $(Au)_r\in F_r(H)$. Similarly, we have $y\in(Bu)_r$, $z\in(Cu)_r$, $w\in(Du)_r$. It follows from Lemma 3.1 that the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u\in H$, $x\in(Au)_r$, $y\in(Bu)_r$, $z\in(Cu)_r$, $w\in(Du)_r$ with $gu-hw\in dom(W(\cdot,z))$. This completes the proof. □

Remark 3.3. Theorem 3.1 extends, improves and unifies Theorem 3.1 in [9] and Theorem 3.6 in [10].

Theorem 3.2. *Let g,h,A,B,C,D,M,W,k and S be as in Theorem 3.1. Suppose that B is F_r-strongly monotone with respect to the second arguments of M with* *constant q. Assume that*

$$\begin{aligned} j &= \sqrt{1-2q+\sigma^2b^2} - \sqrt{1-2\alpha+m^2} \geq 0, \\ L &= \eta^2a^2 - j^2,\ T = s-(1-k)j. \end{aligned}$$

If there exists a positive constant $\rho \in (0,1]$ satisfying (3.4) and one of (3.5)-(3.7), then for given $f \in H$, the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ with $gu - hw \in dow(W(\cdot,z))$ and the sequences $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ generated by Algorithm 3.1 converge strongly to u, x, y, z and w, respectively.

Proof. Because B is F_r- Lipschitz continuous with constant b and F_r-strongly monotone with respect to the second argument of M with constant q, and M is Lipschitz continuous with respect to the second argument with constant σ, we know that

$$\begin{aligned} &\|u_n - u_{n-1} - (M(x_{n-1},y_n) - M(x_{n-1},y_{n-1}))\|^2 \\ &= \|u_n - u_{n-1}\|^2 - 2\langle M(x_{n-1},y_n) - M(x_{n-1},y_{n-1}), u_n - u_{n-1}\rangle \\ &\quad + \|M(x_{n-1},y_n) - M(x_{n-1},y_{n-1})\|^2 \\ &\leq (1-2q+\sigma^2b^2(1+n^{-1})^2)\|u_n - u_{n-1}\|. \end{aligned} \tag{3.12}$$

By making use of the same arguments used for obtaining (3.9), we have

$$\begin{aligned} &\|u_n - u_{n-1} + \rho(M(x_n,y_n) - M(x_{n-1},y_n))\| \\ &\leq \sqrt{1-2\rho s + \rho^2\eta^2a^2(1+n^{-1})^2}\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.13}$$

It follows from (3.2), (3.3), (3.9), (3.10) and Lemma 2.1 that

(3.14)

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq (1-t)\|u_n - u_{n-1}\| + t\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\ &\quad + t\|hw_n - hw_{n-1}\| \\ &\quad + t\|(1-\rho)(gu_n - gu_{n-1}) - (hw_n - hw_{n-1}) \\ &\quad + \rho(M(x_n,y_n) - M(x_{n-1},y_{n-1}))\| \\ &\quad + t\mu\|z_n - z_{n-1}\| \\ &\leq (1-t)\|u_n - u_{n-1}\| + t(2-\rho)\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\ &\quad + 2t\|hw_n - hw_{n-1}\| \\ &\quad + t\|u_n - u_{n-1} + \rho(M(x_n,y_n) - M(x_{n-1},y_n))\| \\ &\quad + t\rho\|u_n - u_{n-1} - (M(x_{n-1},y_n) - M(x_{n-1},y_{n-1}))\| \\ &\quad + t\mu(1+n^{-1})H((Cu_n)_r,(Cu_{n-1})_r) \\ &\leq (1-t+t(2-\rho)\sqrt{1-2\alpha+m^2})\|u_n - u_{n-1}\| \\ &\quad + 2tl(1+n^{-1})H((Du_n)_r,(Du_{n-1})_r) \end{aligned}$$

$$+t(\sqrt{1-2\rho s+\rho^2\eta^2a^2(1+n^{-1})^2}+\rho\sqrt{1-2q+\sigma^2b^2(1+n^{-1})^2}$$
$$+\mu c(1+n^{-1}))\|u_n-u_{n-1}\|$$
$$\leq (1-(1-\theta_n)t)\|u_n-u_{n-1}\|,$$

where

$$\begin{aligned}\theta_n &= (2-\rho)\sqrt{1-2\alpha+m^2}+(2ld+\mu c)(1+n^{-1})+\sqrt{1-2\rho s+\rho^2\eta^2a^2(1+n^{-1})^2}\\ &\quad+\rho\sqrt{1-2q+\rho^2b^2(1+n^{-1})^2}\\ &\to \theta = k+\rho j+\sqrt{1-2\rho s+\rho^2\eta^2a^2}\end{aligned}$$

as $n\to\infty$. Obviously,

$$\theta<1 \Leftrightarrow \sqrt{1-2\rho s+\rho^2\eta^2a^2}<1-k-\rho j \Leftrightarrow L\rho^2-2\rho T<S.$$

The rest of the proof is similar to that of Theorem 3.1. This completes the proof. □

Remark 3.4. Theorem 3.2 is a generalization of Theorem 3.1 in [6] and Theorem 3.3 in [7], [8].

Theorem 3.3. *Let $g: H\to H$ satisfy that $I-g$ is Lipschitz continuous with constant m, $k=2m+2ld+\mu c$ and $j=\sigma b-m\geq 0$. Let h,A,B,C,D,M,W,T,L,S be be as in Theorem 3.1. If there exists a constant $\rho\in(0,1]$ satisfying (3.4) and one of (3.5)-(3.7), then for given $f\in H$, the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u\in H$, $x\in(Au)_r$, $y\in(Bu)_r$, $z\in(Cu)_r$, $w\in(Du)_r$ with $gu-hw\in dom(W(\cdot,z))$ and the sequences $\{u_n\}_{n\geq0}$, $\{x_n\}_{n\geq0}$, $\{y_n\}_{n\geq0}$, $\{z_n\}_{n\geq0}$, $\{w_n\}_{n\geq0}$ generated by Algorithm 3.1 converge strongly to u,x,y,z and w, respectively.*

Proof. Since $I-g$ is Lipschitz continuous with constant m, it follows that

$$\|gu_n-gu_{n-1}-(u_n-u_{n-1})\|\leq m\|u_n-u_{n-1}\|. \tag{3.15}$$

As in the proof of Theorem 3.1, by (3.15) we immediately infer that

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq (1-t)\|u_n-u_{n-1}\|+t(2-\rho)\|gu_n-gu_{n-1}-(u_n-u_{n-1})\|\\ &\quad+2t\|hw_n-hw_{n-1}\|\\ &\quad+t\|(1-\rho)(u_n-u_{n-1})+\rho(M(x_n,y_n)-M(x_{n-1},y_n))\|\\ &\quad+t\rho\|M(x_{n-1},y_n)-M(x_{n-1},y_{n-1})\|+t\mu\|z_n-z_{n-1}\|\\ &\leq (1-(1-\theta_n)t)\|u_n-u_{n-1}\|,\end{aligned}$$

where

$$\begin{aligned}\theta_n &= (2-\rho)m + (2ld+\mu c)(1+n^{-1}) \\ &\quad + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho s + \rho^2\eta^2 a^2(1+n^{-1})^2} + \rho\sigma b(1+n^{-1}) \\ &\to \theta = k + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho s + \rho^2\eta^2 a^2} + \rho j\end{aligned}$$

as $n \to \infty$. By a similar argument used in the proof of Theorem 3.1, the result follows. This completes the proof. □

The proof of the following result goes in a similar fashion as that of Theorems 3.2 and 3.3, so we omit the proof.

Theorem 3.4. *Let $g : H \to H$ satisfy that $I-g$ is Lipschitz continuous with constant m, $k = 2m+2ld+\mu c$, and $j = \sqrt{1-2q+\sigma^2 b^2} - m \geq 0$. Let $h, A, B, C, D, M, W, T, L, S$ be be as in Theorem 3.2. If there exists a constant $\rho \in (0,1]$ satisfying (3.4) and one of (3.5)-(3.7), then for given $f \in H$, the generalized multivalued quasivariational inclusion for fuzzy mappings (2.1) has a solution $u \in H$, $x \in (Au)_r$, $y \in (Bu)_r$, $z \in (Cu)_r$, $w \in (Du)_r$ with $gu - hw \in dom(W(\cdot, z))$ and the sequences $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ generated by Algorithm 3.1 converge strongly to u, x, y, z and w, respectively.*

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