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THE MODIFIED HYERS-ULAM-RASSIAS STABILITY OF A MIXED TYPE FUNCTIONAL EQUATION

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ABSTRACT. We investigate the modified Hyers-Ulam-Rassias stability for the following mixed type functional equation, i.e, cubic or quadratic type functional equation:

$$9f(x+y)-9f(x-y)+f(6y)=3f(x+3y)-3f(x-3y)+9f(2y).$$

1. INTRODUCTION

In 1940, it is well known that the stability of functional equations was first raised by S. M. Ulam [24] as follows: *Under what condition does there is an additive mapping near an approximately additive mapping between a group and a metric group?*

In next year, D. H. Hyers [6] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [19]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instances, [2, 3, 5, 7, 8, 9, 14, 15, 16, 20, 21, 22]). In particular, one of the important functional equations studied is the following functional equation [1, 4, 13, 17]:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x)=qx^2$ is a solution of this functional equation, and so one usually is called the above functional equation to be quadratic.

A Hyers-Ulam stability problem for the quadratic functional equation was first proved by F. Skof [23] for functions $f: X \to Y$, where X is a normed space and Y a Banach space. S. Czerwik [4] generalized the Hyers-Ulam stability of the quadratic functional equation.

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The cubic function $f(x) = cx^3$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1}$$

The equation (1) was solved by K.-W. Jun and H.-M. Kim [10] (see also [18]).

In this note, we promise that the equation (1) is called a cubic functional equation and every solution of the cubic functional equation (1) is said to be a cubic function.

Now, let us introduce the following functional equation:

$$9f(x+y) - 9f(x-y) + f(6y) = 3f(x+3y) - 3f(x-3y) + 9f(2y). \tag{2}$$

It is easy to see that the real-valued function $f(x) = cx^3 + qx^2$ is a solution of the functional equation (2). Our main purpose in this note is to examine the modified Hyers-Ulam-Rassias stability problem (or the stability in the spirit of Găvruta [5]) for the equation (2).

2. Solutions of Eq. (2)

Let X and Y be real vector spaces. In this section we will find out the general solution of (2).

Lemma 1. *A mapping $f : X \to Y$ is cubic if and only if f is odd and satisfies the functional equation $f(x+3y) + 3f(x-y) = f(x-3y) + 3f(x+y) + 48f(y)$ for all $x,\ y \in X$.*

Proof. ($\Rightarrow$) Suppose that f is cubic, that is, the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{3}$$

holds for all $x,\ y \in X$. By putting $x = y = 0$ in (3), we see that $f(0) = 0$, and setting $x = 0$ in (3) yields the fact that f is odd. If we interchange x and y in (3), we have

$$f(x+2y) - f(x-2y) = 2f(x+y) - 2f(x-y) + 12f(y). \tag{4}$$

Let $x := x+y$ and $x := x-y$, respectively, in (4). Then we obtain

$$f(x+3y) - f(x-y) = 2f(x+2y) - 2f(x) + 12f(y)$$

and

$$f(x+y) - f(x-3y) = 2f(x) - 2f(x-2y) + 12f(y).$$

Comparing the above two results, we get

$$f(x+3y) - f(x-3y) - f(x-y) + f(x+y) = 2f(x+2y) - 2f(x-2y) + 24f(y)$$

which, by (4), gives

$$f(x+3y)+3f(x-y)=f(x-3y)+3f(x+y)+48f(y).$$

($\Leftarrow$) Assume that f is odd and satisfies the functional equation

$$f(x+3y)+3f(x-y)=f(x-3y)+3f(x+y)+48f(y) \tag{5}$$

for all $x, y \in X$. By interchanging x and y in (5), we obtain

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \tag{6}$$

and applying [12, Theorem 2.2] to (6), we see that f is cubic. □

The following lemma is due to [11].

Lemma 2. *A mapping $f: X \to Y$ is quadratic if and only if $f(0)=0$, f is even and satisfies the functional equation $f(x+3y)+3f(x-y)=f(x-3y)+3f(x+y)$ for all $x, y \in X$.*

Theorem 1. *A function $f: X \to Y$ satisfies the equation (1.2) for all $x, y \in X$ if and only if there exist a cubic function $C: X \to Y$ and a quadratic function $Q: X \to Y$ such that $f(x)=C(x)+Q(x)$ for all $x \in X$.*

Proof. ($\Rightarrow$) Define the functions $C, Q: X \to Y$ by $C(x)=\frac{1}{2}[f(x)-f(-x)]$ and $Q(x)=\frac{1}{2}[f(x)+f(-x)]$ for all $x, y \in X$, respectively. Then we have $C(0)=0$, $C(-x)=-C(x)$, $Q(-x)=Q(x)$,

$$9C(x+y)-9C(x-y)+C(6y)=3C(x+3y)-3C(x-3y)+9C(2y), \tag{7}$$

and

$$9Q(x+y)-9Q(x-y)+Q(6y)=3Q(x+3y)-3Q(x-3y)+9Q(2y) \tag{8}$$

for all $x, y \in X$.

First, we claim that C is cubic. If we let $x := y$ in (7), we get

$$C(6y)=3C(4y)+3C(2y), \tag{9}$$

and replacing x by $3y$ in (7) gives

$$2C(6y)=9C(4y)-18C(2y). \tag{10}$$

We compare (9) with (10) to obtain $C(4y)=8C(2y)$, and so by putting $y := \frac{y}{2}$, we have $C(2y)=8C(y)$, from which (9) gives $C(6y)=216C(y)$.

Therefore (7) now becomes

$$C(x+3y)-C(x-3y)=3C(x+y)-3C(x-y)+48C(y),$$

and interchanging x and y yields

$$C(3x+y)+C(3x-y)=3C(x+y)+3C(x-y)+48C(x).$$

Therefore C is cubic by [12].

Secondly, we claim that Q is quadratic. By letting $x=y=0$ in (8), we get $Q(0)=0$. If we put $x=0$ in (8), we have

$$Q(6y)=9Q(2y).$$

Hence (8) can be written in the form

$$Q(x+3y)-Q(x-3y)=3Q(x+y)-3Q(x-y)$$

which implies that Q is quadratic in view of [11].

That is, if $f:X\to Y$ satisfies the equation (2), then we have $f(x)=C(x)+Q(x)$ for all $x\in X$.

($\Leftarrow$) Suppose that there exist a cubic mapping $C:X\to Y$ and a quadratic mapping $Q:X\to Y$ such that $f(x)=C(x)+Q(x)$ for all $x\in X$.

Since $C(2x)=8C(x)$, $C(3x)=27C(x)$, $Q(2x)=4Q(x)$ and $Q(3x)=9Q(x)$ for all $x\in X$, it follows from Lemma 1 and Lemma 2 that

$$\begin{aligned}
&9f(x+y)-9f(x-y)+f(6y)-3f(x+3y)+3f(x-3y)-9f(2y)\\
&=9C(x+y)-9C(x-y)+C(6y)-3C(x+3y)+3C(x-3y)-9C(2y)\\
&\quad+9Q(x+y)-9Q(x-y)+Q(6y)-3Q(x+3y)+3Q(x-3y)-9Q(2y)\\
&=-3[C(x+3y)+3C(x-y)-C(x-3y)-3C(x+y)-48C(y)]\\
&\quad-3[Q(x+3y)+3Q(x-y)-Q(x-3y)-3Q(x+y)]=0
\end{aligned}$$

for all $x,\ y\in X$. □

3. STABILITY OF EQ. (2)

In this section, X and Y will be a real normed space and a real Banach space, respectively. Given a function $f:X\to Y$, we set

$$Df(x,y):=9f(x+y)-9f(x-y)+f(6y)-3f(x+3y)+3f(x-3y)-9f(2y)$$

for all $x,\ y\in X$.

Let $\phi:X\times X\to[0,\infty)$ be a function satisfying one of the conditions (11) and (12), and one of the conditions (13) and (14) below:

$$\text{(11)}\qquad \varepsilon_1(x):=\frac{1}{24}\sum_{i=0}^{\infty}\frac{1}{8^i}\alpha(2^ix)<\infty,\quad \frac{\phi(2^nx,2^ny)}{8^n}\to 0\quad\text{as } n\to\infty,$$

(12) $\varepsilon_2(x) := \frac{1}{3}\sum_{i=0}^{\infty} 8^i \alpha(2^{-(i+1)}x) < \infty, \quad 8^n\phi(2^{-n}x, 2^{-n}y) \to 0 \qquad \text{as } n \to \infty,$

where $\alpha(x) := \phi(\frac{x}{2}, \frac{x}{2}) + \phi(-\frac{x}{2}, -\frac{x}{2}) + \frac{1}{2}[\phi(\frac{3}{2}x, \frac{x}{2}) + \phi(-\frac{3}{2}x, -\frac{x}{2})]$ for all $x,\ y \in X$, and

$$\varepsilon_3(x) := \frac{1}{12}\sum_{i=0}^{\infty} \frac{1}{4^i}\beta(2^i x) < \infty, \quad \frac{\phi(2^n x, 2^n y)}{4^n} \to 0 \text{ as } n \to \infty, \tag{13}$$

$$\varepsilon_4(x) := \frac{1}{3}\sum_{i=0}^{\infty} 4^i \beta(2^{-(i+1)}x) < \infty, \quad 4^n\phi(2^{-n}x, 2^{-n}y) \to 0 \text{ as } n \to \infty, \tag{14}$$

where $\beta(x) := \frac{1}{2}[\phi(\frac{x}{2}, \frac{x}{2}) + \phi(-\frac{x}{2}, -\frac{x}{2})] + \frac{1}{2}[\phi(0, \frac{x}{2}) + \phi(0, -\frac{x}{2})]$ for all $x,\ y \in X$.

Theorem 2. *If the function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \phi(x,y) \tag{15}$$

for all $x,\ y \in X$ and $f(0) = 0$, then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - (C(x) + Q(x))\| \le \varepsilon_k(x) + \varepsilon_j(x), \tag{16}$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le \varepsilon_k(x), \tag{17}$$

and

$$\left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\| \le \varepsilon_j(x) \tag{18}$$

for all $x \in X$, where $k = 1$ or 2 and $j = 3$ or 4.

The functions C and Q are given by

$$C(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{8^n} & \text{if } \phi \text{ satisfies (11)} \\ \lim_{n\to\infty} 8^n\left[f(\frac{x}{2^n}) - f(-\frac{x}{2^n})\right] & \text{if } \phi \text{ satisfies (12)} \end{cases} \tag{19}$$

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{4^n} & \text{if } \phi \text{ satisfies (13)} \\ 4^n\left[f(\frac{x}{2^n}) + f(-\frac{x}{2^n})\right] & \text{if } \phi \text{ satisfies (14)} \end{cases} \tag{20}$$

for all $x \in X$.

Proof. Let $g : X \to Y$ be the function defined by $g(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then we have $g(-x) = -g(x)$ and

$$\begin{aligned} \|Dg(x,y)\| &= \|9g(x+y) - 9g(x-y) + g(6y) \\ &\quad - 3g(x+3y) + 3g(x-3y) - 9g(2y)\| \\ &\le \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \end{aligned} \tag{21}$$

for all $x,\ y \in X$. Putting $y := x$ in (21) yields

$$\|g(6x) - 3g(4x) - 2g(2x)\| \le \frac{1}{2}[\phi(x,x) + \phi(-x,-x)], \tag{22}$$

and setting $x := 3y$ in (21) gives

$$\|9g(4y) - 18g(2y) - 3g(6y)\| \le \frac{1}{2}[\phi(3y,y) + \phi(-3y,-y)]$$

which, by letting $y := x$, becomes

$$\|9g(4x) - 18g(2x) - 2g(6x)\| \le \frac{1}{2}[\phi(3x,x) + \phi(-3x,-x)] \tag{23}$$

for all $x \in X$. Using (22) and (23), we get

$$\begin{aligned} \|24g(2x) - 3g(4x)\| &\le 2\|g(6x) - 3g(4x) - 3g(2x)\| \\ &\quad + \|9g(4x) - 18g(2x) - 2g(6x)\| \\ &\le \alpha(2x) \end{aligned} \tag{24}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (24) and then dividing by 24, we obtain

$$\left\| g(x) - \frac{g(2x)}{8} \right\| \le \frac{1}{24}\alpha(x) \tag{25}$$

for all $x \in X$.

Assume that ϕ satisfies the condition (11). Substituting $2x$ for x in (13) and dividing by 8, we get

$$\left\| \frac{g(2x)}{8} - \frac{g(2^2x)}{8^2} \right\| \le \frac{1}{24} \cdot \frac{1}{8}\alpha(2x) \tag{26}$$

for all $x \in X$. An induction argument now implies that

$$\left\| g(x) - \frac{g(2^n x)}{8^n} \right\| \le \frac{1}{24}\sum_{i=0}^{n-1} \frac{1}{8^i}\alpha(2^i x) \tag{27}$$

for all $x \in X$. We claim that $\{8^{-n}g(2^n x)\}$ is a Cauchy sequence in Y.

For $m < n$,

$$\|8^{-n}g(2^n x) - 8^{-m}g(2^m x)\| \le \sum_{i=m}^{n-1} \|8^{-i}g(2^i x) - 8^{-(i+1)}g(2^{i+1}x)\| \le \frac{1}{24}\sum_{i=m}^{n-1}\frac{\alpha(2^i x)}{8^i} \tag{28}$$

for all $x \in X$. Taking the limit as $m \to \infty$ in (28), we get

$$\lim_{m\to\infty} \|8^{-n}g(2^n x) - 8^{-m}g(2^m x)\| = 0$$

for all $x \in X$. Since Y is a Banach space, it follows that the sequence $\{8^{-n}g(2^n x)\}$ converges. We define a function $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty} 8^{-n}g(2^n x) \tag{29}$$

for all $x \in X$. It is clear that $C(-x) = -C(x)$ for all $x \in X$, and it follows from (29) that

$$\begin{aligned}\|DC(x,y)\| &= \lim_{n\to\infty} 8^{-n}\|Dg(2^n x, 2^n y)\| \\ &\le \lim_{n\to\infty} 8^{-n}\frac{1}{2}[\phi(2^n x, 2^n y) + \phi(2^n(-x), 2^n(-y))] = 0\end{aligned}$$

for all $x, y \in X$. Hence C is cubic. To prove the inequality (17), taking the limit in (27) as $n \to \infty$, we have

$$\|g(x) - C(x)\| \le \varepsilon_1(x) \tag{30}$$

for all $x \in X$. It remains to show that C is unique. Suppose now that $\widetilde{C} : X \to Y$ is another cubic function satisfying (30). Then it is obvious that $C(2x) = 8C(x)$ for all $x \in X$, and so it follows from (30) that

$$\begin{aligned}\|\widetilde{C}(x) - C(x)\| &= 8^{-n}\|\widetilde{C}(2^n x) - C(2^n x)\| \\ &\le 8^{-n}(\|\widetilde{C}(2^n x) - g(2^n x)\| + \|g(2^n x) - C(2^n x)\|) \\ &\le 2 \cdot 8^{-n}\varepsilon_1(2^n x)\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, we have $\widetilde{C}(x) = C(x)$ for all $x \in X$

If ϕ satisfies the condition (12), then we replace x by $\frac{x}{4}$ in (24) and divide by 3 to obtain

$$\|g(x) - 8g(2^{-1}x)\| \le \frac{1}{3}\alpha(2^{-1}x)$$

for all $x \in X$. The rest of the proof is similar to the corresponding part of the proof of the case (11). Therefore, $C : X \to Y$ is the unique cubic function defined by

$$C(x) = \lim_{n\to\infty} 8^n g(2^{-n}x)$$

for all $x \in X$ such that

$$\|g(x) - C(x)\| \le \varepsilon_2(x)$$

for all $x \in X$.

Now let $h : X \to Y$ be the function defined by $h(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then we have $h(-x) = h(x)$ and

$$\begin{aligned} \|Dh(x,y)\| &= \|9h(x+y) - 9h(x-y) + h(6y) \\ &\quad - 3h(x+3y) + 3h(x-3y) - 9h(2y)\| \\ &\le \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \end{aligned} \tag{31}$$

for all $x,\ y \in X$. By setting $x := 0$ in (31) and then letting $y := x$, we get

$$\|h(6x) - 9g(2x)\| \le \frac{1}{2}[\phi(0,x) + \phi(0,-x)], \tag{32}$$

and also replacing y by x in (31) yields

$$\|3h(4x) - 3h(2x) - h(6x)\| \le \frac{1}{2}[\phi(x,x) + \phi(-x,-x)]. \tag{33}$$

Utilizing (32) and (33), we get

$$\begin{aligned} \|3h(4x) - 12h(2x)\| &\le \|3h(4x) - 3h(2x) - h(6x)\| + \|h(6x) - 9h(2x)\| \\ &\le \beta(2x) \end{aligned} \tag{34}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (34) and then dividing by 12, we obtain

$$\left\|h(x) - \frac{h(2x)}{4}\right\| \le \frac{1}{12}\beta(x) \tag{35}$$

for all $x \in X$.

Assume that ϕ satisfies the condition (13). Substituting $2x$ for x in (35) and dividing by 4, we get

$$\left\|\frac{h(2x)}{4} - \frac{h(2^2x)}{4^2}\right\| \le \frac{1}{12}\cdot\frac{1}{4}\beta(2x)$$

for all $x \in X$. By induction we see that

$$\left\|h(x) - \frac{h(2^nx)}{4^n}\right\| \le \frac{1}{12}\sum_{i=0}^{n-1}\frac{1}{4^i}\beta(2^ix) \tag{36}$$

for all $x \in X$. We claim that $\{4^{-n}h(2^n)\}$ is a Cauchy sequence in Y.

For $m < n$,

$$\|4^{-n}h(2^n x) - 4^{-m}h(2^m x)\| \leq \sum_{i=m}^{n-1} \|4^{-i}h(2^i x) - 4^{-(i+1)}h(2^{i+1}x)\| \leq \frac{1}{12}\sum_{i=m}^{n-1}\frac{1}{4^i}\beta(2^i x) \tag{37}$$

for all $x \in X$. Taking the limit as $m \to \infty$ in (37), we get

$$\lim_{m\to\infty} \|4^{-n}h(2^n x) - 4^{-m}h(2^m x)\| = 0$$

for all $x \in X$. Since Y is a Banach space, it follows that the sequence $\{4^{-n}h(2^n x)\}$ converges. We define a function $C : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 4^{-n}h(2^n x) \tag{38}$$

for all $x \in X$. It is clear that $Q(-x) = Q(x)$ for all $x \in X$, and it follows from (38) that

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{n\to\infty} 4^{-n}\|Dh(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} 4^{-n}\frac{1}{2}[\phi(2^n x, 2^n y) + \phi(2^n(-x), 2^n(-y))] = 0\end{aligned}$$

for all $x, y \in X$. Hence Q is quadratic. By taking the limit in (36) as $n \to \infty$ to prove the inequality (18), we obtain

$$\|h(x) - Q(x)\| \leq \varepsilon_3(x) \tag{39}$$

for all $x \in X$. To show that Q is unique, let us assume that $\widetilde{Q} : X \to Y$ is another quadratic function satisfying (39). Then it is obvious that $Q(2x) = 4Q(x)$ for all $x \in X$, and so it follows from (39) that

$$\begin{aligned}\|\widetilde{Q}(x) - Q(x)\| &= 4^{-n}\|\widetilde{Q}(2^n x) - Q(2^n x)\| \\ &\leq 4^{-n}(\|\widetilde{Q}(2^n x) - h(2^n x)\| + \|h(2^n x) - Q(2^n x)\|) \\ &\leq 2\cdot 4^{-n}\varepsilon_3(2^n x)\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, we have $\widetilde{Q}(x) = Q(x)$ for all $x \in X$.

If ϕ satisfies the condition (14), then we replace x by $\frac{x}{4}$ in (34) and divide by 3 to obtain

$$\|h(x) - 4h(2^{-1}x)\| \leq \frac{1}{3}\beta(2^{-1}x)$$

for all $x \in X$. The rest of the proof goes through the corresponding part of the proof of the case (13). Consequently, we can obtain the unique quadratic function $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} 4^n h(2^{-n}x)$$

for all $x \in X$ such that

$$\|h(x) - Q(x)\| \le \varepsilon_4(x)$$

for all $x \in X$.

Since we have $f(x) = g(x) + h(x)$ for all $x \in X$, we see that

$$\begin{aligned}\|f(x) - (C(x) + Q(x))\| &\le \|g(x) - C(x)\| + \|h(x) - Q(x)\| \\ &\le \varepsilon_k(x) + \varepsilon_j(x)\end{aligned}$$

for all $x \in X$, where $k = 1$ or 2 and $j = 3$ or 4. We complete the proof of the theorem. □

From Theorem 2, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [19] of the functional equation (2).

Let $p \neq 2, 3$ be any real number. For the convenience, let

$$r_1(p) := \frac{1}{24} \cdot \frac{5 + 3^p}{2^p(1 - 2^{p-3})}, \quad r_2(p) := \frac{1}{24} \cdot \frac{5 + 3^p}{2^p(2^{p-3} - 1)},$$

and

$$r_3(p) := \frac{1}{4 \cdot 2^p(1 - 2^{p-2})}, \quad r_4(p) := \frac{1}{4^p(1 - 2^{2-p})}.$$

Corollary 1. *Let $p \neq 2, 3$ and $\theta > 0$ be real numbers. If the function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all x, $y \in X$ and $f(0) = 0$, then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - (C(x) + Q(x))\| \le r(p)\theta\|x\|^p,$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le r_k(p)\theta\|x\|^p \quad (k = 1 \ \ or \ \ 2),$$

and

$$\left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\| \le r_j(p)\theta\|x\|^p \quad (j = 3 \ \ or \ \ 4)$$

for all $x \in X$*, where*

$$r(p) = \begin{cases} r_2(p) + r_4(p) & \text{if } p > 3 \\ r_1(p) + r_4(p) & \text{if } 2 < p < 3 \\ r_1(p) + r_3(p) & \text{if } p < 2. \end{cases}$$

The functions C *and* Q *are given by*

$$C(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{8^n} & \text{if } p < 3 \\ \lim_{n\to\infty} 8^n \Big[f(\frac{x}{2^n}) - f(-\frac{x}{2^n}) \Big] & \text{if } p > 3, \end{cases}$$

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{8^n} & \text{if } p < 3 \\ \lim_{n\to\infty} 4^n \Big[f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) \Big] & \text{if } p > 2 \end{cases}$$

for all $x \in X$*.*

Proof. Let $\phi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x \in X$. If $p < 3$, then a simple calculation gives $\alpha(2^i x) = (5 + 3^p)2^{(i-1)p}\theta\|x\|^p$, and so we have

$$\varepsilon_1(x) = \frac{1}{24} \sum_{i=0}^{\infty} \frac{1}{8^i} \alpha(2^i x) = r_1(p)\theta\|x\|^p$$

for all $x \in X$. If $p > 3$, then, by considering $\alpha(2^{-(i+1)}x) = (5 + 3^p)2^{-(i+2)p}\theta\|x\|^p$, we obtain

$$\varepsilon_2(x) = \frac{1}{3} \sum_{i=0}^{\infty} 8^i \alpha(2^{-(i+1)}x) = r_2(p)\theta\|x\|^p$$

for all $x \in X$. On the other hand, suppose that $p < 2$. Since $\beta(2^i x) = 3 \cdot 2^{(i-1)p}\theta\|x\|^p$, we see that

$$\varepsilon_3(x) = \frac{1}{12} \sum_{i=0}^{\infty} \frac{1}{4^i} \beta(2^i x) = r_3(p)\theta\|x\|^p$$

for all $x \in X$. Finally, if $p > 2$, then we know that

$$\varepsilon_4(x) = \frac{1}{3} \sum_{i=0}^{\infty} 4^i \beta(2^{-(i+2)}x) = r_4(p)\theta\|x\|^p$$

because of $\beta(2^{-(i+1)}x) = 3 \cdot 2^{-(i+2)p}\theta\|x\|^p$ for all $x \in X$.

Therefore, we deduce that

$$\varepsilon_k(x) + \varepsilon_j(x) := r(p)\theta\|x\|^p = \begin{cases} (r_2(p) + r_4(p))\theta\|x\|^p & \text{if } p > 3 \\ (r_1(p) + r_4(p))\theta\|x\|^p & \text{if } 2 < p < 3 \\ (r_1(p) + r_3(p))\theta\|x\|^p & \text{if } p < 2 \end{cases}$$

for all $x \in X$. □

Corollary 2. *Let $\theta > 0$ be a real number. If the function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta$$

for all $x, y \in X$ and $f(0) = 0$, then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - (C(x) + Q(x))\| \le \frac{23}{63}\theta,$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le \frac{1}{7}\theta,$$

and

$$\left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\| \le \frac{2}{9}\theta$$

for all $x \in X$.

The functions C and Q are given by

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{8^n}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{4^n}$$

for all $x \in X$.

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