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ALTERNATING DIRECTION IMPLICIT METHOD FOR TWO-DIMENSIONAL FOKKER-PLANCK EQUATION OF DENSE SPHERICAL STELLAR SYSTEMS

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ABSTRACT

The Fokker-Planck (FP) model is one of the commonly used methods for studies of the dynamical evolution of dense spherical stellar systems such as globular clusters and galactic nuclei. The FP model is numerically stable in most cases, but we find that it encounters numerical difficulties rather often when the effects of tidal shocks are included in two-dimensional (energy and angular momentum space) version of the FP model or when the initial condition is extreme (e.g., a very large cluster mass and a small cluster radius). To avoid such a problem, we have developed a new integration scheme for a two-dimensional FP equation by adopting an Alternating Direction Implicit (ADI) method given in the Douglas-Rachford split form. We find that our ADI method reduces the computing time by a factor of ~ 2 compared to the fully implicit method, and resolves problems of numerical instability.

Key Words: stellar dynamics — methods:numerical — globular clusters:general

I. INTRODUCTION

The Fokker-Planck (FP) model is a statistical way of describing the time evolution of a probability density function under the effects of drift and diffusion. One of the first uses of FP model in stellar dynamics was by Cohn (1979), who developed a numerical method that directly integrates the two-dimensional (2D) FP equation in energy-angular momentum (E, J) space targeted for dense spherical stellar systems such as globular clusters. But this pioneering attempt suffered a non-negligible numerical problem with the energy conservation, and to eliminate this problem, Cohn (1980) developed a one-dimensional FP model in energy space with an assumption that the velocity distribution of stars in the cluster is isotropic (i.e., the distribution function can be described by energy only). He adopted the finite-difference scheme by Chang and Cooper (1970) and was able to greatly reduce the numerical errors.

As the computing power of workstations greatly increased in 1990's, it became feasible to integrate the 2D-FP equation with a relatively large number of grids on workstations. Takahashi (1995) challenged the 2D-FP model again with the help of the increased computing power, and successfully developed a numerically reliable method for 2D-FP equation by adopting the Chang & Cooper finite-difference scheme. Takahashi found that this scheme greatly reduced the numerical errors when applied to the energy dimension, but it was not so effective when applied to the angular momentum space. Thus he applied the scheme only to the energy space. Although his model was not a full 2D generalization of the Chang & Cooper scheme, it significantly reduced the numerical errors on the energy conservation compared to the 2D model by Cohn (1979).

The 2D-FP model developed by Takahashi (1995) reliably calculates the dynamical evolution of dense stellar systems in most cases, but we find that it encounters a numerical problem when the effect of tidal shocks (disk and/or bulge shocks) is added to the model or when the initial condition is extreme (e.g., a very large cluster mass and a small cluster radius). The Chang & Cooper scheme adopted by Takahashi is an implicit finite-difference method, which involves solving a large matrix. A matrix inversion is a numerically challenging task particularly when the magnitude range of the numbers in the matrix is large, and in such a case, the inversion can result

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in significantly inaccurate answers. We find that when the effect of tidal shocks is added, Takahashi's model encounters a numerical problem in the matrix inversion part of the implicit scheme and results in a partially negative distribution function.

In the present paper, we develop a new numerical method for the 2D-FP equation by adopting an Alternating Direction Implicit (ADI) method, instead of the fully implicit method as in the Chang & Cooper's scheme, to overcome the forementioned numerical problem. Fokker-Planck models are advantageous over N-body simulations particularly when studying the dynamical evolution of a system of globular clusters or galactic nuclei, and the method presented here will be useful as a numerically efficient and stable tool for such studies.

We briefly introduce the formulation of the 2D-FP equation and show its finite-difference expressions in implicit and explicit fashions in Section 2. We present and discuss the ADI formulation of the 2D-FP equation in Section 3, and summarize our findings in Section 4.

II. TWO-DIMENSIONAL FOKKER-PLANCK EQUATION

Here we briefly introduce the formulation of the 2D-FP equation following the discription by Takahashi (1995). In a steady-state spherical system, a distribution function $f(\vec{r}, \vec{v}, t)$ with velocity space $\vec{v}$, volume space $\vec{r}$ at time t is a function of only energy E and angular momentum J per unit mass, and it evolves only due to collisional effects. The evolution of f can be described by the orbit-averaged FP equation in (E,J)-space, because the relaxation (or diffusion) time scale is much longer than the dynamical time scale. The scaled angular momentum R is often used instead of J as a basic variable, and is defined as $J^2/J_c^2(E)$ where $J_c(E)$ is the angular momentum of a circular orbit of energy E. The number density $N(E,R)$ in (E, R)-space is given by

$$\begin{aligned} N(E,R) &= 4\pi^2 P(E,R) J_c^2(E) f(E,R), \\ &\equiv A(E,R) f(E,R), \end{aligned} \tag{1}$$

where $P(E,R)$ is the orbital period. When the gravitational potential is fixed, the 2D-FP equation can be written in a flux-conserving form (Cohn 1979) such that

$$A\frac{\partial f}{\partial t} = -\frac{\partial F_E}{\partial E} - \frac{\partial F_R}{\partial R}, \tag{2}$$

where

$$\begin{aligned} -F_E &= D_{EE}\frac{\partial f}{\partial E} + D_{ER}\frac{\partial f}{\partial R} + D_E f \\ -F_R &= D_{RE}\frac{\partial f}{\partial E} + D_{RR}\frac{\partial f}{\partial R} + D_R f, \end{aligned} \tag{3}$$

and the expressions for the diffusion coefficients D's are given in Appendix C of Cohn (1979).

The implicit version of the finite-difference formulation for the above 2D-FP equation can be written as

$$A^n_{i,j}\frac{f^{n+1}_{i,j} - f^n_{i,j}}{\Delta t} = -\frac{\tilde{F}_{x\,i+\frac{1}{2},j} - \tilde{F}_{x\,i-\frac{1}{2},j}}{\Delta x} - \frac{\tilde{F}_{y\,i,j+\frac{1}{2}} - \tilde{F}_{y\,i,j-\frac{1}{2}}}{\Delta y}, \tag{4}$$

where

$$\begin{aligned} \tilde{F}_{x\,i+\frac{1}{2},j} &= -D^n_{x\,i+\frac{1}{2},j}\tilde{f}_{i+\frac{1}{2},j} - D^n_{xx\,i+\frac{1}{2},j}\frac{\tilde{f}_{i+1,j} - \tilde{f}_{i,j}}{\Delta x} - D^n_{xy\,i+\frac{1}{2},j}\frac{\tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} - \tilde{f}_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} \\ \tilde{F}_{x\,i-\frac{1}{2},j} &= -D^n_{x\,i-\frac{1}{2},j}\tilde{f}_{i-\frac{1}{2},j} - D^n_{xx\,i-\frac{1}{2},j}\frac{\tilde{f}_{i,j} - \tilde{f}_{i-1,j}}{\Delta x} - D^n_{xy\,i-\frac{1}{2},j}\frac{\tilde{f}_{i-\frac{1}{2},j+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta y} \\ \tilde{F}_{y\,i,j+\frac{1}{2}} &= -D^n_{y\,i,j+\frac{1}{2}}\tilde{f}_{i,j+\frac{1}{2}} - D^n_{yy\,i,j+\frac{1}{2}}\frac{\tilde{f}_{i,j+1} - \tilde{f}_{i,j}}{\Delta y} - D^n_{yx\,i,j+\frac{1}{2}}\frac{\tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} \\ \tilde{F}_{y\,i,j-\frac{1}{2}} &= -D^n_{y\,i,j-\frac{1}{2}}\tilde{f}_{i,j-\frac{1}{2}} - D^n_{yy\,i,j-\frac{1}{2}}\frac{\tilde{f}_{i,j} - \tilde{f}_{i,j-1}}{\Delta y} - D^n_{yx\,i,j-\frac{1}{2}}\frac{\tilde{f}_{i+\frac{1}{2},j-\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}. \end{aligned} \tag{5}$$

Here, $f^n_{i,j}$ is the distribution function at the energy and angular momentum mesh of index (i,j) in the n-th time step, and Δx, Δy, and Δt are the intervals of energy mesh, angular momentum mesh, and time, respectively. Takahashi (1995) adopts the Crank-Nicolson scheme for the time advance and the cross terms, i.e.

$$\begin{aligned} \tilde{f}_{i,j} &= \frac{1}{2}(f^n_{i,j} + f^{n+1}_{i,j}) \\ f_{i\pm\frac{1}{2},j\pm\frac{1}{2}} &= \frac{1}{4}(f_{i,j} + f_{i\pm1,j} + f_{i,j\pm1} + f_{i\pm1,j\pm1}), \end{aligned} \tag{6}$$

while he adopts the Chang & Cooper scheme for the energy dimension such that

$$\begin{aligned} f_{i+\frac{1}{2},j} &= \delta_{x\,i,j} f_{i,j} + (1-\delta_{x\,i,j}) f_{i+1,j} \\ \delta_{x\,i,j} &= \frac{1}{w_{x\,i,j}} - \frac{1}{\exp(w_{x\,i,j}) - 1} \\ w_{x\,i,j} &= \Delta x \frac{D_{x\,i+\frac{1}{2},j}}{D_{xx\,i+\frac{1}{2},j}}. \end{aligned} \tag{7}$$

For the angular momentum dimension, δ_y is always set to be 0.5.

Rearranging equation (4) for $f^{n+1}_{i,j}$ results in

$$\Sigma^{+1}_{l=-1}\Sigma^{+1}_{m=-1} B_{i+l,j+m} f^{n+1}_{i+l,j+m} + \frac{A^n_{i,j}}{\Delta t} B_{i,j} f^{n+1}_{i,j} = -\Sigma^{+1}_{l=-1}\Sigma^{+1}_{m=-1} B_{i+l,j+m} f^n_{i+l,j+m} + \frac{A^n_{i,j}}{\Delta t} B_{i,j} f^n_{i,j}, \tag{8}$$

where

$$\begin{aligned} B_{i-1,j-1} &= -\frac{D^n_{xy\,i-\frac{1}{2},j} + D^n_{yx\,i,j-\frac{1}{2}}}{4\Delta x \Delta y} \\ B_{i-1,j} &= \frac{\delta_{x\,i-1,j} D^n_{x\,i-\frac{1}{2},j}}{\Delta x} - \frac{D^n_{xx\,i-\frac{1}{2},j}}{\Delta x \Delta x} - \frac{D^n_{yx\,i,j-\frac{1}{2}} - D^n_{yx\,i,j+\frac{1}{2}}}{4\Delta x \Delta y} \\ B_{i-1,j+1} &= \frac{D^n_{xy\,i-\frac{1}{2},j} + D^n_{yx\,i,j+\frac{1}{2}}}{4\Delta x \Delta y} \\ B_{i,j-1} &= \frac{\frac{1}{2} D^n_{y\,i,j-\frac{1}{2}}}{\Delta y} - \frac{D^n_{yy\,i,j-\frac{1}{2}}}{\Delta y \Delta y} - \frac{D^n_{xy\,i-\frac{1}{2},j} - D^n_{xy\,i+\frac{1}{2},j}}{4\Delta x \Delta y} \\ B_{i,j} &= \frac{(1-\delta_{x\,i-1,j}) D^n_{x\,i-\frac{1}{2},j} - \delta_{x\,i,j} D^n_{x\,i+\frac{1}{2},j}}{\Delta x} + \frac{\frac{1}{2} D^n_{y\,i,j-\frac{1}{2}} - \frac{1}{2} D^n_{y\,i,j+\frac{1}{2}}}{\Delta y} \\ &\quad + \frac{D^n_{xx\,i-\frac{1}{2},j} + D^n_{xx\,i+\frac{1}{2},j}}{\Delta x \Delta x} + \frac{D^n_{yy\,i,j-\frac{1}{2}} + D^n_{yy\,i,j+\frac{1}{2}}}{\Delta y \Delta y} \\ B_{i,j+1} &= -\frac{\frac{1}{2} D^n_{y\,i,j+\frac{1}{2}}}{\Delta y} - \frac{D^n_{yy\,i,j+\frac{1}{2}}}{\Delta y \Delta y} + \frac{D^n_{xy\,i-\frac{1}{2},j} - D^n_{xy\,i+\frac{1}{2},j}}{4\Delta x \Delta y} \\ B_{i+1,j-1} &= \frac{D^n_{xy\,i+\frac{1}{2},j} + D^n_{yx\,i,j-\frac{1}{2}}}{4\Delta x \Delta y} \\ B_{i+1,j} &= -\frac{(1-\delta_{x\,i,j}) D^n_{x\,i+\frac{1}{2},j}}{\Delta x} - \frac{D^n_{xx\,i+\frac{1}{2},j}}{\Delta x \Delta x} + \frac{D^n_{yx\,i,j-\frac{1}{2}} - D^n_{yx\,i,j+\frac{1}{2}}}{4\Delta x \Delta y} \\ B_{i+1,j+1} &= -\frac{D^n_{xy\,i+\frac{1}{2},j} + D^n_{yx\,i,j+\frac{1}{2}}}{4\Delta x \Delta y}. \end{aligned} \tag{9}$$

Equation (8) forms a set of linear equations and its solution can be obtained by inverting the matrix whose components are $B_{i,j}$. Because every component of $B_{i,j}$ is non-zero in general, and because the size of the matrix easily goes over 50 in each dimension, solving equation (8) becomes a numerically challenging task.

On the other hand, the explicit version of the finite-difference formulation for the 2D-FP equation can be written as

$$A^n_{i,j}\frac{f^{n+1}_{i,j}-f^n_{i,j}}{\Delta t} = -\frac{F^n_{x\,i+\frac{1}{2},j}-F^n_{x\,i-\frac{1}{2},j}}{\Delta x} - \frac{F^n_{y\,i,j+\frac{1}{2}}-F^n_{x\,i,j-\frac{1}{2}}}{\Delta y}, \tag{10}$$

where

$$F^n_{x\,i+\frac{1}{2},j} = -D^n_{x\,i+\frac{1}{2},j}f^n_{i+\frac{1}{2},j} - D^n_{xx\,i+\frac{1}{2},j}\frac{f^n_{i+1,j}-f^n_{i,j}}{\Delta x} - D^n_{xy\,i+\frac{1}{2},j}\frac{f^n_{i+\frac{1}{2},j+\frac{1}{2}}-f^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}, \tag{11}$$

and $F^n_{x\,i-\frac{1}{2},j}$, $F^n_{x\,i,j+\frac{1}{2}}$, and $F^n_{x\,i,j-\frac{1}{2}}$ are similarly defined. Since the $f^{n+1}_{i,j}$ term appears only once in the above formulation, an inversion of a matrix is not involved in obtaining the solution at the next step.

III. ALTERNATING DIRECTION IMPLICIT METHOD

As shown in Section 2, implicit finite-difference methods obtain the solution for the next time step from the state of both current and next time steps, while explicit methods obtain the solution from the state of the current time step only. Implicit methods require more computations per step but they can implement longer time intervals without suffering numerical instabilities (note that, however, implicit methods are stable for one-dimensional problems, but not necessarily for multi-dimensional problems). Implicit methods are preferred in most cases because of this benefit, but they involve the inversion of a matrix, which can be numerically problematic in some cases. When such a problem is encountered, one could implement an explicit method instead, but explicit methods require much smaller time intervals than an implicit method to avoid numerical instabilities. We find that the required small time intervals greatly increase the computing time to the degree that the merit of the FP model over direct N-body simulations is lost.

The ADI method is a finite-difference method for solving differential equations in two or more dimensions. For a 2D problem, the ADI method solves the first dimension implicitly and the second dimension explicitly, and in the next step the first dimension explicitly and the second dimension implicitly, and so on. This method is unconditionally stable, and since it applies the implicit scheme to one dimension at a time, the non-zero terms are present only in the three diagonal lines in the matrix, which is considerably simpler to solve compared to the matrix created by the fully implicit method (such as the Chang & Cooper method) in 2D.

In the present paper, we develop an ADI-type finite difference method for solving a 2D-FP equation for dense spherical stellar systems. We adopt an ADI scheme in Douglas & Rachford (1956) split form and write the finite-difference formulation such that

$$\begin{aligned}
\left[\frac{A^n_{i,j}}{\Delta t} - \frac{\delta^2_x}{2(\Delta x)^2} + \frac{\nabla_x}{2\Delta x}\right] f^{n+1^*}_{i,j} &= \left[\frac{A^n_{i,j}}{\Delta t} + \frac{\delta^2_y}{(\Delta y)^2} - \frac{\nabla_y}{\Delta y} + \frac{\delta^2_x}{2(\Delta x)^2} - \frac{\nabla_x}{2\Delta x} + \frac{\delta_{xy}}{\Delta x\Delta y}\right] f^n_{i,j} \\
\left[\frac{A^n_{i,j}}{\Delta t} - \frac{\delta^2_y}{2(\Delta y)^2} + \frac{\nabla_y}{2\Delta y}\right] f^{n+1}_{i,j} &= \frac{A^n_{i,j}}{\Delta t} f^{n+1^*}_{i,j} - \left[\frac{\delta^2_y}{2(\Delta y)^2} - \frac{\nabla_y}{2\Delta y}\right] f^n_{i,j},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\delta^2_x f^n_{i,j} &= D^n_{xx\,i+\frac{1}{2},j}(f^n_{i+1,j}-f^n_{i,j}) - D^n_{xx\,i-\frac{1}{2},j}(f^n_{i,j}-f^n_{i-1,j}) \\
\delta^2_y f^n_{i,j} &= D^n_{yy\,i,j+\frac{1}{2}}(f^n_{i,j+1}-f^n_{i,j}) - D^n_{yy\,i,j-\frac{1}{2}}(f^n_{i,j}-f^n_{i,j-1}) \\
\delta_{xy} f^n_{i,j} &= \frac{1}{4}[D^n_{xy\,i+\frac{1}{2},j}(f^n_{i+\frac{1}{2},j+\frac{1}{2}}-f^n_{i+\frac{1}{2},j-\frac{1}{2}}) - D^n_{xy\,i-\frac{1}{2},j}(f^n_{i-\frac{1}{2},j+\frac{1}{2}}-f^n_{i-\frac{1}{2},j-\frac{1}{2}}) \\
&\quad + D^n_{yx\,i,j+\frac{1}{2}}(f^n_{i+\frac{1}{2},j+\frac{1}{2}}-f^n_{i-\frac{1}{2},j+\frac{1}{2}}) - D^n_{yx\,i,j-\frac{1}{2}}(f^n_{i+\frac{1}{2},j-\frac{1}{2}}-f^n_{i-\frac{1}{2},j-\frac{1}{2}})] \\
\nabla_x f^n_{i,j} &= \frac{1}{2}[D^n_{x\,i+\frac{1}{2},j}(f^n_{i+1,j}+f^n_{i,j}) - D^n_{x\,i-\frac{1}{2},j}(f^n_{i,j}+f^n_{i-1,j})] \\
\nabla_y f^n_{i,j} &= \frac{1}{2}[D^n_{y\,i,j+\frac{1}{2}}(f^n_{i,j+1}+f^n_{i,j}) - D^n_{y\,i,j-\frac{1}{2}}(f^n_{i,j}+f^n_{i,j-1})].
\end{aligned} \tag{13}$$

Here, $f^n_{i+\frac{1}{2},j+\frac{1}{2}}$ and similar expressions are the distribution functions at the center of the four nearby mesh points. For example,

$$f^n_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{4}(f^n_{i,j} + f^n_{i+1,j} + f^n_{i,j+1} + f^n_{i+1,j+1}). \tag{14}$$

For the boundary conditions, we impose $D = 0$ at the boundary meshes. An example of the boundary conditions at $i = 1$ (the first mesh point in the energy dimension) is

$$\begin{aligned}
\delta^2_x f^n_{1,j} &= D^n_{xx\,1+\frac{1}{2},j}(f^n_{2,j} - f^n_{1,j}) \\
\delta^2_y f^n_{1,j} &= D^n_{yy\,1,j+\frac{1}{2}}(f^n_{1,j+1} - f^n_{1,j}) - D^n_{yy\,1,j-\frac{1}{2}}(f^n_{1,j} - f^n_{1,j-1}) \\
\delta_{xy} f^n_{1,j} &= \frac{1}{4}D^n_{xy\,1+\frac{1}{2},j}(f^n_{1+\frac{1}{2},j+\frac{1}{2}} - f^n_{1+\frac{1}{2},j-\frac{1}{2}}) \\
\nabla_x f^n_{1,j} &= \frac{1}{2}D^n_{x\,1+\frac{1}{2},j}(f^n_{2,j} + f^n_{1,j}) \\
\nabla_y f^n_{1,j} &= \frac{1}{2}[D^n_{y\,1,j+\frac{1}{2}}(f^n_{1,j+1} + f^n_{1,j}) - D^n_{y\,1,j-\frac{1}{2}}(f^n_{1,j} + f^n_{1,j-1})].
\end{aligned} \tag{15}$$

The implicit scheme is first applied to the E-direction to obtain $f^{n+1^*}_{i,j}$, then applied to the R-direction to obtain the solution at the next step, $f^{n+1}_{i,j}$, with the information of $f^{n+1^*}_{i,j}$. Solving for $f^{n+1^*}_{i,j}$ and $f^{n+1}_{i,j}$ each requires an inversion of a tridiagonal matrix, which is a numerically straightforward task with only minimal numerical errors. We find that our ADI method requires $\sim$ 50 % less computing time than the fully implicit method by Takahashi (1995) when mesh points of 181, 51, and 151 are used for energy, angular momentum, and radial meshes, respectively.

More importantly, our ADI method perfectly prevents numerical problems encountered by the fully implicit method. We performed 2D-FP calculations for 578 different initial conditions (different cluster masses, galactocentric radii, orbit eccentricities, and orbit inclnations relative to the galactic plane) of globular clusters with the effects of stellar evolution, binary heating, disk/bulge shocks, realistic orbital motions, and dynamical friction using both implicit and ADI methods (the results of these calculations are to be reported elsewhere). We adopted the 2D-FP model by Takahashi et al. (1997) and modified it for tidal binary heating, realistic cluster orbit, dynamical friction, and disk/bulge shocks. For disk/bulge shocks, we adopted the recipes for the heating in energy dimension by Gnedin et al. (1999a,b) and extended them for the energy-angular momentum space (this extention will be reported elsewhere). The original 2D-FP model by Takahashi et al. implements an implicit method (Chang & Cooper scheme) for integrating the FP equation, and we modified their model so that it can implement our ADI method instead of the implicit method as an option. We find that $\sim$ 70 % of the calculations performed with the implicit method (Chang & Cooper scheme) encountered numerical problems (negative distribution functions or crashes during the matrix inversion) when the effects of disk/bulge shocks are included in the calculation. The disk/bulge shocks heat the stars near the tidal boundary the most (see Fig. 1), and it appears that inverting the matrix created by the implicit formulism becomes numerically difficult when the stars near the tidal boundary are heated significantly enough. When the effects of disk/bulge shocks are not included, less than 10 % of the calculations encountered numerical problems, and these happen mostly for clusters with a very large initial mass and/or a small initial radius. On the other hand, none of the calculations performed with our ADI method encountered such problems. This clearly shows that our ADI method not only reduces the computing time but also resolves numerical problems involved in the fully implicit finite-difference method for the 2D-FP equation of dense spherical stellar systems.

As an example, Fig. 2 compares the distribution functions calculated with the ADI and implicit methods at the epoch when the implicit method encounters a numerical problem in one of the 578 calculations discussed above. The distribution function for the next time step obtained with the implicit method has mostly negative values and more importantly, it is significantly different from that of the current step. This indicates that the matrix created by the implicit method is numerically challenging and the matrix inversion results in a considerably incorrect answer. On the other hand, the solution obtained with our ADI method is very close to the value at the current time step and does not have negative values, implying that the ADI method is numerically stable.

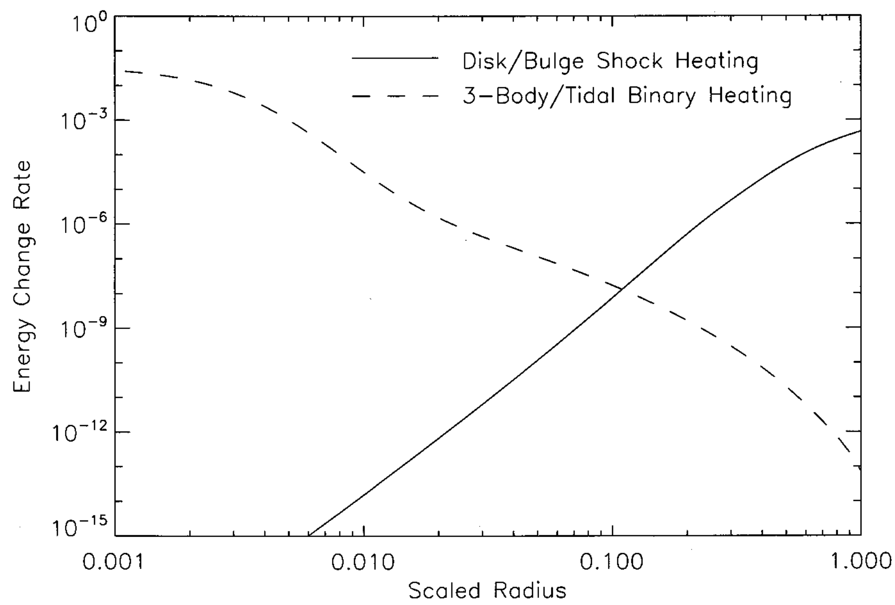

Fig. 1.— Phase-averaged first-order energy change rates $\langle \Delta E \rangle$ by the disk and bulge shocks (solid) and by the three-body and tidal binary heatings (dashed) at the epoch when the implicit method encounters a numerical problem ($t = 8.86 t_{relax}$; t_{relax} is the initial half-mass relaxation time) in one of our 578 calculations for globular clusters with the effects of disk/bulge shocks. Energy change rates are in arbitrary units and the radius is in units of the tidal radius. While the binaries preferentially heat the core of the cluster because of the high density there, the shocks preferentially heat the outskirt of the cluster because the tidal force by the external gravitational field is proportional to the distance from the cluster center.

IV. SUMMARY

We have developed a new integration method for the 2D-FP equation of dense spherical stellar systems by adopting an ADI finite-difference scheme. This method shortens the computing time by a factor of 2 compared to the implicit method, and does not encounter numerical problems such as negative distribution functions or crashes during the matrix inversion that implicit methods suffer when the effects of disk/bulge shocks are included in the calculation or when extreme initial conditions such as very high cluster masses and/or small cluster radii are used. Disk/bulge shocks heat the stars near the tidal boundary of the cluster the most, and it appears that inverting the matrix created by the implicit formulism becomes numerically difficult when the stars near the tidal boundary are heated significantly enough. The ADI method applies the implicit scheme to one dimension of the distribution function at a time and it only needs to solve two tridiagonal matrices each time step, which is a numerically straightforward task. We find that this merit of the ADI method effectively removes the problems involved with the implicit methods such as the Chang & Cooper scheme.

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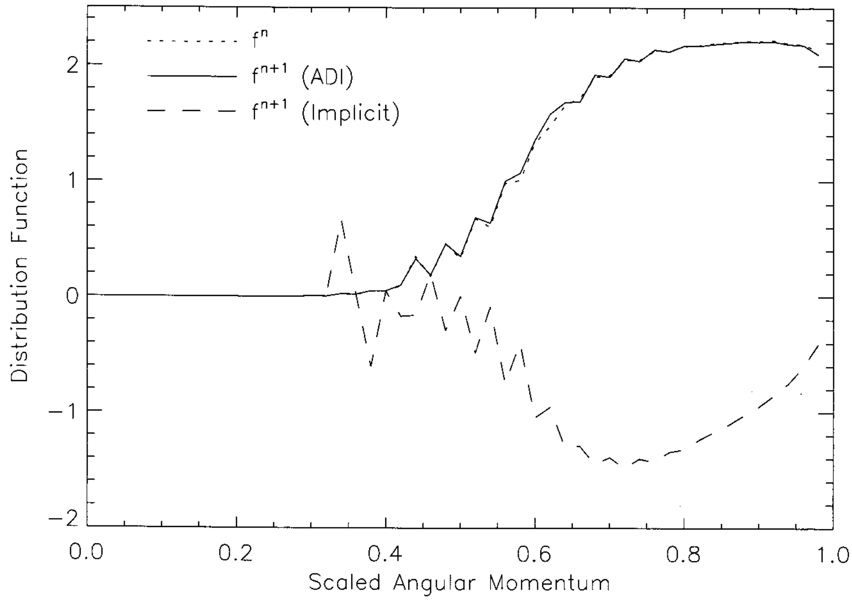

Fig. 2.— Distribution functions of the current step (dotted) and the next step by our ADI method (solid) and by an implicit method (dashed) at $i = 150$ for the calculation shown in Fig. 1. The mesh point $i = 150$ is the 31st-smallest energy mesh among a total of 181 energy meshes. The distribution functions are in arbitrary units.

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