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A CHANGE OF SCALE FORMULA FOR CONDITIONAL WIENER INTEGRALS ON CLASSICAL WIENER SPACE

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Abstract. Let $X_k(x) = (\int_0^T \alpha_1(s)dx(s), \ldots, \int_0^T \alpha_k(s)dx(s))$ and $X_\tau(x) = (x(t_1), \ldots, x(t_k))$ on the classical Wiener space, where $\{\alpha_1, \ldots, \alpha_k\}$ is an orthonormal subset of $L_2[0,T]$ and $\tau : 0 < t_1 < \cdots < t_k = T$ is a partition of $[0,T]$.

In this paper, we establish a change of scale formula for conditional Wiener integrals $E[G_r|X_k]$ of functions on classical Wiener space having the form

$$G_r(x) = F(x)\Psi\Big(\int_0^T v_1(s)dx(s), \ldots, \int_0^T v_r(s)dx(s)\Big),$$

for $F \in \mathcal{S}$ and $\Psi = \psi + \phi\,(\psi \in L_p(\mathbb{R}^r),\ \phi \in \hat{\mathrm{M}}(\mathbb{R}^r))$, which need not be bounded or continuous. Here $\mathcal{S}$ is a Banach algebra on classical Wiener space and $\hat{\mathrm{M}}(\mathbb{R}^r)$ is the space of Fourier transforms of measures of bounded variation over $\mathbb{R}^r$. As results of the formula, we derive a change of scale formula for the conditional Wiener integrals $E[G_r|X_\tau]$ and $E[F|X_\tau]$. Finally, we show that the analytic Feynman integral of F can be expressed as a limit of a change of scale transformation of the conditional Wiener integral of F using an inversion formula which changes the conditional Wiener integral of F to an ordinary Wiener integral of F, and then we obtain another type of change of scale formula for Wiener integrals of F.

1. Introduction and preliminaries

It is well-known that the classical Wiener space $C_0[0,T]$ is the space of real-valued continuous functions on $[0,T]$ which vanish at 0. As mentioned in [12], Wiener measure and Wiener measurability behave badly under change of scale transformation and under translation ([1, 2]). Various kinds of the change of scale formulas for Wiener integrals of bounded functions were developed on the classical and abstract Wiener spaces $\mathbb{B}$ ([5, 10, 11, 13]). But, in [12],

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Chang, Kim, Song and Yoo established a change of scale formula for the Wiener integrals of functions on abstract Wiener space which have the form

$$F_1(x) = G(x)\Psi((e_1, x)^\sim, \dots, (e_n, x)^\sim)$$

for $G \in \mathcal{F}(\mathbb{B})$, the Fresnel class ([6]) and $\Psi = \psi + \phi$, where $\psi \in L_p(\mathbb{R}^n), 1 \le p < \infty$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^n$.

On the other hand, in [9], Park and Skoug introduced a simple formula for conditional Wiener integrals which evaluate the conditional Wiener integral of a function given X_τ as a Wiener integral of the function and in [7], using the formula, they expressed the analytic Feynman integral of the functions in Cameron and Storvick's Banach algebra $\mathcal{S}$ ([3]) as an integral of the conditional analytic Feynman integral of the functions. Further, in [8], they extended the simple formula with more generalized conditioning function X_k and then, evaluated the conditional Wiener integrals of various functions.

In this paper, under the conditioning function X_k, we derive a change of scale formula for the conditional Wiener integrals of possibly unbounded functions on classical Wiener space which have the form

$$G_r(x) = F(x)\Psi\left(\int_0^T v_1(s)dx(s), \dots, \int_0^T v_r(s)dx(s)\right)$$

for $F \in \mathcal{S}$ and $\Psi = \psi + \phi$, where $\psi \in L_p(\mathbb{R}^r), 1 \le p \le \infty$, and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. Note that r is a positive integer and the stochastic integrals mean the Paley-Wiener-Zygmund integrals with an orthonormal subset $\{v_1, \dots, v_r\}$ of $L_2[0,T]$. As corollaries of the formula, we derive a change of scale formula for the conditional Wiener integral of G_r with the conditioning function X_τ. Finally, under the conditioning functions X_k and X_τ, we show that the analytic Feynman integral of F can be expressed as a limit of a change of scale transformation of the conditional Wiener integral of F using an inversion formula which changes the conditional Wiener integral of F to an ordinary Wiener integral of F.

Let $C_0[0,T]$ be the classical Wiener space with the Wiener measure m, and let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the Paley-Wiener-Zygmund integral and the inner product on the real Hilbert space $L_2[0,T]$, respectively. Let k be an arbitrary positive integer, but fixed, and let $\{\alpha_1, \dots, \alpha_k\}$ be an orthonormal subset of $L_2[0,T]$. Define $X_k : C_0[0,T] \to \mathbb{R}^k$ by

$$X_k(x) = ((\alpha_1, x), \dots, (\alpha_k, x)) \tag{1}$$

for $x \in C_0[0,T]$ and let $Z_j(t) = \int_0^t \alpha_j(s)ds$ on $[0,T]$ for $j = 1, \dots, k$. Further, for $v \in L_2[0,T]$, let

$$\mathcal{P}_k v = \sum_{j=1}^k (v, Z_j)\alpha_j \tag{2}$$

be the orthogonal projection from $L_2[0,T]$ onto the subspace generated by $\{\alpha_1, \ldots, \alpha_k\}$, and let $w_k(v) = v - \mathcal{P}_k v$ which is orthogonal to each α_j. For $x \in C_0[0,T]$ and $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k$, let

$$x_k(t) = \sum_{j=1}^{k} (\alpha_j, x) Z_j(t) \tag{3}$$

and

$$\vec{\xi}_k(t) = \sum_{j=1}^{k} \xi_j \langle \alpha_j, I_{[0,t]} \rangle = \sum_{j=1}^{k} \xi_j Z_j(t), \tag{4}$$

where $I_{[0,t]}$ is the indicator function of the interval $[0,t]$.

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and the set of complex numbers with positive real parts, respectively.

Let $F : C_0[0,T] \to \mathbb{C}$ be integrable and let X be a random vector on $C_0[0,T]$. Then, we have the conditional expectation $E[F|X]$ given X from a well-known probability theory. Further, there exists a P_X-integrable function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for m-a.e. $x \in C_0[0,T]$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener integral of F given X and it is also denoted by $E[F|X]$.

Lemma 1 ([8, Theorem 2]). *Let F be integrable on $C_0[0,T]$ and let X_k be given by* (1). *Then we have*

$$E[F|X_k](\vec{\xi}) = E[F(x - x_k + \vec{\xi}_k)]$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$, where x_k and $\vec{\xi}_k$ are given by (3) *and* (4), *respectively.*

A subset N of $C_0[0,T]$ is called a scale-invariant null set if $m(\lambda N) = 0$ for any $\lambda > 0$. A property is said to hold scale-invariant almost everywhere (in abbreviation, s-a.e.) if it holds except for a scale-invariant null set. For a function F defined on $C_0[0,T]$ and for $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-1/2} x)$.

Now, suppose that $E[F^\lambda | X_k^\lambda]$ exists. From Lemma 1, we have

$$E[F^\lambda | X_k^\lambda](\vec{\xi}) = E[F(\lambda^{-1/2}(x - x_k) + \vec{\xi}_k)] \tag{5}$$

for $P_{X_k^\lambda}$-a.e. $\vec{\xi} \in \mathbb{R}^k$, where $P_{X_k^\lambda}$ is the probability distribution of X_k^λ on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. If $E[F(\lambda^{-1/2}(x - x_k) + \vec{\xi}_k)]$ has the analytic extension $J_\lambda^*(F)(\vec{\xi})$ on $\mathbb{C}_+$ as a function of λ, then we call $J_\lambda^*(F)(\vec{\xi})$ the conditional analytic Wiener integral of F given X_k over $C_0[0,T]$ with parameter λ and write

$$E^{anw_\lambda}[F|X_k](\vec{\xi}) = J_\lambda^*(F)(\vec{\xi}).$$

Moreover, for a non-zero real q, if the limit

$$\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_k](\vec{\xi})$$

exists, where λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman integral of F given X_k over $C_0[0,T]$ with parameter q and we write

$$E^{anf_q}[F|X_k](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_k](\vec{\xi}).$$

2. Conditional analytic Wiener and Feynman integrals

Let $\{v_1, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,T]$. For $1 \le p < \infty$, let $\mathcal{A}_r^{(p)}$ be the space of all cylinder functions F_r on $C_0[0,T]$ of the form

$$F_r(x) = f((v_1, x), \ldots, (v_r, x)) \tag{6}$$

for s-a.e. x in $C_0[0,T]$, where $f : \mathbb{R}^r \to \mathbb{R}$ is in $L_p(\mathbb{R}^r)$. Let $\mathcal{A}_r^{(\infty)}$ be the space of all functions of the form (6) with $f \in L_\infty(\mathbb{R}^r)$, the space of essentially bounded functions on $\mathbb{R}^r$. Note that, without loss of generality, we can take f to be Borel measurable.

Let $\mathcal{M} = \mathcal{M}(L_2[0,T])$ be the class of all $\mathbb{C}$-valued Borel measures on $L_2[0,T]$ with bounded variation and let $\mathcal{S}$ be the space of all s-equivalence classes of functions F which for $\sigma \in \mathcal{M}$ has the form

$$F(x) = \int_{L_2[0,T]} \exp\{i(v, x)\} d\sigma(v) \tag{7}$$

for $x \in C_0[0,T]$. Note that $\mathcal{S}$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ ([3]).

Now, let $\{u_1 - \mathcal{P}_k u_1, \ldots, u_{r'} - \mathcal{P}_k u_{r'}\}$ be a maximal independent subset of $\{v_1 - \mathcal{P}_k v_1, \ldots, v_r - \mathcal{P}_k v_r\}$ with $r' \le r$ if it exists, where $\mathcal{P}_k$ is the orthogonal projection given by (2). Let $\{e_1, \ldots, e_{r'}\}$ be the orthonormal set obtained from $\{u_1 - \mathcal{P}_k u_1, \ldots, u_{r'} - \mathcal{P}_k u_{r'}\}$ using Gram-Schmidt orthonormalization process. For convenience, we introduce useful notations from the process. For $v \in L_2[0,T]$, we obtain an orthonormal set $\{e_1, \ldots, e_{r'}, e_{r'+1}\}$ as follows;

$$c_j(v) = \begin{cases} \langle v, e_j \rangle & \text{for } j = 1, \ldots, r' \\ \sqrt{\|v\|_2^2 - \sum_{l=1}^{r'} \langle v, e_l \rangle^2} & \text{for } j = r'+1 \end{cases} \tag{8}$$

and

$$e_{r'+1} = \frac{1}{c_{r'+1}(v)} \left[v - \sum_{j=1}^{r'} c_j(v) e_j \right]$$

if $c_{r'+1}(v) \neq 0$. Then we have

$$v = \sum_{j=1}^{r'+1} c_j(v) e_j \quad \text{and} \quad \|v\|_2^2 = \sum_{j=1}^{r'+1} [c_j(v)]^2. \tag{9}$$

Note that (9) hold trivially for the case $c_{r'+1}(v) = 0$. Further, let

$$
\begin{aligned}
(10) \qquad v_1 - \mathcal{P}_k v_1 &= w_k(v_1) = \sum_{j=1}^{r'} \alpha_{j1} e_j \\
&\vdots \\
v_r - \mathcal{P}_k v_r &= w_k(v_r) = \sum_{j=1}^{r'} \alpha_{jr} e_j
\end{aligned}
$$

be linear combinations of the e_j's and let

$$
(11) \qquad A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{r'1} & \alpha_{r'2} & \cdots & \alpha_{r'r} \end{bmatrix}
$$

be the transpose of coefficient matrix of the combinations in (10). Now, for $v \in L_2[0,T]$ and m-a.e. $x \in C_0[0,T]$, we have

$$
\begin{aligned}
(12) \qquad (v, x - x_k) &= (v, x) - \sum_{j=1}^{k} (v, Z_j)(\alpha_j, x) \\
&= \left(v - \sum_{j=1}^{k} (v, Z_j)\alpha_j,\ x \right) = (w_k(v), x)
\end{aligned}
$$

by the linearity of Paley-Wiener-Zygmund integrals.

The following lemma is useful to prove several results.

Lemma 2. *Let $v \in L_2[0,T]$ and let $a > 0$. Further, let F_r be given by (6) and let*

$$
I(a) = E[F_r(a(x - x_k)) \exp\{ia(v, x - x_k)\}].
$$

Then we have

$$
\begin{aligned}
I(a) &\stackrel{*}{=} \left(\frac{1}{2\pi a^2}\right)^{r'/2} \int_{\mathbb{R}^{r'}} f(\vec{u}_{r'}A) \exp\left\{ \frac{a^2}{2} \left[\sum_{j=1}^{r'} \left[\frac{u_j}{a^2} i + c_j(w_k(v)) \right]^2 \right.\right. \\
&\qquad \left.\left. - \|w_k(v)\|_2^2 \right] \right\} d\vec{u}_{r'}
\end{aligned}
$$

with $\vec{u}_{r'} = (u_1, \ldots, u_{r'})$, where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal, and A is given by (11) and $c_j(w_k(v))$ by (8) with replacing v by $w_k(v)$.

Proof. By (6) and (12), we have

$$\begin{aligned} I(a) &= \int_{C_0[0,T]} f(a((v_1, x - x_k), \ldots, (v_r, x - x_k))) \\ &\quad \times \exp\{ia(v, x - x_k)\}dm(x) \\ &= \int_{C_0[0,T]} f(a((w_k(v_1), x), \ldots, (w_k(v_r), x))) \\ &\quad \times \exp\{ia(w_k(v), x)\}dm(x). \end{aligned}$$

Since $(e_j, \cdot)$'s are mean zero Gaussian with variance 1 and they are independent, we have

$$\begin{aligned} I(a) &= \int_{C_0[0,T]} f\Big(a\Big(\sum_{j=1}^{r'} \alpha_{j1}(e_j, x), \ldots, \sum_{j=1}^{r'} \alpha_{jr}(e_j, x)\Big)\Big) \\ &\quad \times \exp\left\{ia \sum_{j=1}^{r'+1} c_j(w_k(v))(e_j, x)\right\}dm(x) \\ &\overset{*}{=} \left(\frac{1}{2\pi}\right)^{(r'+1)/2} \int_{\mathbb{R}^{r'+1}} f(a\vec{u}_{r'}A) \exp\left\{ia \sum_{j=1}^{r'+1} c_j(w_k(v))u_j \right. \\ &\quad \left. -\frac{1}{2}\sum_{j=1}^{r'+1} u_j^2\right\} d(u_1, \ldots, u_{r'+1}) \end{aligned}$$

by (9) and (10). Using the following well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\}du = \left(\frac{\pi}{a}\right)^{1/2} \exp\left\{-\frac{b^2}{4a}\right\} \tag{13}$$

for $a \in \mathbb{C}_+$ and any real b, we have

$$\begin{aligned} I(a) &= \left(\frac{1}{2\pi a^2}\right)^{r'/2} \int_{\mathbb{R}^{r'}} f(\vec{u}_{r'}A) \exp\left\{\frac{a^2}{2}\left[\sum_{j=1}^{r'}\left[\frac{u_j}{a^2}i + c_j(w_k(v))\right]^2 \right.\right. \\ &\quad \left.\left. -\|w_k(v)\|_2^2\right]\right\} d\vec{u}_{r'} \end{aligned}$$

by the change of variable theorem, where $\vec{u}_{r'} = (u_1, \ldots, u_{r'})$. □

Let $\{v_1, \ldots, v_r\}$ be an orthonormal set in $L_2[0,T]$ such that $\{w_k(v_1), \ldots, w_k(v_r)\}$ is an independent set. Using Gram-Schmidt orthonormalization process we can have the orthonormal set $\{e_1, \ldots, e_r\}$ from $\{w_k(v_1), \ldots, w_k(v_r)\}$. Further, we can take a complete orthonormal set $\{e_j : j = 1, 2, \ldots\}$ containing $\{e_1, \ldots, e_r\}$. In this case, we have $r' = r$ in the equations (8), (9), (10) and (11) so that the matrix A is square and non-singular. From now on, unless otherwise specified, the sets $\{v_1, \ldots, v_r\}$ and $\{e_1, \ldots, e_r, e_{r+1}, \ldots\}$ are the sets as mentioned just before.

Remark 1. A possible example of the above complete orthonormal set can be obtained from the following process. Let $0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0, T]$ and for $l = 1, \ldots, r$, let

$$h_l(t) = \frac{(-1)^{j+1}2^{2l-1}}{(t_j - t_{j-1})^{2l-1}}\left(t - \frac{t_{j-1}+t_j}{2}\right)^{2l-1}$$
$$\text{if } t_{j-1} \le t \le t_j \ (j = 1, \ldots, k).$$

Further, let $\alpha_j = \frac{1}{\sqrt{t_j - t_{j-1}}} I_{[t_{j-1}, t_j]}$ for $j = 1, \ldots, k$. Then, $\{h_1, \ldots, h_r\}$ is independent and $w_k(h_l) = h_l - \mathcal{P}_k h_l = h_l - \sum_{j=1}^k \langle h_l, \alpha_j\rangle \alpha_j = h_l$ for $l = 1, \ldots, r$. Let $\{v_1, \ldots, v_r\}$ be the orthonormal set obtained from $\{h_1, \ldots, h_r\}$ using Gram-Schmidt orthonormalization process. Now, let

$$v_l = \sum_{j=1}^r \beta_{lj} h_j$$

for $l = 1, \ldots, r$. Then we have $w_k(v_l) = v_l - \mathcal{P}_k v_l = \sum_{j=1}^r \beta_{lj}(h_j - \mathcal{P}_k h_j) = \sum_{j=1}^r \beta_{lj} h_j = v_l$. Using Stone-Weierstrass theorem and Gram-Schmidt orthonormalization process again, we can have the desired complete orthonormal set.

For convenience, let

$$\vec{v}(\vec{\xi}) = ((v_1, \vec{\xi}_k), \ldots, (v_r, \vec{\xi}_k)) = \left(\sum_{j=1}^k \xi_j (v_1, Z_j), \ldots, \sum_{j=1}^k \xi_j (v_r, Z_j)\right). \tag{14}$$

Now, we have the following theorem which evaluate the conditional analytic Wiener integral of the product of the function in $\mathcal{S}$ and the cylinder function.

Theorem 3. *For s-a.e.* $x \in C_0[0, T]$, *let*

$$G_r(x) = F(x)F_r(x), \tag{15}$$

where $F_r \in \mathcal{A}_r^{(p)}$ $(1 \le p \le \infty)$ *and* $F \in \mathcal{S}$ *are given by* (6) *and* (7), *respectively. Further, let* X_k *be given by* (1). *Then, for* $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G_r|X_k](\vec{\xi})$ *exists for a.e.* $\vec{\xi} \in \mathbb{R}^k$ *and under the notations given as in Lemma 2, it is given by*

$$E^{anw_\lambda}[G_r|X_k](\vec{\xi}) = \left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \exp\{i(v, \vec{\xi}_k)\} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi})) \times \exp\left\{\frac{1}{2\lambda}\left[\sum_{j=1}^r [\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\right\} d\vec{u}_r \, d\sigma(v),$$

where $\vec{u}_r = (u_1, \ldots, u_r)$ *and* $\vec{v}(\vec{\xi})$ *is given by* (14).

Proof. Using Fubini's theorem, we have for $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^k$

$$\begin{aligned}
&E[G_r(\lambda^{-1/2}(x - x_k) + \vec{\xi}_k)] \\
= &\int_{L_2[0,T]} \exp\{i(v, \vec{\xi}_k)\} E[F_r(\lambda^{-1/2}(x - x_k) + \vec{\xi}_k) \\
&\times \exp\{i\lambda^{-1/2}(v, x - x_k)\}] d\sigma(v) \\
= &\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \exp\{i(v, \vec{\xi}_k)\} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi})) \\
&\times \exp\left\{\frac{1}{2\lambda}\left[\sum_{j=1}^{r} [\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\right\} d\vec{u}_r d\sigma(v)
\end{aligned}$$

by Lemma 2 and (14). Here, the justification of using Fubini's theorem, is contained in the following argument. By Bessel's inequality, we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
I(v, \lambda, \vec{\xi}) \equiv &\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \left|\exp\left\{\frac{1}{2\lambda}\left[\sum_{j=1}^{r} [\lambda i u_j + c_j(w_k(v))]^2 \right.\right.\right. \\
&\left.\left.\left. - \|w_k(v)\|_2^2\right]\right\}\right| d\vec{u}_r \\
= &\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \exp\left\{-\frac{\operatorname{Re}\lambda}{2|\lambda|^2}\left(\|w_k(v)\|_2^2 - \right.\right. \\
&\left.\left.\sum_{j=1}^{r} [c_j(w_k(v))]^2\right) - \frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^{r} u_j^2\right\} d\vec{u}_r \\
\le &\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \exp\left\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^{r} u_j^2\right\} d\vec{u}_r.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))|^p \, d\vec{u}_r = \frac{1}{|\det(A)|} \int_{\mathbb{R}^r} |f(\vec{u}_r)|^p d\vec{u}_r \tag{16}$$

by the change of variable theorem, we have

$$\int_{L_2[0,T]} I(v, \lambda, \vec{\xi}) d|\sigma|(v) < \infty \tag{17}$$

from Hölder's inequality. Now, by the analytic extension, we have the theorem. □

When $p = 1$, we have the following corollary by the dominated convergence theorem.

Corollary 4. *Let the assumptions and notations be given as in Theorem 3 with $F_r \in \mathcal{A}_r^{(1)}$. Then, for a non-zero real q, $E^{anf_q}[G_r|X_k](\vec{\xi})$ exists for a.e.*

$\vec{\xi} \in \mathbb{R}^k$ and it is given by

$$\begin{aligned} &E^{anf_q}[G_r|X_k](\vec{\xi}) \\ = &\left(\frac{q}{2\pi i}\right)^{r/2} \int_{L_2[0,T]} \exp\{i(v,\vec{\xi}_k)\} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi})) \\ &\times \exp\left\{\frac{i}{2q}\left[\sum_{j=1}^{r}[qu_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\right\} d\vec{u}_r d\sigma(v). \end{aligned}$$

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(u_1,\ldots,u_r) = \int_{\mathbb{R}^r} \exp\left\{i\sum_{j=1}^{r} u_j z_j\right\} d\rho(z_1,\ldots,z_r), \tag{18}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. Note that $\langle\cdot,\cdot\rangle$ denotes both the inner product on $L_2[0,T]$ and the dot product on $\mathbb{R}^r$ if any confusions are not occurred.

Theorem 5. *Let ϕ be given by (18). Let $K_r(x) = \phi((v_1,x),\ldots,(v_r,x))$ for s-a.e. $x \in C_0[0,T]$ and let $G_r = FK_r$, where F is given by (7). Further, let X_k be given by (1). Then, for a non-zero real q, $E^{anf_q}[G_r|X_k](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{R}^k$ and it is given by*

$$\begin{aligned} &E^{anf_q}[G_r|X_k](\vec{\xi}) \\ = &\int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\left\{ i[(v,\vec{\xi}_k) + \langle \vec{z}_r, \vec{v}(\vec{\xi})\rangle] + \frac{1}{2qi}\left[\|w_k(v)\|_2^2 \right.\right. \\ &\left.\left. + 2\sum_{j=1}^{r}\sum_{l=1}^{r} c_j(w_k(v)) z_l \alpha_{jl} + \sum_{j=1}^{r}\left(\sum_{l=1}^{r} z_l \alpha_{jl}\right)^2\right]\right\} d\rho(\vec{z}_r) d\sigma(v) \end{aligned}$$

with $\vec{z}_r = (z_1,\ldots,z_r)$, where $\vec{v}(\vec{\xi})$ and $c_j(w_k(v))$ are given by (14) and (8) with replacing v by $w_k(v)$, respectively.

Proof. Since ϕ is bounded, we have $K_r \in \mathcal{A}_r^{(\infty)}$. By Theorem 3, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G_r|X_k](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{R}^k$ and it is given by

$$\begin{aligned} &E^{anw_\lambda}[G_r|X_k](\vec{\xi}) \\ = &\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \exp\{i(v,\vec{\xi}_k)\} \int_{\mathbb{R}^r} \phi(\vec{u}_r A + \vec{v}(\vec{\xi})) \\ &\times \exp\left\{\frac{1}{2\lambda}\left[\sum_{j=1}^{r}[\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\right\} d\vec{u}_r d\sigma(v), \end{aligned}$$

where $\vec{u}_r = (u_1, \ldots, u_r)$. Hence, by (13), (18) and Fubini's theorem, we have

$$\begin{aligned}
&E^{anw_\lambda}[G_r|X_k](\vec{\xi}) \\
=& \left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{i[(v, \vec{\xi}_k) + \langle \vec{z}_r, \vec{v}(\vec{\xi})\rangle]\} \int_{\mathbb{R}^r} \exp\biggl\{ i\langle \vec{z}_r, \vec{u}_r A\rangle \\
&+\frac{1}{2\lambda}\left[\sum_{j=1}^r [\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\biggr\} d\vec{u}_r d\rho(\vec{z}_r) d\sigma(v) \\
=& \left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{i[(v, \vec{\xi}_k) + \langle \vec{z}_r, \vec{v}(\vec{\xi})\rangle]\} \int_{\mathbb{R}^r} \exp\biggl\{ i \sum_{j=1}^r \sum_{l=1}^r \\
& z_l \alpha_{jl} u_j + \frac{1}{2\lambda}\left[\sum_{j=1}^r [\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2\right]\biggr\} d\vec{u}_r d\rho(\vec{z}_r) d\sigma(v) \\
=& \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\biggl\{ i[(v, \vec{\xi}_k) + \langle \vec{z}_r, \vec{v}(\vec{\xi})\rangle] - \frac{1}{2\lambda}\biggl[\|w_k(v)\|_2^2 + 2\sum_{j=1}^r \sum_{l=1}^r \\
& c_j(w_k(v)) z_l \alpha_{jl} + \sum_{j=1}^r \left(\sum_{l=1}^r z_l \alpha_{jl}\right)^2\biggr]\biggr\} d\rho(\vec{z}_r) d\sigma(v).
\end{aligned}$$

Now, the theorem follows from Bessel's inequality and the dominated convergence theorem. □

From the above theorems and corollary, we have the following corollary by the linearity of conditional Wiener and Feynman integrals on classical Wiener space.

Corollary 6. *Let $F_r \in \mathcal{A}_r^{(p)} (1 \le p \le \infty)$ and F be given by (6) and (7), respectively, and K_r be given as in Theorem 5. Further, let X_k be given by (1) and q be a non-zero real number. Then $E^{anw_\lambda}[F(F_r + K_r)|X_k](\vec{\xi})$ exists for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, and it is given by*

$$\begin{aligned}
&E^{anw_\lambda}[F(F_r + K_r)|X_k](\vec{\xi}) \\
=& \int_{L_2[0,T]} \exp\left\{ i(v, \vec{\xi}_k) - \frac{1}{2\lambda}\|w_k(v)\|_2^2\right\} \left[\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi}))\right. \\
&\times \exp\left\{\frac{1}{2\lambda}\sum_{j=1}^r [\lambda i u_j + c_j(w_k(v))]^2\right\} d\vec{u}_r + \int_{\mathbb{R}^r} \exp\biggl\{ i\langle \vec{z}_r, \vec{v}(\vec{\xi})\rangle - \frac{1}{2\lambda} \\
&\times \left.\left[2\sum_{j=1}^r \sum_{l=1}^r c_j(w_k(v)) z_l \alpha_{jl} + \sum_{j=1}^r \left(\sum_{l=1}^r z_l \alpha_{jl}\right)^2\right]\biggr\} d\rho(\vec{z}_r)\right] d\sigma(v),
\end{aligned}$$

where $c_j(w_k(v))$ is given by (8) with replacing v by $w_k(v)$. In particular, if $F_r \in \mathcal{A}_r^{(1)}$, then $E^{anf_q}[F(F_r + K_r)|X_k](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{R}^k$ and it is obtained with replacing λ by $-iq$ in the right-hand side of the above equality.

3. Change of scale formula for conditional Wiener integrals

In this section, we derive a change of scale formula for conditional Wiener integrals of unbounded functions on classical Wiener space. Let $\{v_1, \ldots, v_r\}$ and $\{e_1, \ldots, e_r\}$ be the orthonormal sets and let $\{e_1, \ldots, e_r, e_{r+1}, \ldots\}$ be the complete orthonormal set as mentioned in Section 2.

Lemma 7. *Let $n > r$ and let F_r be given by (6). Further, for $\lambda \in \mathbb{C}_+$ and $v \in L_2[0,T]$, let*

$$\begin{aligned}&\Gamma_\lambda(v)\\ &= \int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2 + i(v, x-x_k)\right\}F_r(x-x_k)dm_{\mathbb{B}}(x).\end{aligned}$$

Then, we have

$$\begin{aligned}\Gamma_\lambda(h) &\stackrel{*}{=} \lambda^{-n/2}\left(\frac{\lambda}{2\pi}\right)^{r/2}\exp\left\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2 - \frac{1}{2}\|w_k(v)\|_2^2\right\}\\ &\quad\times\int_{\mathbb{R}^r} f(\vec{u}_r A)\exp\left\{\frac{1}{2\lambda}\sum_{j=1}^{r}[\lambda i u_j + c_j(w_k(v))]^2\right\}d\vec{u}_r\end{aligned}$$

with $\vec{u}_r = (u_1, \ldots, u_r)$, where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal, and $c_j(w_k(v))$ is given by (8) for $j = 1, \ldots, n+1$ with replacing r' by n.

Proof. By (12), we have

$$\begin{aligned}&\Gamma_\lambda(v)\\ &= \int_{C_0[0,T]} f((v_1, x-x_k), \ldots, (v_r, x-x_k))\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\right.\\ &\quad\left.+i(v, x-x_k)\right\}dm(x)\\ &= \int_{C_0[0,T]} f((w_k(v_1), x), \ldots, (w_k(v_r), x))\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\right.\\ &\quad\left.+i(w_k(v), x)\right\}dm(x).\end{aligned}$$

Since $(e_j, \cdot)$'s are mean zero Gaussian with variance 1 and they are independent, by (9), (10) and (13), we have

$$\Gamma_\lambda(v) \stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{(n+1)/2}\int_{\mathbb{R}^{n+1}} f(\vec{u}_r A)\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}u_j^2\right.$$

$$+i\sum_{j=1}^{n+1}c_j(w_k(v))u_j-\frac{1}{2}\sum_{j=1}^{n+1}u_j^2\Big\}d(u_1,\dots,u_{n+1})$$
$$=\lambda^{-n/2}\Big(\frac{\lambda}{2\pi}\Big)^{r/2}\exp\Big\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2-\frac{1}{2}\|w_k(v)\|_2^2\Big\}$$
$$\times\int_{\mathbb{R}^r}f(\vec{u}_rA)\exp\Big\{\frac{1}{2\lambda}\sum_{j=1}^{r}[\lambda iu_j+c_j(w_k(v))]^2\Big\}d\vec{u}_r$$

which is the desired result. □

Applying the method used in the proof of Theorem 5 to the result in Lemma 7, we have the following lemma.

Lemma 8. *Let $n>r$ and for $\lambda\in\mathbb{C}_+$ and $v\in L_2[0,T]$, let*

$$K_\lambda(v) = \int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2+i(v,x-x_k)+i\sum_{j=1}^{r}z_j(v_j,x-x_k)\Big\}dm(x),$$

where $z_j\in\mathbb{R}$ for $j=1,\dots,r$. Then we have

$$K_\lambda(v) = \lambda^{-n/2}\exp\Big\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2-\frac{1}{\lambda}\sum_{j=1}^{r}\sum_{l=1}^{r}c_j(w_k(v))z_l\alpha_{jl}-\frac{1}{2\lambda}\sum_{j=1}^{r}\Big(\sum_{l=1}^{r}z_l\alpha_{jl}\Big)^2-\frac{1}{2}\|w_k(v)\|_2^2\Big\},$$

where α_{jl}'s are given by (10) and $c_j(w_k(v))$ is given by (8) for $j=1,\dots,n+1$ with replacing r' by n.

Proof. Let $f_1(u_1,\dots,u_r)=\exp\{i\sum_{j=1}^r z_ju_j\}$ on $\mathbb{R}^r$ and let F_r be given by (6) with replacing f by f_1. Then, by (13) and Lemma 7, we have

$$K_\lambda(v) = \lambda^{-n/2}\Big(\frac{\lambda}{2\pi}\Big)^{r/2}\exp\Big\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2-\frac{1}{2}\|w_k(v)\|_2^2\Big\}$$
$$\times\int_{\mathbb{R}^r}f_1(\vec{u}_rA)\exp\Big\{\frac{1}{2\lambda}\sum_{j=1}^{r}[\lambda iu_j+c_j(w_k(v))]^2\Big\}d\vec{u}_r$$
$$=\lambda^{-n/2}\Big(\frac{\lambda}{2\pi}\Big)^{r/2}\exp\Big\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2-\frac{1}{2}\|w_k(v)\|_2^2\Big\}$$

$$\times \int_{\mathbb{R}^r} \exp\left\{ i\sum_{j=1}^{r}\sum_{l=1}^{r} z_l\alpha_{jl}u_j + \frac{1}{2\lambda}\sum_{j=1}^{r}[\lambda i u_j + c_j(w_k(v))]^2 \right\} d\vec{u}_r$$
$$= \lambda^{-n/2}\exp\left\{ \frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2 - \frac{1}{\lambda}\sum_{j=1}^{r}\sum_{l=1}^{r} c_j(w_k(v))z_l\alpha_{jl} \right.$$
$$\left. -\frac{1}{2\lambda}\sum_{j=1}^{r}\left(\sum_{l=1}^{r} z_l\alpha_{jl}\right)^2 - \frac{1}{2}\|w_k(v)\|_2^2 \right\},$$

where $\vec{u}_r = (u_1, \dots, u_r)$. Now, the proof is completed. □

Now we derive a relationship between the Wiener integral and the conditional analytic Wiener integral on classical Wiener space.

Theorem 9. *Let G_r be given by* (15) *and the assumptions be given as in Theorem 3. Then, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, we have*

(19)
$$E^{anw_\lambda}[G_r|X_k](\vec{\xi}) = \lim_{n\to\infty}\left[\lambda^{n/2}\int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j, x)^2 \right\} G_r(x - x_k + \vec{\xi}_k)dm(x) \right].$$

Proof. Let $n \in \mathbb{N}$ with $n > r$ and let

$$\Gamma_n = \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j, x)^2 \right\} G_r(x - x_k + \vec{\xi}_k)dm(x).$$

By Lemma 7 and Fubini's theorem, we have

$$\Gamma_n = \int_{L_2[0,T]} \exp\{i(v, \vec{\xi}_k)\} \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j, x)^2 + i(v, x - x_k) \right\}$$
$$\times F_r(x - x_k + \vec{\xi}_k)dm(x)d\sigma(v)$$
$$= \lambda^{-n/2}\left(\frac{\lambda}{2\pi}\right)^{r/2}\int_{L_2[0,T]} \exp\left\{ i(v, \vec{\xi}_k) + \frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2 \right.$$
$$\left. -\frac{1}{2}\|w_k(v)\|_2^2 \right\} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi}))\exp\left\{ \frac{1}{2\lambda}\sum_{j=1}^{r}[\lambda i u_j + c_j(w_k(v))]^2 \right\}$$
$$d\vec{u}_r d\sigma(v),$$

where $\vec{u}_r = (u_1, \dots, u_r)$ and $c_j(w_k(v))$ is given by (8). Here, the justification of using Fubini's theorem is contained in the following argument. By Bessel's inequality, we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
&\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \left| \exp\left\{ \frac{\lambda - 1}{2\lambda} \sum_{j=1}^{n} [c_j(w_k(v))]^2 - \frac{1}{2}\|w_k(v)\|_2^2 \right.\right. \\
&\left.\left. + \frac{1}{2\lambda} \sum_{j=1}^{r} [\lambda i u_j + c_j(w_k(v))]^2 \right\} \right| d\vec{u}_r \\
= &\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \exp\left\{ \frac{1}{2}\left[\sum_{j=1}^{n} [c_j(w_k(v))]^2 - \|w_k(v)\|_2^2 \right] \right. \\
&\left. - \frac{\operatorname{Re}\lambda}{2|\lambda|^2} \sum_{j=r+1}^{n} [c_j(w_k(v))]^2 - \frac{\operatorname{Re}\lambda}{2} \sum_{j=1}^{r} u_j^2 \right\} d\vec{u}_r \\
\le &\int_{\mathbb{R}^r} |f(\vec{u}_r A + \vec{v}(\vec{\xi}))| \exp\left\{ -\frac{\operatorname{Re}\lambda}{2} \sum_{j=1}^{r} u_j^2 \right\} d\vec{u}_r,
\end{aligned}$$

which is integrable on $L_2[0,T]$ from (6) and (16). Using the dominated convergence theorem, Parseval's relation and Theorem 3, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \lambda^{n/2} \Gamma_n \\
= &\lim_{n\to\infty} \left[\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \exp\left\{ i(v, \vec{\xi}_k) + \frac{\lambda - 1}{2\lambda} \sum_{j=1}^{n} \right.\right. \\
&\left. [c_j(w_k(v))]^2 - \frac{1}{2}\|w_k(v)\|_2^2 \right\} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi})) \\
&\left. \times \exp\left\{ \frac{1}{2\lambda} \sum_{j=1}^{r} [\lambda i u_j + c_j(w_k(v))]^2 \right\} d\vec{u}_r d\sigma(v) \right] \\
= &\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} \exp\{ i(v, \vec{\xi}_k) \} \int_{\mathbb{R}^r} f(\vec{u}_r A + \vec{v}(\vec{\xi})) \\
&\times \exp\left\{ \frac{1}{2\lambda} \left[\sum_{j=1}^{r} [\lambda i u_j + c_j(w_k(v))]^2 - \|w_k(v)\|_2^2 \right] \right\} \\
&d\vec{u}_r d\sigma(v) = E^{anw_\lambda}[G_r | X_k](\vec{\xi}).
\end{aligned}$$

□

Now we derive a relationship between the Wiener integral and the conditional analytic Feynman integral on classical Wiener space. Using the method given as in the proof of Theorem 9, the following corollary immediately follows from Corollary 4, Parseval's relation and the dominated convergence theorem.

Corollary 10. *Under the assumptions given as in Corollary 4, we have for a.e.* $\vec{\xi} \in \mathbb{R}^k$

$$(20) \qquad E^{anf_q}[G_r|X_k](\vec{\xi}) = \lim_{n\to\infty}\left[\lambda_n^{n/2}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda_n}{2}\sum_{j=1}^n (e_j,x)^2\right\}G_r(x-x_k+\vec{\xi}_k)dm(x)\right]$$

for any sequence $\{\lambda_n\}_{n=1}^\infty$ *in* $\mathbb{C}_+$ *with* $\lambda_n \to -iq$ *as* $n \to \infty$.

Theorem 11. *Let* ϕ *be given by* (18). *Let* $K_r(x) = \phi((v_1,x),\ldots,(v_r,x))$ *for s-a.e.* $x \in C_0[0,T]$ *and let* $G_r = FK_r$, *where* F *is given by* (7). *Further, let* X_k *be given by* (1). *Then the equation* (19) *holds for* $\lambda \in \mathbb{C}_+$ *and a.e.* $\vec{\xi} \in \mathbb{R}^k$.

Proof. Let $n \in \mathbb{N}$ with $n > r$ and let

$$\Gamma_n = \int_{C_0[0,T]}\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^n (e_j,x)^2\right\}G_r(x-x_k+\vec{\xi}_k)dm(x).$$

By Fubini's theorem and Lemma 8, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, we have

$$\begin{aligned}
&\Gamma_n \\
= &\int_{L_2[0,T]}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^n (e_j,x)^2 + i(v,x-x_k+\vec{\xi}_k)\right\} \\
&\times K_r(x-x_k+\vec{\xi}_k)dm(x)d\sigma(v) \\
= &\int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\{i[(v,\vec{\xi}_k)+\langle\vec{z}_r,\vec{v}(\vec{\xi})\rangle]\}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^n (e_j,x)^2\right. \\
&\left.+i(v,x-x_k)+i\sum_{j=1}^r z_j(v_j,x-x_k)\right\}dm(x)d\rho(\vec{z}_r)d\sigma(v) \\
= &\lambda^{-n/2}\int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\{i[(v,\vec{\xi}_k)+\langle\vec{z}_r,\vec{v}(\vec{\xi})\rangle]\}\exp\left\{\frac{\lambda-1}{2\lambda}\right. \\
&\times\sum_{j=1}^n [c_j(w_k(v))]^2 - \frac{1}{\lambda}\sum_{j=1}^r\sum_{l=1}^r c_j(w_k(v))z_l\alpha_{jl} - \frac{1}{2\lambda}\sum_{j=1}^r\left(\sum_{l=1}^r z_l\alpha_{jl}\right)^2 \\
&\left.-\frac{1}{2}\|w_k(v)\|_2^2\right\}d\rho(\vec{z}_r)d\sigma(v),
\end{aligned}$$

where $\vec{z}_r = (z_1, \dots, z_r)$ and $c_j(w_k(v))$ is given by (8). Now, we have

$$\begin{aligned}
&\left|\exp\left\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{n}[c_j(w_k(v))]^2 - \frac{1}{\lambda}\sum_{j=1}^{r}\sum_{l=1}^{r} c_j(w_k(v))z_l\alpha_{jl}\right.\right.\\
&\left.\left. -\frac{1}{2\lambda}\sum_{j=1}^{r}\left(\sum_{l=1}^{r} z_l\alpha_{jl}\right)^2 - \frac{1}{2}\|w_k(v)\|_2^2\right\}\right|\\
= &\exp\left\{-\frac{1}{2}\left[\|w_k(v)\|_2^2 - \sum_{j=1}^{n}[c_j(w_k(v))]^2\right] - \frac{\operatorname{Re}\lambda}{2|\lambda|^2}\left[\sum_{j=r+1}^{n}[c_j(w_k(v))]^2\right.\right.\\
&\left.\left. +\sum_{j=1}^{r}\left[\sum_{l=1}^{r} z_l\alpha_{jl} + c_j(w_k(v))\right]^2\right]\right\}\\
\le &\ 1
\end{aligned}$$

by Bessel's inequality. Hence, by the dominated convergence theorem and Parseval's relation, we have the theorem from Theorem 5. □

Corollary 12. *Let q be a non-zero real number and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ with $\lambda_n \to -iq$ as $n \to \infty$. Under the assumptions given as in Theorem 11, for a.e. $\vec{\xi} \in \mathbb{R}^k$, the equation* (20) *holds.*

Corollary 13. *Let $G_r = F(F_r + K_r)$, where F, F_r and K_r are given as in Corollary 6. Then, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, the equation* (19) *holds.*

Corollary 14. *Let q be a non-zero real number and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ with $\lambda_n \to -iq$ as $n \to \infty$. Moreover, let $G_r = F(F_r + K_r)$, where F is given by* (7) *and F_r, K_r are given as in Corollary* 4, *Theorem* 5, *respectively. Then, for a.e. $\vec{\xi} \in \mathbb{R}^k$, the equation* (20) *holds.*

Our main result, namely, a change of scale formula for conditional Wiener integrals on classical Wiener space, follows from Corollary 13.

Theorem 15. *Let the assumptions be given as in Corollary 13. Then, for $\gamma > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, we have*

$$\begin{aligned}
(21)\qquad &E[G_r(\gamma\cdot)|X_k(\gamma\cdot)](\vec{\xi})\\
&= \lim_{n\to\infty}\left[\gamma^{-n}\int_{C_0[0,T]}\exp\left\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^{n}(e_j,x)^2\right\}G_r(x - x_k + \vec{\xi}_k)dm(x)\right].
\end{aligned}$$

Proof. Letting $\lambda = \gamma^{-2}$ in (19), we have (21) from (5). □

Now, letting $F_r = 0$, $K_r = 1$ and $\lambda = \gamma^{-2}$, we have the following corollary from Corollaries 13 and 14. In addition, scrutinizing the proofs of Lemmas 2, 7, 8 and Theorems 3, 5, 9, 11, we can see that the choice of the complete orthonormal set $\{e_j : j = 1, 2, \dots\}$ is independent of the v_j's in this case.

Corollary 16. *Let F be given by (7) and $\{e_j : j = 1, 2, \dots\}$ be any complete orthonormal subset of $L_2[0,T]$. Then, we have for $\lambda \in \mathbb{C}_+$ and a non-zero real q*

$$(22)\qquad E^{anw_\lambda}[F|X_k](\vec{\xi}) = \lim_{n\to\infty}\left[\lambda^{n/2}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\right\}F(x-x_k+\vec{\xi}_k)dm(x)\right]$$

and

$$(23)\qquad E^{anf_q}[F|X_k](\vec{\xi}) = \lim_{n\to\infty}\left[\lambda_n^{n/2}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda_n}{2}\sum_{j=1}^{n}(e_j,x)^2\right\}F(x-x_k+\vec{\xi}_k)dm(x)\right]$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$, where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ with $\lambda_n \to -iq$ as $n \to \infty$. In particular, we have for $\gamma > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^k$

$$E[F(\gamma\cdot)|X_k(\gamma\cdot)](\vec{\xi}) = \lim_{n\to\infty}\left[\gamma^{-n}\int_{C_0[0,T]}\exp\left\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^{n}(e_j,x)^2\right\}F(x-x_k+\vec{\xi}_k)dm(x)\right].$$

Let

$$(24)\qquad \tau : 0 = t_0 < t_1 < \cdots < t_k = T$$

be a partition of $[0,T]$ and let x be in $C_0[0,T]$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$(25)\qquad [x](t) = \sum_{j=1}^{k} I_{(t_{j-1},t_j]}(t)\left[x(t_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j)-x(t_{j-1}))\right]$$

for $t \in [0,T]$. For $\vec{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given by (25) with replacing $x(t_j)$ by ξ_j for $j = 0, 1, \dots, k$ ($\xi_0 = 0$). Let $X_\tau : C_0[0,T] \to \mathbb{R}^k$ be the random variable given by

$$(26)\qquad X_\tau(x) = (x(t_1), \dots, x(t_k)).$$

For $v \in L_2[0,T]$, define the sectional average $\bar{v}$ of v by

$$\bar{v}(t) = \frac{1}{t_j - t_{j-1}}\int_{t_{j-1}}^{t_j} v(t^*)dt^*$$

on the interval $(t_{j-1}, t_j]$ for $j = 1, \dots, k$ and $\bar{v}(0) = 0$. Further, let $\alpha_j = \frac{1}{\sqrt{t_j-t_{j-1}}}I_{(t_{j-1},t_j]}$ for $j = 1, \dots, k$ and $X_k(x) = ((\alpha_1, x), \dots, (\alpha_k, x))$ for $x \in C_0[0,T]$. Then, from [8, p.299], we have $x_k = [x]$ and for an integrable function

F on $C_0[0,T]$, we also have

$$\begin{aligned} E[F|X_\tau](\vec{\xi}) &= E[F|X_k]\Big(\frac{\xi_1-\xi_0}{\sqrt{t_1-t_0}},\dots,\frac{\xi_k-\xi_{k-1}}{\sqrt{t_k-t_{k-1}}}\Big) \\ &= E[F(x-[x]+[\vec{\xi}])] \end{aligned} \tag{27}$$

by Lemma 1. Moreover, since $\mathcal{P}_k v = \bar{v}$, we also have

$$w_k(v) = v - \mathcal{P}_k v = v - \bar{v}$$

for $v \in L_2[0,T]$. With replacing X_k, x_k, $\vec{\xi_k}$ and $w_k(v)$ by X_τ, $[x]$, $[\vec{\xi}]$ and $v-\bar{v}$, respectively, we have all the lemmas, theorems and corollaries of this and previous sections. Note that, in this case, (14) is rewritten by

$$\begin{aligned} \vec{v}(\vec{\xi}) &= ((v_1,[\vec{\xi}]),\dots,(v_r,[\vec{\xi}])) \\ &= \Big(\sum_{j=1}^{k}\bar{v}_1(t_j)(\xi_j-\xi_{j-1}),\dots,\sum_{j=1}^{k}\bar{v}_r(t_j)(\xi_j-\xi_{j-1})\Big). \end{aligned}$$

In particular, from Corollaries 13 and 14 with $K_r = 1$ and $F_r = 0$, (22) and (23) are rewritten by

$$\begin{aligned} &E^{anw_\lambda}[F|X_\tau](\vec{\xi}) \\ &= \lim_{n\to\infty}\Big[\lambda^{n/2}\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\Big\}F(x-[x]+[\vec{\xi}])dm(x)\Big] \end{aligned} \tag{28}$$

and

$$\begin{aligned} &E^{anf_q}[F|X_\tau](\vec{\xi}) \\ &= \lim_{n\to\infty}\Big[\lambda_n^{n/2}\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda_n}{2}\sum_{j=1}^{n}(e_j,x)^2\Big\}F(x-[x]+[\vec{\xi}])dm(x)\Big], \end{aligned} \tag{29}$$

respectively, for a.e. $\vec{\xi} \in \mathbb{R}^k$, where F is given by (7).

4. Change of scale formula for Wiener integrals

Throughout this section, let $\{e_1, e_2, \dots\}$ be any complete orthonormal set in $L_2[0,T]$.

Let F be defined on $C_0[0,T]$. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$ and it has the analytic extension $J_\lambda^*(F)$ on $\mathbb{C}_+$. Then we call $J_\lambda^*(F)$ the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ and write

$$E^{anw_\lambda}[F] = J_\lambda^*(F).$$

Moreover, if for a non-zero real q, $E^{anw_\lambda}[F]$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman integral of F over $C_0[0,T]$ with parameter q and denoted by

$$E^{anf_q}[F] = \lim_{\lambda\to -iq} E^{anw_\lambda}[F].$$

Lemma 17. *Let F be given by (7) and X_k be given by (1). Then, we have for a non-zero real q*

$$E^{anf_q}[F] = \int_{L_2[0,T]} \exp\left\{\frac{1}{2qi}\|v\|_2^2\right\} d\sigma(v) \tag{30}$$

and

$$E^{anf_q}[F|X_k](\vec{\xi}) = \int_{L_2[0,T]} \exp\left\{i(v,\vec{\xi}_k) + \frac{1}{2qi}\|w_k(v)\|_2^2\right\} d\sigma(v) \tag{31}$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$, where $\vec{\xi}_k$ is given by (4).

Proof. It is not difficult to show that we have for $\lambda \in \mathbb{C}_+$

$$E^{anw_\lambda}[F] = \int_{L_2[0,T]} \exp\left\{-\frac{1}{2\lambda}\|v\|_2^2\right\} d\sigma(v) \tag{32}$$

and hence (30) follows from the dominated convergence theorem. Now, we prove for $\lambda \in \mathbb{C}_+$

$$E^{anw_\lambda}[F|X_k](\vec{\xi}) = \int_{L_2[0,T]} \exp\left\{i(v,\vec{\xi}_k) - \frac{1}{2\lambda}\|w_k(v)\|_2^2\right\} d\sigma(v) \tag{33}$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$. To prove (33), let $\lambda > 0$. By Fubini's theorem and (12), we have

$$\begin{aligned} & E[F(\lambda^{-1/2}(x - x_k) + \vec{\xi}_k)] \\ = & \int_{L_2[0,T]} \exp\{i(v,\vec{\xi}_k)\} \int_{C_0[0,T]} \exp\{i\lambda^{-1/2}(w_k(v),x)\} dm(x) d\sigma(v) \\ = & \int_{L_2[0,T]} \exp\left\{i(v,\vec{\xi}_k) - \frac{1}{2\lambda}\|w_k(v)\|_2^2\right\} d\sigma(v) \end{aligned}$$

and hence we have (33) by Morera's theorem. (31) follows from (33) by the dominated convergence theorem. □

Remark 2. Let X_τ be given by (26). If we replace X_k by X_τ, we obtain both (31) and (33) with replacing $\vec{\xi}_k$, $w_k(v)$ by $[\vec{\xi}]$, $v - \bar{v}$, respectively.

For a complex-valued function f on $\mathbb{R}^k$, let

$$\overline{\int_{\mathbb{R}^k}} f(\vec{\xi}) d\vec{\xi} = \lim_{A\to\infty} \int_{\mathbb{R}^k} f(\vec{\xi}) \exp\left\{-\frac{1}{2A}\sum_{j=1}^k \xi_j^2\right\} d\vec{\xi}$$

with $\vec{\xi} = (\xi_1, \ldots, \xi_k)$ whenever the limit exists. Using the notation, we have the following lemma which expresses the analytic Feynman integral of the function in S as an ordinary integral of the conditional analytic Feynman integral of the function.

Lemma 18. *Let F be given by (7) and X_k be given by (1). Then, we have for $\lambda \in \mathbb{C}_+$*

$$\left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} E^{anw_\lambda}[F|X_k](\vec{\xi}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{k} \xi_j^2\right\} d\vec{\xi} = E^{anw_\lambda}[F] \tag{34}$$

and for a non-zero real q

$$\left(\frac{q}{2\pi i}\right)^{k/2} \overline{\int_{\mathbb{R}^k}} E^{anf_q}[F|X_k](\vec{\xi}) \exp\left\{\frac{qi}{2}\sum_{j=1}^{k} \xi_j^2\right\} d\vec{\xi} = E^{anf_q}[F], \tag{35}$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_k)$.

Proof. By (33) and Fubini's theorem, we have for $\lambda \in \mathbb{C}$

$$\begin{aligned}
&\left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} E^{anw_\lambda}[F|X_k](\vec{\xi}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{k} \xi_j^2\right\} d\vec{\xi} \\
= \ &\left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{L_2[0,T]} \int_{\mathbb{R}^k} \exp\left\{i(v, \vec{\xi}_k) - \frac{1}{2\lambda}\|v - \mathcal{P}_k v\|_2^2 - \frac{\lambda}{2}\sum_{j=1}^{k} \xi_j^2\right\} \\
&d\vec{\xi} d\sigma(v) \\
= \ &\left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{L_2[0,T]} \int_{\mathbb{R}^k} \exp\left\{i\sum_{j=1}^{k}(v, Z_j)\xi_j - \frac{1}{2\lambda}[\|v\|_2^2 - \|\mathcal{P}_k v\|_2^2] \right. \\
&\left. -\frac{\lambda}{2}\sum_{j=1}^{k} \xi_j^2\right\} d\vec{\xi} d\sigma(v) \\
= \ &\int_{L_2[0,T]} \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^{k}(v, Z_j)^2 - \frac{1}{2\lambda}\|v\|_2^2 + \frac{1}{2\lambda}\sum_{j=1}^{k}(v, Z_j)^2\right\} d\sigma(v) \\
= \ &E^{anw_\lambda}[F]
\end{aligned}$$

by (13) and (32). To prove (35), let $A > 0$. By (31) and Fubini's theorem, we have

$$\begin{aligned}
&\left(\frac{q}{2\pi i}\right)^{k/2} \int_{\mathbb{R}^k} E^{anf_q}[F|X_k](\vec{\xi}) \exp\left\{-\left(\frac{q}{2i} + \frac{1}{2A}\right)\sum_{j=1}^{k} \xi_j^2\right\} d\vec{\xi} \\
= \ &\left(\frac{q}{2\pi i}\right)^{k/2} \int_{L_2[0,T]} \int_{\mathbb{R}^k} \exp\left\{i\sum_{j=1}^{k}(v, Z_j)\xi_j + \frac{1}{2qi}\|v\|_2^2 - \frac{1}{2qi}\sum_{j=1}^{k}(v, Z_j)^2\right.
\end{aligned}$$

$$-\frac{1-iqA}{2A}\sum_{j=1}^{k}\xi_j^2\Big\}d\vec{\xi}d\sigma(v)$$

$$= \left(\frac{q}{2\pi i}\right)^{k/2}\left(\frac{2\pi A}{1-iqA}\right)^{k/2}\int_{L_2[0,T]}\exp\Big\{-\frac{A}{2(1-iqA)}\sum_{j=1}^{k}(v,Z_j)^2$$

$$+\frac{1}{2qi}\|v\|_2^2-\frac{1}{2qi}\sum_{j=1}^{k}(v,Z_j)^2\Big\}d\sigma(v),$$

where the last equality follows from (13). Letting $A\to\infty$, we have (35) from (30) and the dominated convergence theorem. □

Remark 3. We have another method to prove (34). Indeed, by the definition of conditional expectation, we have for $\lambda>0$

$$\begin{aligned} E[F^\lambda] &= \int_{C_0[0,T]} E[F^\lambda|X_k^\lambda](X_k^\lambda(x))dm(x) \\ &= \left(\frac{\lambda}{2\pi}\right)^{k/2}\int_{\mathbb{R}^k} E^{anw_\lambda}[F|X_k](\vec{\xi})\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{k}\xi_j^2\Big\}d\vec{\xi}\end{aligned}$$

by the change of variable theorem, since $(\alpha_j,\cdot)$'s are mean zero Gaussian with variance 1. By (32) and (33), each side of the equation has the analytic extension on $\mathbb{C}_+$. By the uniqueness of analytic extension, we have (34) for $\lambda\in\mathbb{C}_+$.

Theorem 19. *Let F be given by (7). Further, let q be a non-zero real number and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ with $\lambda_n\to -iq$ as $n\to\infty$. Then, we have*

(36)
$$\begin{aligned} &E^{anw_\lambda}[F] \\ &= \lim_{n\to\infty}\Big[\lambda^{n/2}\left(\frac{\lambda}{2\pi}\right)^{k/2}\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\Big\} \\ &\quad\times\int_{\mathbb{R}^k}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{k}\xi_j^2\Big\}F(x-x_k+\vec{\xi}_k)d\vec{\xi}dm(x)\Big]\end{aligned}$$

for $\lambda\in\mathbb{C}_+$ and

(37)
$$\begin{aligned} &E^{anf_q}[F] \\ &= \lim_{A\to\infty}\lim_{n\to\infty}\Big[\lambda_n^{n/2}\left(\frac{\lambda_n}{2\pi}\right)^{k/2}\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda_n}{2}\sum_{j=1}^{n}(e_j,x)^2\Big\} \\ &\quad\times\int_{\mathbb{R}^k}\exp\Big\{-\left(\frac{\lambda_n}{2}+\frac{1}{2A}\right)\sum_{j=1}^{k}\xi_j^2\Big\}F(x-x_k+\vec{\xi}_k)d\vec{\xi}dm(x)\Big],\end{aligned}$$

where $\vec{\xi}=(\xi_1,\dots,\xi_k)$.

Proof. By Corollary 16 and (34), we have

$$\begin{aligned} E^{anw_\lambda}[F] &= \left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} E^{anw_\lambda}[F|X_k](\vec{\xi}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^k \xi_j^2\right\} d\vec{\xi} \\ &= \left(\frac{\lambda}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} \left[\lim_{n\to\infty}\left[\lambda^{n/2}\int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^n (e_j,x)^2\right.\right.\right. \\ &\quad \left.\left.\left. -\frac{\lambda}{2}\sum_{j=1}^k \xi_j^2\right\} F(x - x_k + \vec{\xi}_k)\, dm(x)\right]\right] d\vec{\xi}. \end{aligned}$$

By an application of the proof of Lemma 7, we also have

$$\begin{aligned} &\int_{\mathbb{R}^k} \left|\lambda^{n/2}\int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^n (e_j,x)^2 - \frac{\lambda}{2}\sum_{j=1}^k \xi_j^2\right\}\right. \\ &\qquad \left.\times F(x - x_k + \vec{\xi}_k)\, dm(x)\right| d\vec{\xi} \\ &= \int_{\mathbb{R}^k} \exp\left\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^k \xi_j^2\right\} \left|\int_{L_2[0,T]} \exp\{i(v,\vec{\xi}_k)\}\right. \\ &\qquad \left.\times \exp\left\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^n [c_j(w_k(v))]^2 - \frac{1}{2}\|w_k(v)\|_2^2\right\} d\sigma(v)\right| d\vec{\xi} \\ &\le \int_{\mathbb{R}^k} \exp\left\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^k \xi_j^2\right\} \int_{L_2[0,T]} \exp\left\{-\frac{1}{2}\left[\|w_k(v)\|_2^2\right.\right. \\ &\qquad \left.\left. -\sum_{j=1}^n [c_j(w_k(v))]^2\right] - \frac{\operatorname{Re}\lambda}{2|\lambda|^2}\sum_{j=1}^n [c_j(w_k(v))]^2\right\} d|\sigma|(v)\, d\vec{\xi} \\ &\overset{(*)}{\le} \|\sigma\| \int_{\mathbb{R}^k} \exp\left\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^k \xi_j^2\right\} d\vec{\xi} < \infty, \end{aligned}$$

where $(*)$ follows from Bessel's inequality. Now, (36) follows from the dominated convergence theorem and Fubini's theorem.

It remains to prove (37). Again, by Corollary 16 and (35), we have

$$E^{anf_q}[F] = \left(\frac{q}{2\pi i}\right)^{k/2} \overline{\int_{\mathbb{R}^k}} E^{anf_q}[F|X_k](\vec{\xi}) \exp\left\{\frac{qi}{2}\sum_{j=1}^k \xi_j^2\right\} d\vec{\xi}$$

$$= \lim_{A\to\infty}\int_{\mathbb{R}^k}\Bigg[\lim_{n\to\infty}\Bigg[\lambda_n^{n/2}\Big(\frac{\lambda_n}{2\pi}\Big)^{k/2}\exp\Big\{-\Big(\frac{\lambda_n}{2}+\frac{1}{2A}\Big)\sum_{j=1}^k\xi_j^2\Big\}$$

$$\times\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda_n}{2}\sum_{j=1}^n(e_j,x)^2\Big\}F(x-x_k+\vec{\xi}_k)dm(x)\Bigg]\Bigg]d\vec{\xi}$$

$$= \lim_{A\to\infty}\lim_{n\to\infty}\Bigg[\lambda_n^{n/2}\Big(\frac{\lambda_n}{2\pi}\Big)^{k/2}\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda_n}{2}\sum_{j=1}^n(e_j,x)^2\Big\}$$

$$\times\int_{\mathbb{R}^k}\exp\Big\{-\Big(\frac{\lambda_n}{2}+\frac{1}{2A}\Big)\sum_{j=1}^k\xi_j^2\Big\}F(x-x_k+\vec{\xi}_k)d\vec{\xi}dm(x)\Bigg]$$

by the dominated convergence theorem and Fubini's theorem, which completes the proof. □

If we take $\lambda = \gamma^{-2}$ for $\gamma > 0$, then we have the following theorem from Theorem 19.

Theorem 20. *Under the assumptions given as in Theorem 19, we have*

$$\int_{C_0[0,T]}F(\gamma x)dm(x)$$

$$= \lim_{n\to\infty}\Bigg[\gamma^{-n}\Big(\frac{1}{2\pi\gamma^2}\Big)^{k/2}\int_{C_0[0,T]}\exp\Big\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^n(e_j,x)^2\Big\}$$

$$\times\int_{\mathbb{R}^k}\exp\Big\{-\frac{1}{2\gamma^2}\sum_{j=1}^k\xi_j^2\Big\}F(x-x_k+\vec{\xi}_k)d\vec{\xi}dm(x)\Bigg]$$

for $\gamma > 0$.

Lemma 21 ([7, Theorem 10]). *Let F be given by (7) and X_τ be given by (26). Then, for $\lambda \in \mathbb{C}_+$ and a non-zero real q, we have*

(38)
$$E^{anw_\lambda}[F]$$

$$= \Bigg[\prod_{j=1}^k\frac{\lambda}{2\pi(t_j-t_{j-1})}\Bigg]^{1/2}\int_{\mathbb{R}^k}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^k\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\Big\}$$

$$\times E^{anw_\lambda}[F|X_\tau](\vec{\xi})d\vec{\xi}$$

and

(39)
$$E^{anf_q}[F]$$

$$= \Bigg[\prod_{j=1}^k\frac{q}{2\pi i(t_j-t_{j-1})}\Bigg]^{1/2}\int_{\mathbb{R}^k}\exp\Big\{\frac{qi}{2}\sum_{j=1}^k\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\Big\}$$

$$\times E^{anf_q}[F|X_\tau](\vec{\xi})d\vec{\xi},$$

where $\vec{\xi} = (\xi_1,\dots,\xi_k)$ and $t_0 = \xi_0 = 0$.

Remark 4. Comparing (34), (35) with (38), (39), respectively, we note that they are different in the expressions of $E^{anw_\lambda}[F]$ and $E^{anf_q}[F]$. Indeed, the random variables X_k and X_τ do not have the same distribution scrutinizing their probability densities. We also emphasize that (27) does not mean $E[F|X_k](\vec{\xi})$.

Let X_τ be given by (26). Then, for $\lambda > 0$, the probability density of X_τ^λ is $[\prod_{j=1}^k \frac{\lambda}{2\pi(t_j-t_{j-1})}]^{1/2}\exp\{-\frac{\lambda}{2}\sum_{j=1}^k \frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\}$. By the same method used in the proof of Theorem 19, we have the following theorem from (28), (29), (38) and (39).

Theorem 22. *Let a partition of* $[0,T]$ *be given by* (24). *Under the assumptions given as in Theorem 19, we have*

$$\begin{aligned} &E^{anw_\lambda}[F] \\ &= \lim_{n\to\infty}\Bigg[\lambda^{n/2}\Bigg[\prod_{j=1}^k \frac{\lambda}{2\pi(t_j-t_{j-1})}\Bigg]^{1/2}\int_{C_0[0,T]}\exp\Bigg\{\frac{1-\lambda}{2}\sum_{j=1}^n(e_j,x)^2\Bigg\} \\ &\quad\times\int_{\mathbb{R}^k}\exp\Bigg\{-\frac{\lambda}{2}\sum_{j=1}^k\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\Bigg\}F(x-[x]+[\vec{\xi}])d\vec{\xi}dm(x)\Bigg] \end{aligned}$$

for $\lambda\in\mathbb{C}_+$ *and*

$$\begin{aligned} &E^{anf_q}[F] \\ &= \lim_{A\to\infty}\lim_{n\to\infty}\Bigg[\lambda_n^{n/2}\Bigg[\prod_{j=1}^k\frac{\lambda_n}{2\pi(t_j-t_{j-1})}\Bigg]^{1/2}\int_{C_0[0,T]}\exp\Bigg\{\frac{1-\lambda_n}{2}\sum_{j=1}^n(e_j, \\ &\quad x)^2\Bigg\}\int_{\mathbb{R}^k}\exp\Bigg\{-\frac{\lambda_n}{2}\sum_{j=1}^k\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}-\frac{1}{2A}\sum_{j=1}^k\xi_j^2\Bigg\}F(x-[x]+[\vec{\xi}]) \\ &\quad d\vec{\xi}dm(x)\Bigg], \end{aligned}$$

where $\vec{\xi}=(\xi_1,\ldots,\xi_k)$ *and* $t_0=\xi_0=0$.

If we take $\lambda=\gamma^{-2}$ for $\gamma>0$, then we have the following theorem from the first equation in Theorem 22.

Theorem 23. *Under the assumptions given as in Theorem 22, we have*

$$\begin{aligned} &\int_{C_0[0,T]}F(\gamma x)dm(x) \\ &= \lim_{n\to\infty}\Bigg[\gamma^{-n}\Bigg[\prod_{j=1}^k\frac{1}{2\pi\gamma^2(t_j-t_{j-1})}\Bigg]^{1/2}\int_{C_0[0,T]}\exp\Bigg\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^n(e_j,x)^2\Bigg\} \\ &\quad\times\int_{\mathbb{R}^k}\exp\Bigg\{-\frac{1}{2\gamma^2}\sum_{j=1}^k\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\Bigg\}F(x-[x]+[\vec{\xi}])d\vec{\xi}dm(x)\Bigg] \end{aligned}$$

for $\gamma > 0$.

Remark 5. Let $\phi = 1$ in [12, Theorem 3.10]. From this theorem, for $F \in \mathcal{S}$ and $\lambda \in \mathbb{C}_+$, we have

$$E^{anw_\lambda}[F] = \lim_{n\to\infty}\left[\lambda^{n/2}\int_{C_0[0,T]}\exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{n}(e_j,x)^2\right\}F(x)dm(x)\right]$$

so that we have another type of the change of scale formula([5, Theorem 2] and [12, Corollary 4.5]);

$$\begin{aligned}&\int_{C_0[0,T]}F(\gamma x)dm(x)\\ =\ &\lim_{n\to\infty}\left[\gamma^{-n}\int_{C_0[0,T]}\exp\left\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^{n}(e_j,x)^2\right\}F(x)dm(x)\right]\end{aligned}$$

for $\gamma > 0$.

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