J. Korean Math. Soc. **44** (2007), No. 4, pp. 1013–1024

ON UNIFORM DECAY OF WAVE EQUATION OF CARRIER MODEL SUBJECT TO MEMORY CONDITION AT THE BOUNDARY

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ABSTRACT. In this paper we consider the uniform decay for the wave equation of Carrier model subject to memory condition at the boundary. We prove that if the kernel of the memory decays exponentially or polynomially, then the solutions for the problems have same decay rates.

1. Introduction

In this paper, we are concerned with the mixed problem for the Carrier model subject to memory condition at the boundary given by

$$u'' - M\left(\int_\Omega |u|^2 dx\right)\Delta u + |u|^\alpha u = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$u + \int_0^t g(t-s) M\left(\int_\Omega |u(s)|^2 dx\right)\frac{\partial u}{\partial \nu}(s)ds = 0 \text{ on } \Gamma_1 \times (0,\infty) \tag{1.3}$$

$$u(0) = u_0, \quad u_t(0) = u_1 \quad \text{on } \Omega, \tag{1.4}$$

where Ω is a bounded domain in R^n with C^2 boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures, ν denotes the unit outer normal vector pointing towards the exterior of Ω and $g \in W^{1,2}(0,\infty)$ is a non-increasing, positive function.

We note that the integral equation (1.3) describes the memory effect which can be caused by the interaction with another viscoelastic element. Indeed, boundary condition (1.3) means that Ω is clamped in a rigid body in the portion Γ_0 of its boundary and in a body in the portion Γ_1 with viscoelastic properties.

Carrier model $u'' - M(\int_\Omega u^2 dx)\Delta u = f$ was derived in [2] to model vibrations of an elastic string with fixed ends when the changes in tension are not small. For the global solvability of Carrier model, we refer [4, 5, 8]. When $\Gamma_1 = \emptyset$,

Received March 14, 2006; Revised December 15, 2006.
2000 *Mathematics Subject Classification.* 35B35, 35B40, 35L70.
Key words and phrases. existence of solution, uniform decay, wave equation, carrier model, boundary value problem, a priori estimates.

author [8] considered the global solvability for the Carrier model with nonlinear damping $|u'|^\alpha u'$ and source term $|u|^\rho u$.

On the other hand, when $\Gamma_1 \neq \emptyset$, Cavalcanti et al. [3] have studied the uniform decay of solutions of coupled linear wave equations with $M(s) = 1$ and boundary conditions (1.2)-(1.3). Santos et al. [10] considered the decay rate for Kirchhoff type wave equation with strong damping $u'' - M(\|\nabla u\|^2)\Delta u - \Delta u_t + f(u) = 0$ on $\Omega \times (0,\infty)$ subject to memory condition at the boundary. Author [1] has proved the decay rates for the coupled wave equation of Kirchhoff type. For the global existence of Kirchhoff type model, we refer [1, 6, 9].

In this paper, we will study the existence of solutions for the Carrier model (1.1) subject to memory condition on the boundary. Moreover, we consider if the memory terms g decay exponentially or polynomially, then the solutions for the Carrier model with nonlinear damping have same decay rates.

2. Statement of results

Throughout this paper we define

$$V := \{u \in H^1(\Omega);\ u = 0 \quad \text{on} \quad \Gamma_0\}, \quad (u,v) := \int_\Omega u(x)v(x)dx,$$

$$(u,v)_\Gamma = \int_\Gamma u(x)v(x)d\Gamma,\ \|u\|_{p,\Gamma}^p = \int_\Gamma |u(x)|^p dx \ \text{ and } \ \|u\|_\infty = \|u\|_{L^\infty(\Omega)}.$$

Now, we shall assume that $\Gamma_0 = \{x \in \Gamma \mid (x - x_0)\cdot\nu(x) \le 0\}$ and $\Gamma_1 = \{x \in \Gamma \mid (x - x_0)\cdot\nu(x) > 0\}$. Denoting by $(g * \phi)(t) = \int_0^t g(t-s)\phi(s)ds$ the convolution product operator and differentiating the equation (1.3), and then applying the Volterra's inverse operator, we get $M(\|u\|^2)\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)}(u'(t) + (k * u')(t))$ on $\Gamma_1 \times (0,\infty)$. Here, the resolvent kernel satisfies $k(t) + \frac{1}{g(0)}(g' * k)(t) = -\frac{1}{g(0)}g'(t)$. Denoting by $\eta_1 = \frac{1}{g(0)}$, we obtain on $\Gamma_1 \times (0,\infty)$,

$$M(\|u\|^2)\frac{\partial u}{\partial \nu} = -\eta_1[u'(t) + k(0)u(t) - k(t)u_0 + (k' * u)(t)]. \tag{2.1}$$

Now, we need the following assumptions:

(A_1) Let us consider $u_0,\ u_1 \in V \cap H^2(\Omega)$ verifying the compatibility conditions

$$M(\|u_0\|^2)\frac{\partial u_0}{\partial \nu} + \frac{1}{g(0)}u_1 = 0 \quad \text{on} \quad \Gamma_1.$$

(A_2) The function $k \in W^{1,\infty}(0,\infty) \cap W^{2,1}(0,\infty)$ satisfies that there exist positive constants m_i, $i = 1,2$, such that for all $t \ge 0$ and $p > 1$

$$k(0) > 0,\ k'(t) \le -m_1 k(t),\ k''(t) \ge -m_2 k'(t), \tag{2.2}$$

or

$$k(0) > 0,\ k'(t) \le -m_1 k(t)^{1+\frac{1}{p}},\ k''(t) \ge m_2(-k'(t))^{1+\frac{1}{p+1}}. \tag{2.3}$$

(A_3) $M(\cdot)$ is a nondecreasing $C^1(0,\infty)$ function with $0 < m_0 \le M(s)$ for every $s \ge 0$.

Let us denote by $(g\Box\phi)(t) := \int_0^t g(t-s)|\phi(t)-\phi(s)|^2 ds$. Then we state our main result.

Theorem 2.1. *Under the assumptions* (A_1)-(A_3), *if* $0 < \alpha \le \frac{1}{n-2}$ *if* $n \ge 3$, *or* $\alpha > 0$ *if* $n = 1, 2$, *then problem* (1.1)-(1.3) *has a unique solution* $u : \Omega \to R$ *such that* $u \in L^\infty(0,\infty; V \cap H^2(\Omega))$, $u' \in L^\infty(0,\infty; V)$, $u'' \in L^\infty(0,\infty; L^2(\Omega))$. *Moreover, there exist positive constants* C_1 *and* C_2 *such that*

$$E(t) \le C_1 E(0)\exp(-C_2 t) \quad or \quad E(t) \le C_1 E(0)(1+t)^{-(p+1)},$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}\{\|u'(t)\|^2 + M(\|u(t)\|^2)\|\nabla u(t)\|^2 + \frac{2}{\alpha+2}\|u(t)\|_{\alpha+2}^{\alpha+2} - \eta_1(k'\Box u)(t)\} \\ &\quad + \frac{\eta_1}{2}k(t)\|u(t)\|_{\Gamma_1}^2. \end{aligned}$$

3. Proof of Theorem 2.1

Without loss of generality, we consider $\eta_1 = 1$. For each $M > 0$, we put

(3.1)
$$\begin{aligned} W(M,T) = \{v \in L^\infty(0,T; V\cap H^2(\Omega)); v_t \in L^\infty(0,T;V), v_{tt} \in L^2(\Omega\times(0,T)), \\ \|v\|_{L^\infty(0,T;V\cap H^2(\Omega))} + \|v_t\|_{L^\infty(0,T;V)} + \|v_{tt}\|_{L^2(\Omega\times(0,T))} \le M\}, \end{aligned}$$

$$W_1(M,T) = \{v \in W(M,T);\ v_{tt} \in L^\infty(0,T;L^2(\Omega))\},$$

$$K_0 = \max_{0\le s\le M^2}|M(s)| \quad \text{and} \quad K_1 = \max_{0\le s\le M^2}|M'(s)|.$$

Suppose that $u_{m-1} \in W_1(M,T)$, then we associate the problem (1.1)-(1.3) with the following problem:

$$\begin{aligned} &(u_m''(t), w) + b_m(t)(\nabla u_m(t), \nabla w) + (|u_m(t)|^\alpha u_m(t), w) + (u_m'(t), w)_{\Gamma_1} \\ (3.2)\quad &+ \int_0^t k'(t-s)(u_m(s), w)_{\Gamma_1} ds + k(0)(u_m(t), w)_{\Gamma_1} \\ &= (k(t)u_{0m}, w)_{\Gamma_1}, \quad w \in V_m, \end{aligned}$$

(3.3) $\quad u_m(0) = u_0, \quad u_m'(0) = u_1,$

where $b_m(t) = M(\|u_{m-1}(t)\|^2)$, then we find $u_m \in W_1(M,T)$ which satisfies the problem (3.2)-(3.3). Using usual Faedo-Galerkin's approximation and multiplier method, we can obtain the following proposition.

Proposition 3.1. *Under the assumption of Theorem 2.1, there exist positive constants* M *and* T *and the recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.2)-(3.3).

Proposition 3.2. *Under the assumption of Theorem 2.1, there exist positive constants* M *and* T *such that the problem* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$. *On the other hand, the linear recurrent sequence* $\{u_m\}$ *defined by* (3.2)-(3.3) *converges to the solution* u *strongly in the space* $W_1(T) = \{v \in L^\infty(0,T;V) \mid v' \in L^\infty(0,T;L^2(\Omega))\}$. *Furthermore, we have*

$$\|u_m - u\|_{L^\infty(0,T;V)} + \|u_m' - u'\|_{L^\infty(0,T;L^2(\Omega))} \le Ck_T^m, \quad for\ all\ m,$$

where $k_T < 1$ is some positive constant and C is a constant depending only on T, u_0, u_1.

Proof. First, we note that $W_1(T)$ is a Banach space with respect to the norm $\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2(\Omega))}$. [7]. We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Consider $z_m = u_{m+1} - u_m$. Then z_m satisfies the variational problem

$$\begin{aligned} (3.4)\qquad & (z_m''(t), w) + b_{m+1}(t)(\nabla z_m(t), \nabla w) + (b_{m+1}(t) - b_m(t))(\nabla u_m(t), \nabla w) \\ & +(|u_{m+1}(t)|^\alpha u_{m+1}(t) - |u_m(t)|^\alpha u_m(t), w) + (z_m'(t), w)_{\Gamma_1} \\ & + \int_0^t k'(t-s)(z_m(s), w)_{\Gamma_1} ds + k(0)(z_m(t), w)_{\Gamma_1} = 0, \quad w \in V, \end{aligned}$$

$$(3.5)\qquad z_m(0) = z_m'(0) = 0 \quad \text{in } V.$$

Now, we can write

$$\begin{aligned} (3.6)\qquad & \frac{d}{dt}E_{2m}(t) + \|z_m'(t)\|_{\Gamma_1}^2 \\ = \ & \frac{1}{2}b_{m+1}'(t)\|\nabla z_m(t)\|^2 + (|u_m(t)|^\alpha u_m(t) - |u_{m+1}(t)|^\alpha u_{m+1}(t), z_m'(t)) \\ & +(b_m(t) - b_{m+1}(t))(\nabla u_m(t), \nabla z_m'(t)) - \int_0^t k'(t-s)(z_m(s), z_m'(t))_{\Gamma_1} ds, \end{aligned}$$

where $E_{2m}(t) = \frac{1}{2}\{\|z_m'(t)\|^2 + k(0)\|z_m(t)\|_{\Gamma_1}^2 + b_{m+1}(t)\|\nabla z_m(t)\|^2\}$. From mean value theorem, we obtain

$$\begin{aligned} |b_{m+1}(t) - b_m(t)| &= |M(\|u_m(t)\|^2) - M(\|u_{m-1}(t)\|^2)| \\ &\le \int_{\|u_{m-1}(t)\|^2}^{\|u_m(t)\|^2} |M'(\xi)| d\xi \\ &\le K_1(\|u_{m-1}(t)\| + \|u_m(t)\|)\|z_{m-1}(t)\|_{W_1(T)} \\ &\le 2MK_1\|z_{m-1}(t)\|_{W_1(T)}, \end{aligned}$$

and so Green identity and boundary condition (2.1) imply

$$\begin{aligned} & |(b_{m+1}(t) - b_m(t))(\nabla u_m(t), \nabla z_m'(t))| \\ & \le 2MK_1\|z_{m-1}(t)\|_{W_1(T)}[\|\Delta u_m(t)\|\|z_m'(t)\| + \|\frac{\partial u_m}{\partial \nu}(t)\|_{\Gamma_1}\|z_m'(t)\|_{\Gamma_1}] \\ & \le M^2K_1[\|z_{m-1}(t)\|_{W_1(T)}^2 + \|z_m'(t)\|^2] \\ & \quad + 2\epsilon^{-1}M^2K_1^2\|z_{m-1}(t)\|_{W_1(T)}^2\|\frac{\partial u_m}{\partial \nu}(t)\|_{\Gamma_1}^2 + \epsilon\|z_m'(t)\|_{\Gamma_1}^2 \\ & \le M^2K_1[\|z_{m-1}(t)\|_{W_1(T)}^2 + \|z_m'(t)\|^2] + \epsilon\|z_m'(t)\|_{\Gamma_1}^2 \\ & \quad + 4(\epsilon m_0^2)^{-1}M^4K_1^2\|z_{m-1}(t)\|_{W_1(T)}^2 \\ & \quad \times (|k(0)|^2 + \|k(t)\|_{L^\infty(0,\infty)}^2 + \|k(t)\|_{L^1(0,\infty)}^2). \end{aligned}$$

Also, we have

$$(|u_m(t)|^\alpha u_m(t) - |u_{m+1}(t)|^\alpha u_{m+1}(t), z_m'(t))$$

$$\begin{aligned} &\leq C_1(\|u_m(t)\|_{\alpha n}^{\alpha} + \|u_{m+1}(t)\|_{\alpha n}^{\alpha})\|z_m(t)\|_{\frac{2n}{n-2}}\|z_m'(t)\| \\ &\leq C_2 M^{\alpha}(\|\nabla z_m(t)\|^2 + \|z_m'(t)\|^2). \end{aligned} \tag{3.7}$$

Thus we arrive at

$$\begin{aligned} &\frac{d}{dt}E_{2m}(t) + (1-2\epsilon)\|z_m'(t)\|_{\Gamma_1}^2 \\ \leq\ & C_{M,K_1}\|z_{m-1}(t)\|_{W_1(T)}^2 + \frac{1}{4\epsilon}\|k'\|_{L^1(0,\infty)}\int_0^t |k'(t-s)|\|z_m(s)\|_{\Gamma_1}^2 ds \\ & +[(\frac{1}{2}+M)MK_1 + C_2M^{\alpha}][\|z_m'(t)\|^2 + \|\nabla z_m(t)\|^2], \end{aligned} \tag{3.8}$$

here

$$C_{M,K_1} = M^2K_1[1+4(\epsilon m_0^2)^{-1}M^2K_1(|k(0)|^2 + \|k(t)\|_{L^\infty(0,\infty)}^2 + \|k(t)\|_{L^1(0,\infty)}^2)].$$

Integrating (3.8) over $[0,t]$, choosing $\epsilon > 0$ sufficiently small and employing Gronwall's lemma we obtain

$$E_{2m}(t) + \int_0^t \|z_m'(s)\|_{\Gamma_1}^2 ds \leq \tilde{k}_T\|z_{m-1}(t)\|_{W_1(T)}^2, \tag{3.9}$$

where $\tilde{k}_T = C_{M,K_1}Te^{[2\max\{1,m_0^{-1}\}((\frac{1}{2}+M)MK_1+C_2M^{\alpha})+\frac{1}{2k(0)\epsilon}\|k'\|_{L^1(0,\infty)}^2]T}$, that is,

$$\|z_m(t)\|_{W_1(T)} \leq k_T\|z_{m-1}(t)\|_{W_1(T)}, \tag{3.10}$$

where $k_T = \sqrt{2}[\min\{1, m_0, k(0)\}]^{-\frac{1}{2}}(\tilde{k}_T)^{\frac{1}{2}} < 1$ Hence

$$\|u_{m+p} - u_m\|_{W_1(T)} \leq \|u_1 - u_0\|_{W_1(T)}\frac{k_T^m}{1-k_T} \quad \text{for all} \quad m,\ p. \tag{3.11}$$

It follows from (3.11) that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Therefore there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in} \quad W_1(T). \tag{3.12}$$

We also note that $u_m \in W_1(M,T)$, then from the sequence $\{u_m\}$ we can deduce a subsequence $\{u_{m_j}\}$ such that

$$\begin{aligned} u_{m_j} &\to u \quad \text{weak star in} \quad L^\infty(0,T;V), \\ u_{m_j}' &\to u' \quad \text{weak star in} \quad L^\infty(0,T;V), \\ u_{m_j}'' &\to u'' \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \end{aligned} \tag{3.13}$$

where $u \in W(M,T)$. The above convergence is sufficient to pass to the limit in the linear terms of (3.2). Taking into account the continuity of trace operator $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$, we have

$$\begin{aligned} u_{m_j} &\to u \quad \text{in} \quad L^2(0,T;L^2(\Gamma_1)), \\ u_{m_j}' &\to u' \quad \text{in} \quad L^2(0,T;L^2(\Gamma_1)). \end{aligned}$$

Sobolev imbedding and (3.13) imply that for every $\phi \in L^2(\Omega)$,

$$\begin{aligned}&|(|u_{m_j}(t)|^\alpha u_{m_j}(t) - |u(t)|^\alpha u(t), \phi)| \\ \leq\ & C(\|u_{m_j}(t)\|^\alpha_{\alpha n} + \|u(t)\|^\alpha_{\alpha n})\|u_{m_j}(t) - u(t)\|_{\frac{2n}{n-2}} \|\phi(t)\| \\ \leq\ & 2CM^\alpha \|u_{m_j}(t) - u(t)\|_{W_1(T)} \|\phi(t)\| \to 0 \quad \text{as} \quad j \to \infty.\end{aligned}$$

Now, we notice that

$$\begin{aligned}&\int_0^T b_m(t)(\nabla u_m(t), \nabla v(t)) - b(t)(\nabla u(t), \nabla v(t))dt \\ =\ & \int_0^T b_m(t)(\nabla u_m(t) - \nabla u(t), \nabla v(t)) + (b_m(t) - b(t))(\nabla u(t), \nabla v(t))dt \\ \leq\ & C_1[K_0 + 2K_1 M]\|u_m - u\|_{W_1(T)} \|v\|_{L^1(0,T;H^1)} \quad \text{for all} \quad v \in L^1(0,T;H^1).\end{aligned}$$

Thus we can pass to the limit with $m = m_j \to \infty$ to obtain

$$u'' - M(\|u\|^2)\Delta u + |u|^\alpha u = 0 \quad \text{in} \quad D'(0,\infty; L^2(\Omega)).$$

Since u, $u'' \in L^2_{loc}(0,\infty; L^2(\Omega))$,

$$u'' - M(\|u\|^2)\Delta u + |u|^\alpha u = 0 \quad \text{in} \quad L^2_{loc}(0,\infty; L^2(\Omega)).$$

Returning to the approximate problem, making use of Green formula, we have

$$M(\|u\|^2)\frac{\partial u}{\partial \nu} + u'(t) + k(0)u(t) - k(t)u_0 + (k' * u)(t) = 0 \text{ on } D'(0,\infty; H^{-\frac{1}{2}}(\Gamma_1)).$$

Since u', u, $k' * u \in L^2_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1))$,

$$M(\|u\|^2)\frac{\partial u}{\partial \nu} + u'(t) + k(0)u(t) - k(t)u_0 + (k' * u)(t) = 0 \text{ on } L^2_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1)).$$

This completes the proof of existence of solutions in Theorem 2.1. The uniqueness of solution is proved by applying the similar course to Proposition 3.2. We omit it. □

4. Uniform decay

4.1. Exponentially decay

In this section we shall show the asymptotic behavior of solutions for the problem (1.1)-(1.3) when the resolvent kernel k decays exponentially. Let us assume k satisfies the condition (2.2). Note that the condition (2.2) implies $k(t) \leq k(0)e^{-m_1 t}$ for all $t > 0$. At first, we begin with the following Lemma.

Lemma 4.1 ([3]). *Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and c_0 such that $f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t}$, then there exist positive constants γ and c such that $f(t) \leq (f(0) + c)e^{-\gamma t}$.*

Now, we define

$$(k\Box u)(t) := \int_0^t k(t-r)\|u(t) - u(r)\|^2_{\Gamma_1} dr, \tag{4.1}$$

and define energy by

$$\begin{aligned} E(t) &= \frac{1}{2}[\|u'(t)\|^2 + b(t)\|\nabla u(t)\|^2 - \eta_1(k'\Box u)(t) + \eta_1 k(t)\|u(t)\|^2_{\Gamma_1}] \\ &+\frac{1}{\alpha+2}\|u(t)\|^{\alpha+2}_{\alpha+2}, \end{aligned} \tag{4.2}$$

where $b(t) = M(\|u(t)\|^2)$. Then we have

$$\begin{aligned} \frac{d}{dt}E(t) &\leq b'(t)\|\nabla u(t)\|^2 - \frac{\eta_1}{2}\|u'(t)\|^2_{\Gamma_1} + \frac{\eta_1}{2}|k(t)|^2\|u_0\|^2_{\Gamma_1} \\ &-\frac{\eta_1}{2}(k''\Box u)(t) + \frac{\eta_1}{2}k'(t)\|u(t)\|^2_{\Gamma_1}. \end{aligned} \tag{4.3}$$

For $u \in W_1(M,T)$, we get

$$\begin{aligned} \frac{d}{dt}E(t) &\leq 2K_1M^2m_0^{-1}b(t)\|\nabla u(t)\|^2 - \frac{\eta_1}{2}\|u'(t)\|^2_{\Gamma_1} + \frac{\eta_1}{2}|k(t)|^2\|u_0\|^2_{\Gamma_1} \\ &+\frac{\eta_1 m_2}{2}(k'\Box u)(t) - \frac{\eta_1 m_1}{2}k(t)\|u(t)\|^2_{\Gamma_1}. \end{aligned} \tag{4.4}$$

Let us define the perturbed modified energy by

$$\begin{aligned} N(t) &= \|u'(t)\|^2 + b(t)\|\nabla u(t)\|^2 + \|u(t)\|^{\alpha+2}_{\alpha+2}, \\ \psi(t) &= \int_\Omega [m\cdot\nabla u(t) + (\frac{n}{2} - \theta)u(t)]u'(t)dx, \end{aligned}$$

where θ is a small positive constant.

Consider the following operator

$$(k\Diamond h)(t) = \int_0^t k(t-s)(h(t)-h(s))ds.$$

Lemma 4.2. *For any strong solution of the problem* (1.1)-(1.3), *we get for small* $\epsilon > 0$

$$\begin{aligned} \frac{d}{dt}\psi(t) &\leq C\int_{\Gamma_1}(|u'(t)|^2 + |k(t)u(t)|^2 + |(k'\Diamond u)(t)|^2 + |k(t)u_0|^2)d\Gamma \\ &-\frac{\theta}{2}N(t) - (1-2\theta-\epsilon m_0^{-1})b(t)\int_\Omega|\nabla u(t)|^2dx. \end{aligned} \tag{4.5}$$

Proof. From the equation (1.1) and integration by parts, we have

$$\begin{aligned} \frac{d}{dt}\psi(t) &\leq \frac{1}{2}\int_{\Gamma_1}(m\cdot\nu)|u'(t)|^2d\Gamma - \theta\int_\Omega|u'(t)|^2dx \\ &-(1-\theta)b(t)\int_\Omega|\nabla u(t)|^2dx - \frac{1}{2}b(t)\int_{\Gamma_1}(m\cdot\nu)|\nabla u(t)|^2d\Gamma \\ &+b(t)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\{m\cdot\nabla u(t) + (\frac{n}{2}-\theta)u(t)\}d\Gamma \\ &-\int_\Omega|u(t)|^\alpha u(t)[m\cdot\nabla u(t) + (\frac{n}{2}-\theta)u(t)]dx, \end{aligned}$$

where we have used $\int_{\Gamma_0}(m\cdot\nu)|\frac{\partial u}{\partial\nu}|^2 d\Gamma \le 0$ and $u|_{\Gamma_0}=0$. Taking θ small enough, we get

$$\begin{aligned}
&\frac{d}{dt}\psi(t)\\
\le\ & -\theta N(t) - (1-2\theta)b(t)\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}\int_{\Gamma_1}(m\cdot\nu)|u'(t)|^2 d\Gamma\\
&-\frac{1}{2}b(t)\int_{\Gamma_1}(m\cdot\nu)|\nabla u(t)|^2 d\Gamma + b(t)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\{m\cdot\nabla u(t) + (\frac{n}{2}-\theta)u(t)\}d\Gamma\\
&-\int_\Omega |u(t)|^\alpha u(t)[m\cdot\nabla u(t) + (\frac{n}{2}-2\theta)u(t)]dx.
\end{aligned}$$

Applying Young and Poincaré inequality, we have for $\epsilon > 0$,

$$\begin{aligned}
(4.6)\qquad & b(t)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\{m\cdot\nabla u(t)+(\frac{n}{2}-\theta)u(t)\}d\Gamma\\
\le\ & \epsilon b(t)\int_{\Gamma_1}\{|m\cdot\nabla u(t)|^2 + (\frac{n}{2}-\theta)^2|u(t)|^2\}d\Gamma + C_1(\epsilon)b(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2 d\Gamma\\
\le\ & \epsilon C_2[b(t)\int_{\Gamma_1}(m\cdot\nu)|\nabla u(t)|^2 d\Gamma + N(t)] + C_1(\epsilon)b(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2 d\Gamma.
\end{aligned}$$

Also, we get

$$\int_\Omega |u(t)|^\alpha u(t) m\cdot\nabla u(t)dx \le \max_{x\in\bar\Omega}\|x-x_0\|\|\nabla u(t)\|\|u(t)\|^{\alpha+1}_{2(\alpha+1)}.$$

Gagliardo-Nirenberg inequality implies

$$\|u(t)\|^{\alpha+1}_{2(\alpha+1)} \le C_3\|\nabla u(t)\|^{(\alpha+1)\theta_1}\|u(t)\|^{(\alpha+1)(1-\theta_1)}_{\alpha+2},$$

where $\theta_1 = \frac{\alpha n}{(\alpha+1)(4-\alpha(n-2))}$. Thus we have

$$\begin{aligned}
(4.7)\qquad & \int_\Omega |u(t)|^\alpha u(t) m\cdot\nabla u(t)dx\\
\le\ & C_3\max_{x\in\bar\Omega}\|x-x_0\|\|\nabla u(t)\|^{1+(\alpha+1)\theta_1}\|u(t)\|^{(\alpha+1)(1-\theta_1)}_{\alpha+2}\\
\le\ & \epsilon\|\nabla u(t)\|^2 + \frac{C_3^2}{4\epsilon}[\max_{x\in\bar\Omega}\|x-x_0\|]^2\|\nabla u(t)\|^{2(\alpha+1)\theta_1}\|u(t)\|^{2(\alpha+1)(1-\theta_1)}_{\alpha+2}\\
\le\ & \epsilon\|\nabla u(t)\|^2 + \frac{C_3^2}{4\epsilon}[\max_{x\in\bar\Omega}\|x-x_0\|]^2\|\nabla u(t)\|^{2(\alpha+1)\theta_1}\|u(t)\|^{\alpha-2\theta_1(\alpha+1)}_{\alpha+2}N(t)\\
\le\ & \epsilon\|\nabla u(t)\|^2 + C_4(\epsilon)E(0)^{\frac{\alpha}{2}}N(t).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\frac{d}{dt}\psi(t)\ \le\ & -\theta(1-C(\epsilon,\theta))N(t) - (1-2\theta-\epsilon m_0^{-1})b(t)\int_\Omega|\nabla u(t)|^2 dx\\
&+\int_{\Gamma_1}(m\cdot\nu)|u'(t)|^2 d\Gamma - \frac{1}{2}b(t)\int_{\Gamma_1}(m\cdot\nu)|\nabla u(t)|^2 d\Gamma\\
&+\epsilon C_2 b(t)\int_{\Gamma_1}(m\cdot\nu)|\nabla u(t)|^2 d\Gamma + C_1(\epsilon)b(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2 d\Gamma,
\end{aligned}$$

where $C(\epsilon,\theta) = \frac{1}{\theta}\{\epsilon C_2 + C_4(\epsilon)E(0)^{\frac{\alpha}{2}}\}$. Choosing small initial data with $C(\epsilon,\theta) < \frac{1}{2}$ in the above inequality. Then we have

$$\begin{aligned} \frac{d}{dt}\psi(t) \;\le\; & -\frac{\theta}{2}N(t) - (1-2\theta-\epsilon m_0^{-1})b(t)\int_\Omega |\nabla u(t)|^2 dx \\ & + C_5 \int_{\Gamma_1} |u'(t)|^2 + |\frac{\partial u}{\partial \nu}|^2 d\Gamma. \end{aligned} \tag{4.8}$$

Now, we note that the boundary condition (2.1) can be written as

$$b(t)\frac{\partial u}{\partial \nu} = -[u'(t) + k(t)u(t) - k(t)u_0 - (k'\Diamond u)(t)] \text{ on } \Gamma_1 \times (0,\infty). \tag{4.9}$$

Taking into account the above estimates with boundary conditions, we get Lemma. □

On the other hand, using Hölder's inequality, we get

$$|(k'\Diamond u)(t)|^2 \le (k(t)-k(0))(k'\Box u)(t) \le -k(0)(k'\Box u)(t). \tag{4.10}$$

Thus we have

$$\begin{aligned} \frac{d}{dt}\psi(t) \le\; & C\int_{\Gamma_1} (|u'(t)|^2 + |k(t)u(t)|^2 - k(0)(k'\Box u)(t) + |k(t)u_0|^2) d\Gamma \\ & -\frac{\theta}{2}N(t) - (1-2\theta-\epsilon m_0^{-1})b(t)\int_\Omega |\nabla u(t)|^2 dx. \end{aligned} \tag{4.11}$$

We choose a positive ϵ and $\bar{n}$ with $K_1M^2 < \frac{m_0(1-2\theta-\epsilon m_0^{-1})}{2\bar{n}}$ and $\epsilon < m_0(1-2\theta)$. Then let us introduce the Lyapunov functional $L(t) := NE(t) + \psi(t)$ with $N > 0$. Then we have

$$\begin{aligned} \frac{d}{dt}L(t) \;\le\; & -\frac{\theta}{2}E(t) - [1-2\theta-\epsilon m_0^{-1} - 2K_1M^2m_0^{-1}N]b(t)\int_\Omega |\nabla u(t)|^2 dx \\ & -\frac{(\alpha+1)\theta}{2(\alpha+2)}\|u(t)\|_{\alpha+2}^{\alpha+2} + (\frac{\eta_1 m_2 N}{2} - \frac{\eta_1\theta}{4} - Ck(0))(k'\Box u)(t) \\ & +(\frac{\eta_1\theta}{4} - \frac{m_1 N\eta_1}{2} + C)k(t)\|u(t)\|_{\Gamma_1}^2 + (C - \frac{\eta_1 N}{2})\|u'(t)\|_{\Gamma_1}^2 \\ & +(\frac{\eta_1 N}{2} + C)|k(t)|^2\|u_0\|_{\Gamma_1}^2. \end{aligned} \tag{4.12}$$

Taking N large with $N > \bar{n}$, then from (2.3), we have

$$\begin{aligned} \frac{d}{dt}L(t) \;&\le\; -\frac{\theta}{2}E(t) + 2N(\frac{\eta_1}{2}|k(t)|^2\|u_0\|_{\Gamma_1}^2) \\ &\le\; -\frac{\theta}{2}E(t) + 2N(\frac{\eta_1}{2}|k(0)|^2\|u_0\|_{\Gamma_1}^2)e^{-2m_1 t} \\ &\le\; -\frac{\theta}{2}E(t) + 2NE(0)e^{-2m_1 t}. \end{aligned}$$

Using Young inequality and taking N large we find that

$$\frac{N}{2}E(t) \le L(t) \le 2NE(t), \tag{4.13}$$

and so we get

$$\frac{d}{dt}L(t) \le -\frac{\theta}{4N}L(t) + 2NE(0)e^{-2m_1t}.$$

Thus from Lemma 4.1, we have

$$L(t) \le (L(0)+C)e^{-\gamma t} \quad \text{for some positive constants } C,\ \gamma.$$

Considering the inequality (4.13), we get the decay estimates of energy.

4.2. Polynomially decay

Now, we consider the uniform decay rates when the resolvent kernel k decay polynomially like $(1+t)^{-p}$. Let us assume that k satisfies the condition (2.3). To obtain our result, we use the following Lemma.

Lemma 4.3 ([3]). *Let $f \ge 0$ be a differentiable function satisfying*

$$f'(t) \le -\frac{c_1}{f(0)^{\frac{1}{\alpha}}}f(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\beta}}f(0), \quad t \ge 0,$$

for some positive constants c_1, c_2, α and β such that $\beta \ge \alpha+1$, then there exist a positive constant c such that

$$f(t) \le \frac{c}{(1+t)^{\alpha}}f(0), \quad t \ge 0.$$

Lemma 4.4 ([3]). *Let $u=\phi$ be a solution of problem* (1.1)-(1.3). *Then for $p>1$, $0<r<1$ and $t \ge 0$, we have*

$$\begin{aligned}
&\Big(\int_{\Gamma_1}(|k'|\Box\phi)(t)d\Gamma\Big)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \\
\le\ & 2\Big(\int_0^t |k'(s)|^r ds \|\phi\|^2_{L^\infty(0,t;L^2(\Gamma_1))}\Big)^{\frac{1}{(1-r)(p+1)}} \int_{\Gamma_1}(|k'|^{\frac{p+2}{p+1}}\Box\phi_i)(t)d\Gamma, \\
&\Big(\int_{\Gamma_1}(|k'|\Box\phi_i)(t)d\Gamma\Big)^{\frac{p+2}{p+1}} \\
\le\ & 2\Big(\int_0^t |\phi(s)|^2_{L^2(\Gamma_1)}ds + t\|\phi\|^2_{L^2(\Gamma_1)}\Big)^{\frac{1}{p+1}} \int_{\Gamma_1}(|k'|^{\frac{p+2}{p+1}}\Box\phi)(t)d\Gamma.
\end{aligned}$$

Using (4.4) and conditions on k, we get

$$\begin{aligned}
\frac{d}{dt}E(t) \ \le\ & -\frac{\eta_1}{2}\|u'(t)\|^2_{\Gamma_1} + \frac{\eta_1}{2}|k(t)|^2\|u_0\|^2_{\Gamma_1} - \frac{\eta_1 m_2}{2}((-k')^{1+\frac{1}{p+1}}\Box u)(t) \\
& -\frac{\eta_1 m_1}{2}k(t)^{1+\frac{1}{p}}\|u(t)\|^2_{\Gamma_1} + 2K_1M^2m_0^{-1}b(t)\|\nabla u(t)\|^2.
\end{aligned}$$

Using Hölder's inequality and condition on k, we get

(4.14)

$$|(k'\Diamond u)(t)|^2 \le \int_0^t (-k'(s))^{\frac{p}{p+1}}ds \cdot ((-k')^{\frac{p+2}{p+1}}\Box u)(t) \le C((-k')^{\frac{p+2}{p+1}}\Box u)(t).$$

Thus we have

$$\begin{aligned}&\frac{d}{dt}\psi(t)\\ \le\ & -\frac{\theta}{2}N(t)+C\int_{\Gamma_1}(|u'(t)|^2+k(t)|u(t)|^2+|k(t)u_0|^2+((-k')^{\frac{p+2}{p+1}}\Box u)(t))d\Gamma\\ & -(1-2\theta-\epsilon m_0^{-1})b(t)\int_\Omega|\nabla u(t)|^2dx.\end{aligned}$$

Also, for large $N>0$, Lyapunov functional $L(t)$ satisfies

$$\frac{d}{dt}L(t)\ \le\ -\frac{\theta}{2}N(t)+2N|k(t)|^2E(0)-\frac{c_1N}{2}\int_{\Gamma_1}((-k')^{\frac{p+2}{p+1}}\Box u)(t)d\Gamma.$$

Condition on k and Lemma 4.4 imply

$$\frac{d}{dt}L(t)\ \le\ -cL(0)^{-\frac{1}{p+1}}L(t)^{1+\frac{1}{p+1}}+2N|k(t)|^2E(0).$$

Thus from Lemma 4.3, we have

$$L(t)\le\frac{C}{(1+t)^{p+1}}L(0)\quad\text{for some}\quad C>0.$$

Therefore, (4.13) yields

$$E(t)\le\frac{C}{(1+t)^{p+1}}E(0).$$

This completes the proof of Theorem. □

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