

NONRELATIVISTIC LIMIT IN THE SELF-DUAL ABELIAN CHERN-SIMONS MODEL

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ABSTRACT. We consider the nonrelativistic limit in the self-dual Abelian Chern-Simons model, and give a rigorous proof of the limit for the radial solutions to the self-dual equations with the nontopological boundary condition when there is only one-vortex point. By keeping the shooting constant of radial solutions to be fixed, we establish the convergence of radial solutions in the nonrelativistic limit.

1. Introduction

The (2+1) dimensional self-dual Chern-Simons theories involve charged scalar fields minimally coupled to gauge fields whose dynamics is provided by a Chern-Simons term. In Chern-Simons theories, the vortex field is charged both electrically and magnetically, and can carry a fractional electric charge proportional to the coefficient of the Chern-Simons term. Chern-Simons theories have attracted much interest in various areas of anyonic quantum physics such as fractional quantum Hall effects, anyonic superconductivity, and Aharonov-Bohm scattering. One of the important features of the self-dual Chern-Simons theories is that in the static case, the self-dual structure allows solutions to describe the existence of stable multi-vortex points. Moreover, the self-dual Chern-Simons theories permit a realization with either relativistic or nonrelativistic dynamics for the scalar fields. We refer to [6] for a survey on the self-dual Chern-Simons theories.

In this paper, we are interested in Abelian Chern-Simons theories. The relativistic Abelian Chern-Simons model was suggested by Hong-Kim-Pac [7] and Jackiw-Weinberg [10], where they considered a model of charged vortices with gauge field dynamics governed only by the Chern-Simons term without the Maxwell term. The self-dual structure is attained with a special choice of the Higgs potential by the 6th order. The relativistic Abelian Chern-Simons model enjoys very rich mathematical structures, and has been widely studied related

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to blow-up analysis. See [15] for more information on recent mathematical results about the self-dual equations.

On the other hand, one can consider the nonrelativistic Abelian Chern-Simons model which was introduced by Jackiw-Pi in [8, 9]. This model describes a nonrelativistic field theory for the second-quantized N -body system of point particles with Chern-Simons interaction. In [9] the authors formally found that the relativistic model is reduced to the nonrelativistic model in the limit $c \rightarrow \infty$, which we call *the nonrelativistic limit*. Here, c denotes the velocity of light. It is quite interesting to verify the nonrelativistic limit rigorously by mathematical arguments.

The nonrelativistic limit problem can be considered for three cases: the self-dual equations, the nonself-dual static equations, and the time-dependent equations. The purpose of this paper is to prove the nonrelativistic limit for the solutions to the self-dual equations.

The outline of this paper is as follows. In the next section, we review the relativistic and nonrelativistic Abelian Chern-Simons models, and derive the nonrelativistic limit formally in the Lagrangian level following the argument of [9]. The main difference of our derivation from the argument of [9] is that we vary the Chern-Simons coupling constant as $c \rightarrow \infty$, which enables us to obtain the nonrelativistic model without any terms containing c . In Section 3, we consider radially symmetric nontopological solutions to the self-dual equations when there is only one-vortex point. We find a condition which guarantees the convergence of these solutions in the nonrelativistic limit, and state the main theorem. The proof of the main theorem is given in Section 4.

2. Models

Let us consider $(2 + 1)$ -dimensional Minkowski space $\mathbb{R}^{1,2}$ with the metric $\text{diag}(1, -1, -1)$. In other words, the metric is given by

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2, \quad x_0 = t/c,$$

where $c > 0$ is the velocity of light. The metric is used to raise or lower indices. The Lagrangian density of the relativistic Abelian Chern-Simons model is given by

$$(2.1) \quad \mathcal{L}^R = -\frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma + \hbar^2 D_\alpha \phi \overline{D^\alpha \phi} - \frac{\hbar^2 q^4}{\kappa^2 c^4} |\phi|^2 (|\phi|^2 - \sigma^2)^2,$$

where all the Greek indices run over 0,1,2. Here $\kappa > 0$ is the Chern-Simons coupling constant representing the strength of the Chern-Simons term, \hbar is the Plank constant, q is the charge of electron, $\sigma > 0$ is the symmetry breaking parameter, $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ is the Higgs field, $A_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the gauge field, $D_\alpha = \partial_\alpha - i(q/c\hbar)A_\alpha$ is the covariant derivative with $i = \sqrt{-1}$, and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength.

We say that (ϕ, A_α) is gauge equivalent to $(\tilde{\phi}, B_\alpha)$ if there exists a smooth function χ such that

$$(\tilde{\phi}, B_\alpha) = (e^{i\chi}\phi, A_\alpha + \partial_\alpha\chi).$$

It is easily verified that the Lagrangian \mathcal{L}^R and its Euler-Lagrange equations are invariant under the gauge transformation. We are interested in the stationary solutions of the Euler-Lagrange equations for (2.1). The static variational equation for A_0 , often called the Gauss constraint equation, gives

$$(2.2) \quad \kappa F_{12} - 2\frac{q^2}{c^2}|\phi|^2 A_0 = 0.$$

Using (2.2), we obtain the static relativistic Abelian Chern-Simons energy functional

$$(2.3) \quad \mathcal{E}^R = \int_{\mathbb{R}^2} \hbar^2 |D_1\phi|^2 + \hbar^2 |D_2\phi|^2 + \frac{\kappa^2 c^2}{4q^2} \frac{F_{12}^2}{|\phi|^2} + \frac{q^4 \hbar^2}{\kappa^2 c^4} |\phi|^2 (|\phi|^2 - \sigma^2)^2.$$

Under suitable decay assumptions, we can rewrite the energy functional as

$$\mathcal{E}^R = \int_{\mathbb{R}^2} \left(\hbar^2 |D_1\phi \pm iD_2\phi|^2 + \left(\frac{\kappa c}{2q} \frac{F_{12}}{|\phi|} \pm \frac{\hbar q^2}{\kappa c^2} |\phi| (|\phi|^2 - \sigma^2) \right)^2 \right) dx \pm q\hbar\sigma^2 \Phi^R,$$

where

$$(2.4) \quad \Phi^R = \frac{1}{c} \int_{\mathbb{R}^2} F_{12}$$

is the magnetic flux. If Φ^R is positive (negative), then we choose the upper (lower) sign. Hence we have the energy lower bound

$$\mathcal{E}^R \geq q\hbar\sigma^2 |\Phi^R|,$$

which is achieved by the self-dual equations

$$(2.5) \quad D_1\phi \pm iD_2\phi = 0,$$

$$(2.6) \quad F_{12} \pm \frac{2\hbar q^3}{\kappa^2 c^3} |\phi|^2 (|\phi|^2 - \sigma^2) = 0.$$

The boundary conditions for ϕ are obtained in two ways from the finite energy condition:

$$(2.7) \quad \begin{cases} |\phi(x)| \rightarrow \sigma & \text{as } |x| \rightarrow \infty, \\ |\phi(x)| \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

The former is called topological and the latter nontopological.

Let us take the upper signs in (2.5) and (2.6). The lower sign case can be obtained by the conjugate transformation $(\phi, A) \rightarrow (\bar{\phi}, -A)$. To examine the self-dual equations further, we employ the classical Jaffe-Taubes arguments [11]. In fact, (2.5) implies that ϕ is holomorphic up to a nonvanishing multiple

factor and has exactly N zeros allowing multiplicities. Thus in the light of the gauge invariance we may assume that ϕ takes the form

$$(2.8) \quad \phi(x) = \exp\left(\frac{1}{2}u(x) + i \sum_{j=1}^k n_j \arg(x - p_j)\right),$$

where the points p_1, \dots, p_k , called the vortex points, are the distinct zeros of ϕ with multiplicities n_1, \dots, n_k , respectively. Obviously, $n_1 + \dots + n_k = N$. Then the equation (2.6) reduces to

$$(2.9) \quad \Delta u = \frac{4q^4}{\kappa^2 c^4} e^u (e^u - \sigma^2) + 4\pi \sum_{j=1}^k n_j \delta_{p_j},$$

where δ_{p_j} denotes the Dirac measure concentrated on the point p_j . The boundary conditions of ϕ are rephrased by

$$\begin{aligned} \text{topological:} \quad & u(x) \rightarrow 2 \ln \sigma \quad \text{as } |x| \rightarrow \infty, \\ \text{nontopological:} \quad & u(x) \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Conversely, once we find a solution u of (2.9), we recover A from (2.5) by the formula

$$(2.10) \quad A_1 + iA_2 = -\frac{2c\hbar i}{q} \bar{\partial} \ln \phi,$$

where $\bar{\partial} = (\partial_1 + i\partial_2)/2$. Thus (2.5) and (2.6) are equivalent to (2.9) via the relations (2.8) and (2.10), and we will focus on the equation (2.9). One can find some results on the topological and nontopological solutions in [1, 2, 5, 13, 14]. See also [15] for a survey of recent results on (2.9).

We now turn to the nonrelativistic limit of the Lagrangian density \mathcal{L}^R developed in [9]. To begin with, let us investigate the matter part of \mathcal{L}^R :

$$\mathcal{L}_{matter}^R = \hbar^2 D_\alpha \phi \overline{D^\alpha \phi} - \frac{\hbar^2 q^4}{\kappa^2 c^4} |\phi|^2 (|\phi|^2 - \sigma^2)^2.$$

We first observe that the quadratic term in the potential of \mathcal{L}_{matter}^R defines the mass of the scalar ϕ equal to $m^2 c^2$. This gives the identity

$$m = \hbar q^2 \sigma^2 / \kappa c^3.$$

We will accompany the limit $c \rightarrow \infty$ with fixed m . To this end, we set $\mu = \kappa c$ to be fixed. Then

$$(2.11) \quad \kappa = \frac{\mu}{c} \quad \text{and} \quad \sigma = c \sqrt{\frac{m\mu}{\hbar q^2}},$$

which give a difference from the method in [9]. Indeed, in [9], not only m but also κ is kept fixed and only σ varies in the limit $c \rightarrow \infty$. As a consequence, the matter part of the resulting nonrelativistic Lagrangian contains the constant c , which gives rise to difficulties in proving the nonrelativistic limit rigorously by mathematical arguments. To overcome this difficulty, we will vary both κ and σ as (2.11) in the limit $c \rightarrow \infty$, which is compatible with (2.12) below.

Then the resulting nonrelativistic Lagrangian involves no terms containing c . Moreover, the constant μ becomes the Chern-Simons coupling constant in the nonrelativistic model.

Now using (2.11), we can rewrite the matter Lagrangian density as

$$\begin{aligned} \mathcal{L}_{matter}^R &= \frac{1}{c^2} |\partial_t \phi - i \frac{q}{\hbar} A_0 \phi|^2 - \hbar^2 \sum_{j=1}^2 |\partial_j \phi - i \frac{q}{\hbar c} A_j \phi|^2 - m^2 c^2 |\phi|^2 \\ &\quad + \frac{2m\hbar q^2}{\mu} |\phi|^4 - \frac{\hbar^2 q^4}{\mu^2 c^2} |\phi|^6. \end{aligned}$$

Let

$$(2.12) \quad \phi(t, x) = \frac{1}{\sqrt{2m}} \exp(-imc^2 t / \hbar) \psi(t, x), \quad a_0 = A_0, \quad a_j = A_j / c.$$

A simple computation yields that

$$\mathcal{L}_{matter}^R = \hbar \text{Re} \{ i \bar{\psi} (\partial_t \psi - i \frac{q}{\hbar} a_0 \psi) \} - \frac{\hbar^2}{2m} \sum_{j=1}^2 |\partial_j \psi - i \frac{q}{\hbar} a_j \psi|^2 + \frac{\hbar q^2}{2m\mu} |\psi|^4 + \Lambda_c,$$

where

$$\Lambda_c = \frac{\hbar^2}{2mc^2} |\partial_t \psi - i \frac{q}{\hbar} a_0 \psi|^2 - \frac{\hbar^2 q^4}{(2m)^3 \mu^2 c^2} |\psi|^6.$$

Since Λ_c vanishes in the limit $c \rightarrow \infty$, we obtain the matter Lagrangian density of the nonrelativistic Abelian Chern-Simons model:

$$\mathcal{L}_{matter}^{NR} = \hbar \text{Re} \{ i \bar{\psi} (\partial_t \psi - i \frac{q}{\hbar} a_0 \psi) \} - \frac{\hbar^2}{2m} \sum_{j=1}^2 |\partial_j \psi - i \frac{q}{\hbar} a_j \psi|^2 + \frac{\hbar q^2}{2m\mu} |\psi|^4.$$

Furthermore, it is easily verified that the Chern-Simons term turns out to be

$$\mathcal{L}_{CS}^R = \frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma = \frac{\mu}{4} \epsilon^{\alpha\beta\gamma} f_{\alpha\beta} a_\gamma = \mathcal{L}_{CS}^{NR}$$

with the notation: $f_{12} = \partial_1 a_2 - \partial_2 a_1$ and $f_{0j} = \partial_t a_j - \partial_j a_0$ for $j = 1, 2$. Consequently, the Lagrangian density \mathcal{L}^R becomes in the limit $c \rightarrow \infty$

$$\mathcal{L}^{NR} = -\frac{\mu}{4} \epsilon^{\alpha\beta\gamma} f_{\alpha\beta} a_\gamma + \hbar \text{Re} \{ i \bar{\psi} (\partial_t \psi - i \frac{q}{\hbar} a_0 \psi) \} - \frac{\hbar^2}{2m} d_j \psi \overline{d_j \psi} + \frac{\hbar q^2}{2m\mu} |\psi|^4,$$

where $d_j = \partial_j - i \frac{q}{\hbar} a_j$ with Greek indices 0,1,2, and $j = 1, 2$. We notice that the constant μ plays the role of the Chern-Simons coupling constant in the nonrelativistic Abelian Chern-Simons model. The Gauss equation is given by

$$(2.13) \quad -\mu f_{12} + |\psi|^2 q = 0,$$

from which we obtain the magnetic flux

$$(2.14) \quad \Phi^{NR} = \int_{\mathbb{R}^2} f_{12} = \frac{q}{\mu} \int_{\mathbb{R}^2} |\psi|^2.$$

Using (2.13), we obtain the static energy functional

$$\mathcal{E}^{NR} = \int_{\mathbb{R}^2} \frac{\hbar^2}{2m} (|d_1\psi|^2 + |d_2\psi|^2) - \frac{\hbar q^2}{2m\mu} |\psi|^4 = \int_{\mathbb{R}^2} \frac{\hbar^2}{2m} |d_1\psi + id_2\psi|^2.$$

Hence the energy minimum $\mathcal{E}^{NR} = 0$ can be achieved if and only if (ψ, a_j) satisfies the self-dual equation

$$(2.15) \quad d_1\psi + id_2\psi = 0$$

with the nontopological boundary condition: $|\psi(x)| \rightarrow 0$ as $|x| \rightarrow \infty$. As in the relativistic case, we may assume by the Jaffe and Taubes argument [11] that ψ takes the form

$$(2.16) \quad \psi(x) = \exp\left(\frac{1}{2}w(x) + i \sum_{j=1}^k n_j \arg(x - p_j)\right),$$

where the points p_1, \dots, p_k are the distinct zeros of ψ with multiplicities n_1, \dots, n_k respectively and $n_1 + \dots + n_k = N$. Then the equation (2.13) is equivalent to

$$(2.17) \quad \Delta w = -\frac{2q^2}{\hbar\mu} e^w + 4\pi \sum_{j=1}^k n_j \delta_{p_j}$$

with the nontopological boundary condition: $w(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. This is the well-known Liouville equation with singular sources. If we find a solution w to (2.17), then we can recover a from (2.15) by the formula

$$(2.18) \quad a_1 + ia_2 = -\frac{2\hbar i}{q} \bar{\partial} \ln \psi.$$

Hence (2.15) is equivalent to (2.17) via the relations (2.16) and (2.18)

We now consider the problem of justifying the nonrelativistic limit for the self-dual equations, that is, proving that the solutions of (2.5) and (2.6) converge to solutions of (2.15) as $c \rightarrow \infty$. This problem is equivalent to showing that the solutions of (2.9) converge to solutions of (2.17) as $c \rightarrow \infty$. If we set $v = u + \ln 2m$, then (2.9) becomes

$$(2.19) \quad \begin{aligned} \Delta v &= \frac{q^4}{m^2 \kappa^2 c^4} e^v (e^v - 2m\sigma^2) + 4\pi \sum_{j=1}^k n_j \delta_{p_j} \\ &= \frac{q^4}{m^2 \kappa^2 c^4} e^{2v} - \frac{2q^2}{\hbar\mu} e^v + 4\pi \sum_{j=1}^k n_j \delta_{p_j}. \end{aligned}$$

Letting $c \rightarrow \infty$, we may derive formally that v converges to a solution of (2.17). However, it is known that there are infinitely many nontopological solutions of (2.19). See [1, 2, 4, 13] for example. Therefore it is not surprising that there may be a sequence of solutions to (2.19) which blows up as $c \rightarrow \infty$ instead of converging to a solution to (2.17). In this point of view, it is important to find a sequence of solutions to (2.19) converging to a solution to (2.17). This

means we need a kind of conditions for solutions of (2.19) to make sure the convergence in the limit $c \rightarrow \infty$. In this paper, we will find such a condition when there is only one-vortex point. In fact, if we choose a sequence of radial solutions to (2.19) with a common shooting constant, then it converges to a solution to (2.17). In the next section, we state the main theorems of this paper and in section 4, their proofs are given.

3. Main Theorem

In this section, we assume that there is only one-vortex point. In this case, under the nontopological boundary condition, the equations (2.17) and (2.19) are rewritten as

$$(3.1) \quad \begin{cases} \Delta w = -\frac{2q^2}{\hbar\mu} e^w + 4\pi N \delta_0, \\ w \rightarrow -\infty \text{ as } |x| \rightarrow \infty, \end{cases}$$

and

$$(3.2) \quad \begin{cases} \Delta v = \frac{q^4}{m^2 \kappa^2 c^4} e^v (e^v - 2m\sigma^2) + 4\pi N \delta_0, \\ v \rightarrow -\infty \text{ as } |x| \rightarrow \infty. \end{cases}$$

We observe that the equation (3.1) is the Liouville equation with a singular source at the origin. We are interested in the radial solutions to (3.1) and (3.2).

To investigate (3.1) and (3.2) further, we first transform the equation (3.1) into

$$(3.3) \quad w_{rr} + \frac{1}{r} w_r = -\frac{2q^2}{\hbar\mu} e^w, \quad r = |x| > 0$$

with the constraint

$$(3.4) \quad \lim_{r \rightarrow 0} \frac{w(r)}{\ln r} = \lim_{r \rightarrow 0} r w_r(r) = 2N, \quad \lim_{r \rightarrow \infty} w(r) = -\infty.$$

It follows from the result of [3, 12] that every radial solution to (3.3) and (3.4) is of the form

$$(3.5) \quad w(r) = \ln \frac{8\lambda(N+1)^2 r^{2N}}{(\lambda + r^{2N+2})^2} - \ln \frac{2q^2}{\hbar\mu},$$

where λ is any positive constant.

Similarly, for (3.2) we get

$$(3.6) \quad v_{rr} + \frac{1}{r} v_r = \frac{q^4}{m^2 \kappa^2 c^4} e^v (e^v - 2m\sigma^2) =: g(v)$$

with the constraints

$$(3.7) \quad \lim_{r \rightarrow 0} \frac{v(r)}{\ln r} = \lim_{r \rightarrow 0} r v_r(r) = 2N$$

and

$$(3.8) \quad \lim_{r \rightarrow \infty} v(r) = -\infty.$$

It is not difficult to see that v is a solution to (3.6) and (3.7) if and only if there exists a constant a satisfying

$$(3.9) \quad v(r) = a + 2N \ln r + \int_0^r \frac{1}{s} \int_0^s \tau g(v(\tau)) d\tau ds.$$

See [4] for instance. Let us denote this solution by $v(r; a)$.

In order to classify the solutions to (3.6) and (3.7), let us keep all the constants appearing in $g(v)$ to be fixed and decompose the set of real numbers as $\mathbb{R} = \mathcal{A}^+ \cup \mathcal{A}^0 \cup \mathcal{A}^-$, where

$$\begin{aligned} \mathcal{A}^+ &= \{a | v(r_0; a) > \ln(2m\sigma^2) \ \exists r_0 > 0\}, \\ \mathcal{A}^0 &= \{a | v(r; a) \leq \ln(2m\sigma^2) \ \forall r > 0 \text{ and } v_r(r; a) \geq 0 \ \forall r > 0\}, \\ \mathcal{A}^- &= \{a | v(r; a) \leq \ln(2m\sigma^2) \ \forall r > 0 \text{ and } v_r(r_0; a) < 0 \ \exists r_0 > 0\}. \end{aligned}$$

Then, we have the following.

Theorem 3.1 ([4]). (i) *There exists a constant $a_* \in \mathbb{R}$ such that*

$$\mathcal{A}^+ = (a_*, \infty), \quad \mathcal{A}^0 = \{a_*\}, \quad \mathcal{A}^- = (-\infty, a_*).$$

(ii) *For $a \in \mathcal{A}^+$, let $r_1 = r_1(a)$ be the first r satisfying $v(r; a) = \ln(2m\sigma^2)$. Then $v_r(r; a) > 0$ for $0 < r \leq r_1$.*

(iii) *If $a = a_*$, then $v(\infty) = \ln(2m\sigma^2)$ and $v_r(r; a) > 0$ for all $r > 0$.*

(iv) *$v(r; a)$ is a solution to (3.6)-(3.8) if and only if $a \in \mathcal{A}^-$. In this case, $v(r; a) < \ln(2m\sigma^2)$ for all $r > 0$. Moreover, v has a unique maximum point r_0 such that $v_r(r; a) > 0$ for $r < r_0$ and $v_r(r; a) < 0$ for $r > r_0$.*

Theorem 3.1 says that for each c there are infinitely many nontopological solutions to (3.6)-(3.8). Thus, as mentioned at the end of the previous section, if we choose an arbitrary sequence of solutions as $c \rightarrow \infty$, then it may diverge. See the Remark 4.3 in the next section for an example. Hence, we need an additional condition to make sure the convergence of solutions in the nonrelativistic limit. We will achieve the convergence of solutions by keeping the shooting constant a of solutions when we take the limit $c \rightarrow \infty$.

We proceed in the proof of nonrelativistic limit for the solutions given by Theorem 3.1. To this aim, let us vary the constant c and adopt the relation (2.11). Denote the solutions to (3.6)-(3.8) by $v(r, c; a)$ for $a < a_* = a_*(c)$. Let us begin with the following lemma.

Lemma 3.2. $a_*(c) \rightarrow \infty$ as $c \rightarrow \infty$.

Proof. Suppose that $a \in \mathcal{A}^+$. Let $r_1 = r_1(a, c)$ be the first r satisfying $v(r, c; a) = \ln(2m\sigma^2)$. Then $v_r(r, c; a) > 0$ for $0 < r \leq r_1$ by Theorem 3.1.

Let $v(r_2, c; a) = \ln(m\sigma^2)$ and $v(r_3, c; a) = \ln(m\sigma^2) - \ln 2$. Then $r_3 < r_2 < r_1$. Since $g(v(r, c; a)) < 0$ for $0 < r < r_1$, it comes from (3.9) that

$$(3.10) \quad a \geq \ln(m\sigma^2) - \ln 2 - 2N \ln r_3 = 2 \ln c + \ln \frac{m^2 \mu}{2\hbar q^2} - 2N \ln r_3.$$

Since $rv_r < 2N$ for $0 < r < r_1$, it follows that

$$\ln 2 = v(r_2, c; a) - v(r_3, c; a) < \int_{r_3}^{r_2} \frac{2N}{r} dr = 2N \ln \frac{r_2}{r_3}.$$

Namely, $r_3 < 2^{1/2N} r_3 < r_2$. Since $g(v)$ is decreasing on $\ln(m\sigma^2) - \ln 2 \leq v \leq \ln(m\sigma^2)$, we have

$$\begin{aligned} 0 < r_2 v_r(r_2, c; a) &= r_3 v_r(r_3, c; a) + \int_{r_3}^{r_2} r g(v) dr \\ &< 2N + g(v(r_3, c; a)) \int_{r_3}^{2^{1/2N} r_3} r dr \\ &= 2N - \frac{3q^4 \sigma^4}{8\kappa^2 c^4} (2^{1/N} - 1) r_3^2. \end{aligned}$$

Thus by the relation (2.11),

$$-2 \ln r_3 \geq 2 \ln c + \ln \frac{(2^{1/N} - 1)}{2N} + \ln \frac{3m^2}{8\hbar^2},$$

which yields together with (3.10) that for any $a \in \mathcal{A}^+$

$$a > 2(N + 1) \ln c + N \ln \frac{(2^{1/N} - 1)}{2N} + N \ln \frac{3m^2}{8\hbar^2} + \ln \frac{m^2 \mu}{2\hbar q^2}.$$

Consequently,

$$a_*(c) \geq 2(N + 1) \ln c + N \ln \frac{(2^{1/N} - 1)}{2N} + N \ln \frac{3m^2}{8\hbar^2} + \ln \frac{m^2 \mu}{2\hbar q^2},$$

and the proof is completed. □

For any fixed $a \in \mathbb{R}$, Lemma 3.2 implies that there exists a solution $v(r, c; a)$ to (3.6)-(3.8) for all sufficiently large c . The following theorem is our first main result.

Theorem 3.3. *Let $\hbar, q, \mu, m > 0$ be fixed, N be a positive integer, and $a \in \mathbb{R}$ be given. Let $v(r, c; a)$ be a solution to (3.6)-(3.8) which is given by the integral formula (3.9). Then, as $c \rightarrow \infty$, $v(r, c; a)$ converges to $w(r)$ which is a solution to (3.3) and (3.4). The function $w(r)$ is explicitly given by (3.5) with $\lambda = \lambda(a)$ defined by*

$$(3.11) \quad \lambda(a) = 4(N + 1)^2 \hbar \mu q^{-2} e^{-a}.$$

Moreover, if we set

$$\tilde{v}(r, c; a) = v(r, c; a) - 2N \ln r, \quad \tilde{w}(r) = w(r) - 2N \ln r,$$

then for any nonnegative integers k

$$(3.12) \quad \|\tilde{v} - \tilde{w}\|_{C^k(B_R)} \leq C_{k,R} c^{-2}$$

as $c \rightarrow \infty$, where B_R is the ball of radius R centered at the origin.

Remark 3.4. We observe that λ is a decreasing function of a in (3.11), which implies that Theorem 3.3 completely characterizes the nonrelativistic limit for radial solutions of a one-vortex case. In other words, for each solution $w(r)$ to the nonrelativistic equations (3.3) and (3.4), we can find one parameter family of solutions $v(r, c; a)$ to the relativistic equations (3.6)-(3.8) such that $v(r, c; a) \rightarrow w(r)$ as $c \rightarrow \infty$. Indeed, $w(r)$ is determined by λ via (3.5) and the corresponding $v(r, c; a)$ converging to $w(r)$ can be realized by the common shooting constant a given by (3.11).

We give the proof of Theorem 3.3 in the next section. Let us keep the notations in Theorem 3.3. Given a smooth function $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}$, let

$$(3.13) \quad \begin{aligned} \phi(z, c; a) &= \frac{1}{\sqrt{2m}} z^N \exp\left(\frac{1}{2} \tilde{v}(r, c; a) + i\chi\right), \\ A_1(z, c; a) + iA_2(z, c; a) &= -\frac{2c\hbar i}{q} \bar{\partial} \ln \phi(z, c; a), \end{aligned}$$

where $z = x_1 + ix_2$ and $r = |z|$. We note that for a solution pair (ϕ, A_1, A_2) to the self-dual equations (2.5) and (2.6), if ϕ has only one zero at the origin of multiplicity N and $|\phi|$ is radially symmetric about the origin, then (ϕ, A_1, A_2) is given by the form (3.13). Similarly, let

$$(3.14) \quad \begin{aligned} \psi(z) &= z^N \exp\left(\frac{1}{2} \tilde{w}(r) + i\chi\right), \\ a_1(z) + ia_2(z) &= -\frac{2\hbar i}{q} \bar{\partial} \ln \psi(z) \end{aligned}$$

be a solution to (2.15). Now we are in a position to state the second main result of this paper.

Theorem 3.5. *Under the notations as above, it holds that for each nonnegative integer k*

$$(3.15) \quad \left\| \phi(z, c; a) - \frac{1}{\sqrt{2m}} \psi(z) \right\|_{C^k(B_R)}, \quad \left\| \frac{A_j(z, c; a)}{c} - a_j(z) \right\|_{C^k(B_R)} \leq C_{k,R} c^{-2}$$

as $c \rightarrow \infty$. Moreover, for the magnetic fluxes (2.4) and (2.14) we have

$$(3.16) \quad \Phi^R \rightarrow \Phi^{NR} = \frac{4\pi\hbar(N+1)}{q},$$

as $c \rightarrow \infty$.

The proof of this theorem is given in the next section.

4. Proof of Theorem 3.3 and Theorem 3.5

This section is devoted to the proof of Theorem 3.3. Throughout this section, let $a \in \mathbb{R}$ be fixed. By Lemma 3.2, if c is large enough, there exists one parameter family of solutions $v(r, c; a)$ to (3.6)-(3.8). It is easily shown that if v is a solution to (3.6)-(3.8), then $v < \ln(2m\sigma^2)$ by the maximum principle.

Lemma 4.1. *Let $\alpha(c)$ be the maximum value of $v(r, c; a)$. Then there exists a constant $\varepsilon > 0$ independent of c such that*

$$\alpha(c) \leq \ln(2m\sigma^2) - \varepsilon$$

as $c \rightarrow \infty$.

Proof. Assume the contrary. Then there would be a sequence $c_n \rightarrow \infty$ such that $\ln(2m\sigma_n^2) - \alpha_n \rightarrow 0$, where $\sigma_n = c_n \sqrt{m\mu/\hbar q^2}$ and $\alpha_n = \alpha(c_n)$. Let $v_n(r) = v(r, c_n; a)$, $\alpha_n = v_n(t_n)$, and $r_n > 0$ such that $v_n(r_n) = \ln(m\sigma_n^2)$. Since $\alpha_n > \ln(m\sigma_n^2)$, we have $r_n < t_n$. Since $g(v_n) < 0$ for $0 < r < t_n$, it comes from (3.9) that

$$(4.1) \quad r_n \geq (m\sigma_n^2)^{1/2N} \exp(-a/2N).$$

On the other hand, integrating $(r(v_n)_r)_r = rg(v_n)$ on $(0, t_n)$, we get

$$\begin{aligned} 2N &= - \int_0^{t_n} rg(v_n)dr \geq - \int_{r_n}^{t_n} \frac{r^2}{r(v_n)_r} g(v_n)(v_n)_r dr \\ &\geq - \frac{r_n^2}{2N} \int_{\ln(m\sigma_n^2)}^{\alpha_n} g(v)dv, \end{aligned}$$

where the last inequality follows from the fact that $r(v_n)_r < 2N$ for $0 < r < t_n$. Hence,

$$4N^2 \geq \frac{q^4 r_n^2}{m^2 \kappa_n^2 c_n^4} \left(2m\sigma_n^2 e^{\alpha_n} - \frac{1}{2} e^{2\alpha_n} - \frac{3}{2} m^2 \sigma_n^4 \right).$$

Here, $\kappa_n = \mu/c_n$. Since $\ln(2m\sigma_n^2) - \alpha_n \rightarrow 0$, if c_n is large enough, then

$$\begin{aligned} 4N^2 &\geq \frac{q^4 r_n^2}{m^2 \kappa_n^2 c_n^4} \cdot \frac{1}{4} m^2 \sigma_n^4 = \frac{1}{4\hbar^2} m^2 c_n^2 r_n^2 \\ &\geq \frac{1}{4\hbar^2} m^2 \exp(-a/N) \left(\frac{m^2 \mu}{\hbar q^2} \right)^{1/N} \cdot c_n^{2+2/N} \rightarrow \infty, \end{aligned}$$

which is a contradiction. Here, the last inequality follows from (4.1). □

In the remaining part of this section, let $r_0 = r_0(c)$ be a unique maximum point of $v(r, c; a)$ and $\alpha = \alpha(c)$ be the maximum value of $v(r, c; a)$. Then

$$v(r_0, c; a) = \alpha < \ln(2m\sigma^2), \quad v_r(r_0, c; a) = 0.$$

Lemma 4.2. *As $c \rightarrow \infty$,*

$$(4.2) \quad a - C_1 e^{-a/(N+1)} \leq \alpha(c) \leq \frac{a}{N+1} + C_2,$$

and

$$(4.3) \quad \frac{\alpha - a}{N} \leq 2 \ln r_0(c) \leq -\alpha + C_3,$$

where C_j 's are independent of a and c .

Proof. Since $g(v) < 0$, it follows from (3.9) that

$$(4.4) \quad \alpha < a + 2N \ln r_0.$$

To estimate $\ln r_0$, let us choose a constant $\delta > 0$ such that $(1 - \delta)^{2N} - e^{-\epsilon} > 0$. Integrating (3.6), we have

$$v((1 - \delta)r_0, c; a) = \alpha + 2N \ln(1 - \delta) - \int_{(1-\delta)r_0}^{r_0} \frac{1}{s} \int_0^s \tau g(v) \, d\tau ds,$$

which implies that

$$\alpha + 2N \ln(1 - \delta) \leq v(r, c; a) \leq \alpha, \quad \forall r \in [(1 - \delta)r_0, r_0].$$

This leads us by Lemma 4.1 that

$$\begin{aligned} 2N &= - \int_0^{r_0} r g(v) \, dr \\ &\geq \int_{(1-\delta)r_0}^{r_0} \frac{q^4}{m^2 \kappa^2 c^4} (2m\sigma^2 e^{\alpha+2N \ln(1-\delta)} - e^{2\alpha}) r \, dr \\ &\geq \frac{2q^2}{\hbar\mu} e^\alpha \cdot \frac{(1 - (1 - \delta)^2)r_0^2}{2} \cdot ((1 - \delta)^{2N} - e^{-\epsilon}). \end{aligned}$$

Thus

$$(4.5) \quad 2 \ln r_0 \leq -\alpha + C,$$

where C is independent of c . This inequality together with (4.4) shows the second inequality of (4.2).

On the other hand, from(3.9),

$$\begin{aligned} \alpha &\geq v(1, c; a) \geq a - \int_0^1 \frac{1}{s} \int_0^s \frac{2q^4 m \sigma^2}{m^2 \kappa^2 c^4} e^{v\tau} \tau \, d\tau ds \\ &\geq a - \frac{2q^2}{\hbar\mu} e^\alpha \int_0^1 \frac{1}{s} \int_0^s \tau \, d\tau ds \geq a - C e^{-a/(N+1)}, \end{aligned}$$

where the last inequality follows from the second inequality of (4.2). This establishes the first inequality of (4.2). Finally, (4.3) comes from (4.4) and (4.5). □

Remark 4.3. Given a sequence $c_n \rightarrow \infty$, if we choose a sequence $a_n \rightarrow -\infty$, then $\alpha(c_n) \rightarrow -\infty$. This means the solutions $v(r, c_n, ; a_n) \rightarrow -\infty$, that is, the sequence $v(r, c_n, ; a_n)$ diverges. As pointed out in the previous section, an arbitrary sequence of solutions may diverge when $c \rightarrow \infty$.

Proof of Theorem 3.3. Let c_n be an arbitrary sequence such that $c_n \rightarrow \infty$. Set $\sigma_n = c_n \sqrt{m\mu/\hbar q^2}$, $\kappa_n = \mu/c_n$, $v_n(r) = v(r, c_n; a)$, $\alpha_n = \sup v_n(r)$, and $t_n = r_0(c_n)$. Hence $\alpha_n = v_n(t_n)$. We note that

$$|g(v_n)| \leq \frac{q^4}{m^2 \kappa_n^2 c_n^4} (e^{2\alpha_n} + 2m\sigma_n^2 e^{\alpha_n}) = \frac{q^4}{m^2 \mu^2 c_n^2} e^{2\alpha_n} + \frac{2q^2}{\hbar \mu} e^{\alpha_n},$$

which means by Lemma 4.2 that g is uniformly bounded for all r as $c_n \rightarrow \infty$. Therefore for any given $R > 0$, if $0 \leq r \leq R$, then

$$|\tilde{v}_n(r)| \leq |a| + \left| \int_0^r \frac{1}{s} \int_0^s \tau g(v_n(\tau)) d\tau ds \right| \leq C_R$$

for some constant C_R dependent only on R . Thus we have $\sup_{[0, R]} |\tilde{v}_n(r)| \leq C_R$. Since

$$r(\tilde{v}_n)_r = \int_0^r s g(v_n(s)) ds,$$

we obtain $\sup_{[0, R]} |r(\tilde{v}_n)_r| \leq C_R$. As a consequence, we see that $\sup_{B_R} |\nabla \tilde{v}| \leq C_R$ and thus $\|\tilde{v}_n\|_{C^\beta(B_R)} \leq C_{R, \beta}$ for $0 < \beta < 1$. Since

$$\|\Delta \tilde{v}_n\|_{C^\beta(B_R)} = \|g(v_n)\|_{C^\beta(B_R)} \leq C_{R, \beta},$$

we conclude that

$$\|\tilde{v}_n\|_{C^{2, \beta}(B_R)} \leq C_{R, \beta}.$$

This implies that there exist a subsequence, denoted by the same notation, \tilde{v}_n and a function $\tilde{w} \in C^{2, \beta}(B_R)$ such that $\tilde{v}_n \rightarrow \tilde{w} \in C^{1, \beta}(B_R)$ for any $\beta \in (0, 1)$ as $c_n \rightarrow \infty$. It follows from the bootstrap argument that $\tilde{v}_n \rightarrow \tilde{w}$ in $C^k(B_R)$ for all nonnegative integers k . Moreover, we may assume by Lemma 4.2 that $\alpha_n \rightarrow \alpha_*$ and $t_n \rightarrow t_*$.

Now let us determine the function \tilde{w} . Since

$$g(v_n) = \frac{q^4}{m^2 \mu^2 c_n^2} r^{2N} e^{2\tilde{v}_n} - \frac{2q^2}{\hbar \mu} r^{2N} e^{\tilde{v}_n} \rightarrow -\frac{2q^2}{\hbar \mu} r^{2N} e^{\tilde{w}},$$

$w(r) = \tilde{w}(r) + 2N \ln r$ satisfies

$$w(r) = a + 2N \ln r + \int_0^r \frac{1}{s} \int_0^s \tau \left(\frac{-2q^2}{\hbar \mu} \right) e^{w(\tau)} d\tau ds, \quad r > 0,$$

which yields (3.3) with

$$\alpha_* = w(t_*) = \sup_{\mathbb{R}^2} w_*.$$

Next, we verify that w satisfies (3.4). Integrating (3.6) on $(0, t_n)$, we get

$$-2N = \int_0^{t_n} \frac{q^4}{m^2 \kappa_n^2 c_n^4} r e^{v_n} (e^{v_n} - 2m\sigma_n^2) dr.$$

Taking the limit, we obtain

$$2N = \int_0^{t_*} \frac{2q^2}{\hbar \mu} r e^w dr,$$

which implies $\lim_{r \rightarrow 0} r w_r = 2N$. Since $w_r < 0$ for $r > r_*$, there exists

$$\gamma = \inf_{r > t_*} w(r) = \lim_{r \rightarrow \infty} w(r) \geq -\infty.$$

If $\gamma > -\infty$, we arrive at a contradiction:

$$\infty > \lim_{r \rightarrow \infty} |w(r)| \geq -|\alpha_*| + \frac{2q^2 e^\gamma}{\hbar \mu} \lim_{r \rightarrow \infty} \int_{t_*}^r \frac{1}{s} \int_{t_*}^s \tau d\tau ds = \infty.$$

Hence $w(\infty) = -\infty$ and (3.4) is proved. As a consequence, w is a radial solution to (3.3) and (3.4). Therefore, there exists $\lambda > 0$ such that w is of the form (3.5). Since

$$\tilde{w}(0) = \lim_{n \rightarrow \infty} \tilde{v}_n(0) = a,$$

we arrive at (3.11).

On the other hand, the uniqueness of w implies that the convergence holds true for the whole sequence c_n . Since $\{c_n\}$ was an arbitrary sequence, we conclude that $\tilde{v}(r, c) \rightarrow \tilde{w}(r)$ as $c \rightarrow \infty$ in $C^k(B_R)$ for any $R > 0$.

It remains to show (3.12). For simplicity, we write $v(r) = v(r, c; a)$, $\alpha = \alpha(c)$, and so on. We first consider the case $k = 0$. Let $R > 0$. For $0 \leq r \leq R$,

$$\begin{aligned} |\tilde{v}(r) - \tilde{w}(r)| &\leq \frac{q^4}{m^2 \mu^2 c^2} e^{2\alpha} + \left| \frac{2q^2}{\hbar \mu} \int_0^r \frac{1}{s} \int_0^s (e^{\tilde{v}} - e^{\tilde{w}}) \tau^{2N+1} d\tau ds \right| \\ &\leq C \cdot c^{-2} + C \sup_{[0, R]} |\tilde{v} - \tilde{w}| \int_0^r \frac{1}{s} \int_0^s \tau^{2N+1} d\tau ds. \end{aligned}$$

Thus for sufficiently small $R > 0$, we have $\sup_{[0, R]} |\tilde{v} - \tilde{w}| \leq C_R \cdot c^{-2}$. Let

$$R_0 = \sup\{R > 0 : (3.12) \text{ holds for } k = 0\}.$$

Suppose that $R_0 < \infty$. For $R > R_0$, if $R_0 < r < R$, then

$$\begin{aligned} |\tilde{v}(r) - \tilde{w}(r)| &\leq |\tilde{v}(R_0) - \tilde{w}(R_0)| + |(\tilde{v}(r) - \tilde{v}(R_0)) - (\tilde{w}(r) - \tilde{w}(R_0))| \\ &\leq C \cdot c^{-2} + C \sup_{[R_0, R]} |\tilde{v} - \tilde{w}| \int_{R_0}^r \frac{1}{s} \int_0^s \tau^{2N+1} d\tau ds. \\ &\leq C \cdot c^{-2} + C(R^{2N+2} - R_0^{2N+2}) \sup_{[R_0, R]} |\tilde{v} - \tilde{w}|. \end{aligned}$$

Thus if R is close enough to R_0 , then $\sup_{[R_0, R]} |\tilde{v} - \tilde{w}| \leq C \cdot c^{-2}$, which is a contradiction. Hence $R_0 = \infty$ and (3.12) is achieved for $k = 0$.

The general cases are given by induction. We note that

$$\Delta(\tilde{v} - \tilde{w}) = \frac{q^4}{m^2 \mu^2 c^2} r^{2N} e^{2\tilde{v}} - \frac{2q^2}{\hbar \mu} r^{2N} (e^{\tilde{v}} - e^{\tilde{w}}) =: G(\tilde{v}, \tilde{w}).$$

Suppose that $\|\partial^l(\tilde{v} - \tilde{w})\|_{C^0(B_R)} \leq C_{k,R} c^{-2}$ for $0 \leq l \leq k$. Then, it is obvious that

$$\|\partial^k G(\tilde{v}, \tilde{w})\|_{C^0(B_R)} \leq C_{k,R} c^{-2}.$$

Since $\Delta \partial^k(\tilde{v} - \tilde{w}) = \partial^k G(\tilde{v}, \tilde{w})$, the standard elliptic regularity implies that

$$\|\partial^{k+1}(\tilde{v} - \tilde{w})\|_{C^0(B_R)} \leq C_{k,R} c^{-2}.$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.5. The asymptotic behavior (3.15) is an immediate consequence of Theorem 3.3. For the proof of (3.16), we note that by (3.5)

$$\lim_{r \rightarrow \infty} r w_r = -2N - 4 \leq -6$$

and

$$\Phi^{NR} = \frac{q}{\mu} \int_{\mathbb{R}^2} e^w dx = \frac{4\pi\hbar(N+1)}{q}.$$

It is proved in [4, 13] that

$$\lim_{r \rightarrow \infty} r v_r(r, c, ; a) < -2N - 4.$$

Let $r_1 = r_1(c)$ and r_2 be numbers such that $r_1 v_r(r_1, c; a) = -3$ and $r_2 w_r(r_2) = -4$. Since

$$r_2 w_r(r_2) = \lim_{c \rightarrow \infty} r_2 v_r(r_2, c, ; a),$$

we get $r_2 v_r(r_2, c, ; a) < -3$ for all large c . Since $r v_r$ is decreasing, we obtain $r_1 < r_2$. This means that $r_1(c)$ is bounded in the limit $c \rightarrow \infty$.

Let us write $v(r) = v(r, c, ; a)$ for simplicity. Now for $r_1 \leq r < \infty$, we have $r v_r \leq -3$ and hence for $r \geq r_1$

$$e^{v(r)} \leq e^{v(r_1)} \left(\frac{r_1}{r}\right)^3 \leq e^\alpha \left(\frac{r_1}{r}\right)^3.$$

This implies that as $c \rightarrow \infty$,

$$\int_0^\infty r e^v dr \leq r_1^2 e^\alpha + \int_{r_1}^\infty r_1^3 e^\alpha r^{-2} dr = 2r_1^2 e^\alpha \leq C.$$

Thus, it follows from the Lebesgue Convergence Theorem that

$$\int_0^\infty r e^v dr \rightarrow \int_0^\infty r e^w dr,$$

as $c \rightarrow \infty$. Furthermore,

$$\int_0^\infty r e^{2v} dr \leq e^\alpha \int_0^\infty r e^v dr \leq C.$$

In the sequel, it follows from (2.6) that

$$\Phi^R = \frac{2\pi q}{\mu} \int_0^\infty r e^v dr - \frac{\pi\hbar q^3}{m^2 \mu^2 c^2} \int_0^\infty r e^{2v} dr \rightarrow \frac{2\pi q}{\mu} \int_0^\infty r e^w dr = \Phi^{NR},$$

which finishes the proof. □

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